KOSZUL DUALITY AND VERDIER DUALITY

JEFF GIANSIRACUSA

<u>REF</u>: LAZAREV – VORONOV QA/070231

OPERAD $\mathcal{O}$ $\rightsquigarrow$ SHEAF $\mathcal{F}^{\mathcal{O}}$ ON THE MODULI SPACE OF $\overset{\text{METRIC}}{\text{THE}}$

CYCLIC OPERAD $\rightsquigarrow$ SHEAF ON THE MODULI SPACE OF GRAPHS

MODULAR OPERAD $\rightsquigarrow$ ———————— STABLE GRAPHS

(AS IN JOHANNES'S TALK – WITH GENUS LABELS, CAN COLLAPSE LOOPS)

"<u>THM</u>: KOZSUL DUALITY CORRESPONDS TO VERDIER DUALITY" (TWISTED)

WORK IN THE CAT OF CHAIN CPX OVER A FIELD OF CHAR. 0 (BOUNDED).

<u>LINEAR ALGEBRA</u>: $V_0 \leftarrow V_1 \leftarrow V_2$ – – – CHAIN CPX

$\longrightarrow \quad \longrightarrow$ COCHAIN CPX

$V[i]$ = "SHIFT TO THE LEFT BY i"

$V[i]_n = V_{i+n}$

$V \mapsto V^\vee$ LINEAR DUAL TERMWISE CHAINS $\rightleftarrows$ COCHAINS

$V \mapsto V^*$ + REVERSING GRADING ———— CHAINS ⟲ ⟲ COCHAINS

V A VECTOR SPACE OF dim n

$\mathrm{Det}(V) = \Lambda^n V[-n]$ 1-dim ~~[illegible]~~

$\mathrm{Det}^{-\ell}(V) = (\mathrm{Det}(V)^*)^{\otimes \ell}$

① $\bigotimes_{i \in I} V_i[-1] \cong \mathrm{Det}(I) \otimes \bigotimes_{i \in I}^{\mathbb{F}^I} V_i$

② $\mathrm{Det}^2(I) \cong \mathbb{F}[-2|I|]$

DEF: $\mathcal{F}$ A SHEAF ON A SIMPLICIAL CPX X IS CALLED CONSTRUCTIBLE IF THE RESTRICTION TO A NBHD OF EACH OPEN FACE OF A SIMPLEX IS CONSTANT.

σ A SIMPLEX, $st(\sigma)$ = OPEN STAR OF σ

FACT: IF $\mathcal{F}$ HAS CONSTRUCTIBLE COHOMOLOGY, THEN IN $D^b(X)$, $\mathcal{F} \cong$ A CONSTRUCTIBLE SHEAF.

i.e. q-ISO

↳ SHEAVES OF BOUNDED CHAIN CPX'S, q-ISO'S INVERTED.

PROP: $\mathcal{F}$ CONSTRUCTIBLE ON X. THEN

$\mathcal{F} : \sigma \xmapsto{\text{CONTRAVARIANT}} \mathcal{F}(st(\sigma))$

POSET
(SIMPLICES, RESTR.)
IIS
(OPEN STARS, INCLUSIONS) $\longrightarrow$ (CHAIN CPX'S)

PROP: $\mathcal{F}$ CONSTRUCTIBLE, $R\Gamma(X, \mathcal{F}) \simeq \bigoplus_{\sigma \in X} \underset{\mathcal{F}(st(\sigma))}{\overset{\parallel}{\mathcal{F}_\sigma}} \otimes \underset{\mathrm{Det}(\mathbb{F}^{\mathrm{VERT}(\sigma)})}{\overset{\parallel}{\mathrm{Det}(\sigma)}}[1]$

deg 0: $\bigoplus_v \mathcal{F}(st(v))$ $\qquad \bigoplus_{\text{EDGES}} \mathcal{F}(st(e))$

TAKES CARE OF THE DEGREE AND THE SIGN IN $\mathcal{F}_\sigma \xrightarrow{\text{res}} \mathcal{F}_{\text{FACE}(\sigma)}$

$\mathrm{Det}(\sigma) \xrightarrow[\text{EVEN OR ODD FACE}]{\pm 1} \mathrm{Det}(\text{FACE } \sigma)$

PROOF: THIS IS THE ČECH CPX FOR THE COVERING $\{St(v)\}$

VERDIER DUALITY

$X \xrightarrow{f} Y$

$$\mathcal{F} \in \begin{tikzcd} Sh(X) \arrow[r, bend left, "f_* \text{ DIRECT IMAGE}"] \arrow[r, "f_! \text{ PROPER DIR. IM.}"'] & Sh(Y) \arrow[l, bend left, "f^{-1} \text{ INVERSE IMAGE}"] \end{tikzcd} \ni \mathcal{G}$$

$f^{-1}\mathcal{G}(U) = \varinjlim_{V \supset f(U)} \mathcal{G}(V)$ $\qquad (f^{-1}, f_*)$ ADJOINT PAIR.

$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(V))$

$f_!\mathcal{F}(V) = \{ s \in \mathcal{F}(f^{-1}(V)) \text{ s.t. } f|_{supp(s)} : supp(s) \to Y \text{ IS PROPER} \}$

IN THE DERIVED CATEGORY, $Rf_!$ HAS A RIGHT ADJOINT, $f^!$
[VERDIER DUALITY]

CONSTRUCTION FOR $f: X \to pt$, $f^!: D^b(pt) \to D^b(X)$

$\mathcal{G} \in$

$f^!(\mathcal{G})$ = sheafification of $U \longmapsto Hom_{\mathbb{F}}(\Gamma_c(U, \tilde{\mathbb{F}}_X), \tilde{\mathcal{G}})$

COMPACTLY SUPPORTED

SOFT AND FLAT RESOLUTION OF CONSTANT SHEAF $\mathbb{F}_X$ ON X

INJECTI CPX REPLACI $\mathcal{G}$

$\mathbb{F}$ CST SHEAF ON A pt

$f^!\mathbb{F} = U \longmapsto \Gamma_c(U, \tilde{\mathbb{F}}_X)^*$

ω_X = THE DUALIZING SHEAF

IF X MFD, $\tilde{\mathbb{F}}_X$ = DE RHAM CPX

DEF: $\mathcal{F} \longmapsto$ VERDIER DUAL $D\mathcal{F} := R\mathcal{H}om(\mathcal{F}, \omega_X)$

PROPERTIES: ① $\mathcal{F}$ WITH CONSTRUCTIBLE COHOMOLOGY
$D\mathcal{F}$ DOES TOO. $D^2(\mathcal{F}) \cong \mathcal{F}$

② $D(Rf_*\mathcal{F}) \cong Rf_! D\mathcal{F}$

③ $R\Gamma_c(D\mathcal{F}) \cong R\Gamma(\mathcal{F})^*$

$$H^*_c(X, D\mathcal{F}) \cong \left(H^*(X, \mathcal{F})\right)^* \sim \text{POINCARÉ DUALITY}$$

ω_X ON X^n MFLDS
$\shortparallel$
$\mathcal{L}_{or}[n]$
ORIENTATION

④ $f : U_{\text{OPEN}} \hookrightarrow X$, $f^! = f^{-1}$

PROP: $\mathcal{F}$ CONSTRUCTIBLE SHEAF $D\mathcal{F}$ IS REPRESENTED BY THE SHEAF

$\longleftrightarrow$ FUNCTOR

$$\sigma \mapsto \bigoplus_{\tau \supset \sigma} \mathcal{F}_\tau^* \otimes \underbrace{\mathrm{Det}^{-1}(\tau)[-1]}_{\deg = -\dim \tau}$$

PROOF: $i : st(\sigma) \hookrightarrow \overline{st(\sigma)}$

$$R\Gamma_c(st(\sigma), \mathcal{F}) \cong R\Gamma_c(\overline{st(\sigma)}, i_! \mathcal{F}|_{st(\sigma)})$$

$$\cong R\Gamma(\overline{st(\sigma)}, i_! \mathcal{F}|_{st(\sigma)})$$

$R\Gamma(D\mathcal{F})^* \cong R\Gamma_c(\mathcal{F})$

$$\tau \longmapsto \begin{cases} \mathcal{F}_\tau & \text{if } \tau \supset \sigma \\ 0 & \text{OTHERWISE} \end{cases}$$

CYCLIC $\mathcal{O}$ OPERAD $\rightsquigarrow$ $\mathcal{F}^{\mathcal{O}}$ ON $\mathcal{M}_{\text{GRAPHS}} = \coprod \mathcal{M}_{b_1}$

$\simeq \coprod B\,\mathrm{Out}(F_{b_1})$

σ SIMPLEX $\longleftrightarrow$ Γ_σ GRAPH

$\dim \sigma = \#$ EDGES

VALENCE OF v

$$\mathcal{F}^{\mathcal{O}}_\sigma = \mathcal{F}^{\mathcal{O}}(st(\sigma)) = (\Gamma_\sigma^{\mathcal{O}})^\vee = \bigotimes_{v \in \mathrm{Vert}(\Gamma_\sigma)} \mathcal{O}((H(v)))^\vee$$

GRAPH REPR σ

OF CYCLIC ENTRIES

TWISTING: $\mathcal{H} : \sigma \longrightarrow \mathcal{H}_\sigma = \underbrace{\mathrm{Det}^{-1} H_1 \Gamma_\sigma [-b_1]}_{\deg 0}$

THM: $D(\mathcal{F}^{\mathcal{O}}) \simeq \mathcal{F}^{D\mathcal{O}} \otimes \mathcal{H}[3b_1 - 4]$

KOZSUL DUAL OPERAD

PROOF: $$D\mathcal{O}((n)) = \bigoplus_{\substack{\text{UNROOTED} \\ n\text{-TREES } T}} \left(T^{s\mathcal{O}[-1]}\right)^*$$

SHIFT OF deg $n-2$

$s\mathcal{O}((n)) = \mathrm{Det}^{-1}(n)[-2] \otimes \mathcal{O}((n))$

$$(\mathcal{F}^{D\mathcal{O}} \otimes \mathcal{H})_\sigma = \bigotimes_{v \in V(\Gamma_\sigma)} \bigoplus_{H(v)\text{-TREES}} \left(T^{s\mathcal{O}[-1]}\right)^{*\vee} \otimes \mathrm{Det}^{-1} H_1 \Gamma_\sigma [-b_1]$$

$$= \bigoplus_{\tau \supset \sigma} \left(\Gamma_\tau^{s\mathcal{O}[-1]}\right)^{*\vee} \otimes \mathrm{Det}\ \_\_\_\_$$

$$\left(\Gamma_\tau^{s\mathcal{O}[-1]}\right)^{*\vee} = \bigotimes_{v \in \Gamma_\tau} \left(\mathrm{Det}^{-1} H(v)[-2] \otimes \mathcal{O}((H(v)))[1]\right)^{*\vee}$$

$$\simeq \bigotimes \mathrm{Det}\, H(v)[3] \otimes \mathcal{O}((H(v)))$$

$$\simeq \Gamma_\tau^{\mathcal{O}} \otimes \mathrm{Det}^{-3}(\mathrm{Vert}(\Gamma_\tau)) \otimes \bigotimes_v \mathrm{Det}(H(v))$$

$$\simeq \Gamma_\tau^{\mathcal{O}} \otimes \mathrm{Det}^{-1}(\mathrm{Vert}(\Gamma_\tau))[2|V|] \otimes \mathrm{Det}^2(\mathrm{Edges})$$