MODULAR OPERADS

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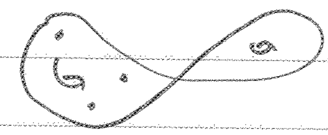
REF: GETZLER - KAPRANOV
MARKL - SHNIDER - STASHEFF

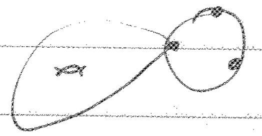
MAIN EXAMPLE: $\overline{\mathcal{M}}_{g,n}$ = MODULI SPACES OF n POINTED STABLE CURVES OF GENUS g

STABLE CURVE
WITH SMOOTH MARKED POINTS



COMPACT SINGULAR CPX 1-dim ~~RIEMANN~~ SURFACE, NBHD OF SINGULAR PT $\cong \{(z_1,z_2)\in\mathbb{C}^2 \mid z_1 z_2 = 0\}$
AS ANALYTIC SPACE

STABILITY CONDITION: $\mathrm{Aut}(C,P)$ FINITE $\Leftrightarrow$ ANY IRREDUCIBLE COMPONENT OF C WHICH IS $\cong \mathbb{P}^1$ HAS AT LEAST 3 MARKED POINTS OR SINGUL.



OR  HAS AT LEAST 1 ~~PT~~.

GENUS = GENUS OF THE "SMOOTHED" SURFACE

$\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$

$\overline{\mathcal{M}}_{g,n}$ IS A SMOOTH ORBIFOLD OF CPX-dim. $3g-3+n$. IT IS PROPER (i.e. "COMPACT")
$\partial\overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n} - \mathcal{M}_{g,n}$ IS A DIVISOR WITH NORMAL CROSSINGS.
codim 1 SUB-ORBIFOLD
COMPONENTS INTERSECT TRANSVERSALLY

S = FINITE SET $\quad |S| = n$

$\overline{\mathcal{M}}_{g,S}$ = MODULI SPACE OF st CURVES WITH PUNCTURES LABELLED BY S.

$S_0 \xrightarrow{f} S_1$ BIJECTION $\rightsquigarrow \overline{\mathcal{M}}_{g,S_0} \longrightarrow \overline{\mathcal{M}}_{g,S_1}$

$s \in S,\ t \in T$ $\quad \overline{\mathcal{M}}_{g,S} \times \overline{\mathcal{M}}_{h,T} \xrightarrow{s \circ t} \overline{\mathcal{M}}_{g+h,\, S \setminus \{s\} \cup T \setminus \{t\}}$

GLUING

$i \neq j \in S$

$$\xi_{ij} : \overline{\mathcal{M}}_{g,S} \longrightarrow \overline{\mathcal{M}}_{g+1,\, S - \{i,j\}}$$

6 OBVIOUS COMPATIBILITIES –

Def: A MODULAR OPERAD in $(\mathcal{C}, \otimes)$ SYMM. MONOIDAL CAT IS A COLLECTION $A(g, -) : \text{fin. sets} \longrightarrow \mathcal{C}$ + MAPS ξ_{ij}, $s \circ t$ AS ABOVE, WHICH SATISFY THE COMPATIBILITY CONDITIONS.

GRAPHS

A GRAPH Γ IS A FINITE SET $Flag(\Gamma)$ OF "FLAGS" (HALF-EDGE), σ = INVOLUTION ON Γ, λ PARTITION OF $Flag(\Gamma)$

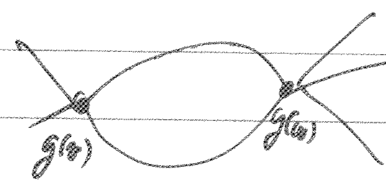
VERTICES = BLOCKS OF PARTITION λ

TWO FLAGS MEET IN v IF THEY LIE IN v

AN EDGE IS $\{i, \sigma(i)\}$ if $\neq \sigma(i)$

A LEG IS i WITH $\sigma(i) = i$.

S FINITE SET. AN S-LABELLED GRAPH IS $\left(\Gamma,\ g : \text{vert}(\Gamma) \to \mathbb{N}_0,\ f : \text{Leg}(\Gamma) \xrightarrow{\cong} S\right)$



$$g(\Gamma) = \sum_{v \in \text{Vert}(\Gamma)} (g(v) - 1) + |\text{EDGES}(\Gamma)| + 1$$

$=$ GENUS OF Γ.

Γ IS STABLE IF $\quad g(v) = 0 \Rightarrow |\text{Leg}(v)| \geq 3$ AND

$\qquad g(v) = 1 \Rightarrow |\text{Leg}(v)| \geq 1$

C STABLE CURVE $\longrightarrow$ GRAPH VERTICES = IRR. COMPONENTS

FLAGS = SPECIAL POINTS

LEGS = MARKED POINTS

GENUS = GENUS OF THE NORMALIZATION, i.e. CUTTING ALONG THE POINTS.

Σ STABLE MODULAR Σ-MODULE :

$\Sigma(g,-)$: fin. Sets $\longrightarrow \mathcal{C}$

$\mathcal{C}$ HAS FINITE COLIMITS, COLIMITS COMMUTE WITH $\otimes$

$$\Sigma(\Gamma) := \bigotimes_{v \in \mathrm{Vert}(\Gamma)} \Sigma(g(v), \mathrm{flag}(v))$$

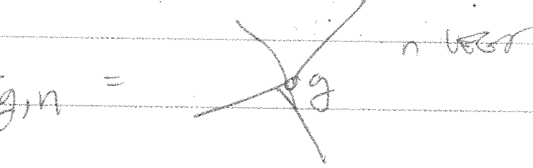
$$\mathbb{M}\Sigma(g,S) := \operatorname*{colim}_{\Gamma \in \mathrm{Iso}\,\Gamma(g,S)} \Sigma(\Gamma)$$

CAT. OF S-LABELLED GRAPHS OF GENUS = g.

$\mathbb{M}$: {Stable modular Σ-Modules} $\circlearrowleft$

<u>PROP</u>: $\mathbb{M}$ IS A TRIPLE

<u>THM</u>: A MODULAR OPERAD IS THE SAME AS AN ALGEBRA OVER THIS TRIPLE.



The Feynman Transform

$\mathcal{D}$ COEFFICIENT SYSTEM / COCYCLE / HYPEROPERAD

$\mathcal{D}$: Iso $\Gamma(g,S) \longrightarrow \mathcal{C}$ $\qquad \mathcal{D}(*_{g,n}) \simeq \mathbb{1}$

$f : \Gamma_0 \to \Gamma_1$ MORPHISM OF GRAPH (CAN COLLAPSE EDGES, ALSO LOOPS BUT ADAPT THE GENUS g)

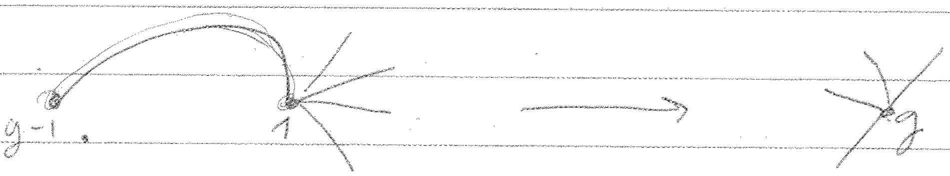
IN THE NEIGHBOURHOOD OUT OF VERTEX

$$\nu_f : \mathcal{D}(\Gamma_1) \otimes \bigotimes_{v \in \mathrm{Vert}(\Gamma)} \mathcal{D}(f^{-1}(v)) \xrightarrow{\cong} \mathcal{D}(\Gamma_0)$$

+ COMPATIBILITIES

ALL $\mathcal{D}(\Gamma)$ ARE INVERTIBLE IN $\mathcal{C}$

EX:



EX: ℓ INVERTIBLE MODULAR STABLE Σ-MODULE

$$\mathcal{D}_\ell(\Gamma) = \ell(g(\Gamma), \mathrm{Leg}(\Gamma)) \otimes \bigotimes_{v \in \mathrm{Vert}(\Gamma)} \ell(g(v), \mathrm{leg}(v))^{-1}$$

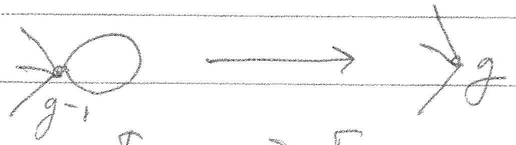
EX: $\mathfrak{K}(\Gamma) \doteq \mathrm{Det}(\mathrm{Edge}(\Gamma)) := \uparrow^{-\mathrm{edge}\,\Gamma} \Lambda^{|\mathrm{edge}(\Gamma)|} k^{\mathrm{edge}\,\Gamma}$

"DUALIZING COCYCLE"

SHIFT DEGREE DOWN BY |EDGE Γ|

$$\mathbb{M}_{\mathcal{D}} \Sigma(g, S) = \operatorname*{colim}_{\Gamma \in \mathrm{Iso}(g,S)} \Sigma(\Gamma) \otimes \mathcal{D}(\Gamma)$$

Def: A MODULAR $\mathcal{D}$-OPERAD A IS AN ALGEBRA OVER THE MONAD $\mathbb{M}_{\mathcal{D}}$



$\Gamma_0 \longrightarrow \Gamma_1$

$$\begin{array}{ccc} \mathcal{D}(\Gamma_0) \otimes A(\Gamma_0) & \longrightarrow & \mathcal{D}(\Gamma_1) \otimes A(\Gamma_1) \\ \| & & \| \\ \mathcal{D}(\Gamma_0) \otimes A(g-1, n+2) & \longrightarrow & A(g, n) \end{array}$$

FEYNMAN TRANSFORM:

$F_{\mathcal{D}}$: {MODULAR $\mathcal{D}$-OPERAD} $\longrightarrow$ {MODULAR $\mathcal{D}^{\vee}$-OPERAD}

$\mathcal{D}^{\vee} = \mathcal{D}^{-1} \otimes k$ (k: DET-COCYCLE)

$\mathcal{C}$ = CHAIN CPX'S OVER A FIELD OF char 0, FINITELY GEN. IN ANY DEGREE.

$F_{\mathcal{D}} A := M_{\mathcal{D}^{\vee}} A^{\#}$ ($A^{\#}$: LINEAR DUAL) $\qquad A: \{A(g,-): fin \to \mathcal{C}\}$

DIFFERENTIAL ON $M_{\mathcal{D}^{\vee}} A^{\#} = \delta = \partial_I + \partial_E$

∂_I: COMES FROM DIFF ON $A(g,S)$

∂_E: $M_{\mathcal{D}^{\vee}} A^{\#}(g,S) \cong \bigoplus_{\Gamma \in Iso(g,S)} \left(\mathcal{D}^{-1}(\Gamma) \otimes k(\Gamma) \otimes A^{\#}(\Gamma) \right)_{Aut \Gamma}$

∂_E RAISES THE NUMBER OF EDGES BY 1.

$\left(\mathcal{D}^{-1}(\Gamma) \otimes k(\Gamma) \otimes A^{\#}(\Gamma)\right)_{Aut\,\Gamma} \longrightarrow \left(\mathcal{D}^{-1}(\Delta) \otimes k(\Delta) \otimes A^{\#}(\Delta)\right)_{Aut\,\Delta}$

- 0 IF $\nexists$ EDGE $e \in \Delta$ s.t. $\Delta/e \cong \Gamma$
- OTHERWISE $f: \Delta \to \Gamma$

$\mathcal{D}(\Delta) \otimes A(\Delta) \to \mathcal{D}(\Gamma) \otimes A(\Gamma)$

$\to$ INDUCED BY DUAL –

$\mathcal{D}^{-1}(\Gamma) \otimes A^{\#}(\Gamma) \to \mathcal{D}^{-1}(\Delta) \otimes A^{\#}(\Delta)$

NEEDED FOR $\partial_E^2 = 0$: $+ \quad k(\Gamma) \to k(\Delta)$

$k(\Gamma) = \det(\text{edge}\,\Gamma)$ [?]

THM: F is HOMOTOPY INVARIANT: $A \xrightarrow{h\text{-equi}} B \Rightarrow F_{\mathcal{D}} B \xrightarrow{h\text{-equi}} F_{\mathcal{D}} A$

$\exists\, F_{\mathcal{D}^{\vee}} F_{\mathcal{D}} \to \mathbb{1}$, $\quad F_{\mathcal{D}^{\vee}} F_{\mathcal{D}} A \xrightarrow{h\text{-equi}} A$

GRAVITY OPERAD

$\overline{\mathcal{M}}_{g,n}$ LOGARITHMIC de Rham CPX

$\cup$

$D_{g,n} = \partial \overline{\mathcal{M}}_{g,n}$ $\mathcal{E}^{\cdot}(\overline{\mathcal{M}}_{g,n}, \log D_{g,n})$

$\alpha \wedge \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_\ell}{f_\ell}$

SMOOTH FORM ON $\overline{\mathcal{M}}_{g,n}$; f_i HOLOMORPHIC ~~EQUATION~~ FOR A COMPONENT OF $D_{g,n}$

$\mathcal{M}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$

THM (DELIGNE): $\mathcal{E}^{\cdot}(\overline{\mathcal{M}}_{g,n}, \log D_{g,n}) \hookrightarrow j^* \mathcal{E}^{\cdot}(\mathcal{M}_{g,n})$ IS A W.E. OF COMPLEXES OF SHEAVES.

$\Rightarrow H^{\cdot}(\mathcal{E}(\overline{\mathcal{M}}_{g,n}, \log D_{g,n})) = H^{\cdot}(\mathcal{M}_{g,n})$

(SIMPLE EXAMPLE: $(\mathbb{C}, \{0\})$

$$C^\infty(\mathbb{C}) \to C^\infty(\mathbb{C}) \cdot \frac{dz}{z} \oplus \Omega^1(\mathbb{C})$$
$$\downarrow$$
$$\Omega^2(\mathbb{C})$$
)

$$\mathcal{E}^{\cdot}(\overline{\mathcal{M}}_{g,n}, \log D_{g,n}) \xrightarrow{\text{Residue}} \mathcal{E}^{\cdot-1}(\tilde{D}_{g,n}, \log \tilde{D}^2_{g,n})$$

$\uparrow$
MODULAR CO-OPERAD

$\mathcal{E}^{\cdot}(\overline{\mathcal{M}}, \log D_{g,n})^{\#} = \mathrm{Grav}(g,n)$ MODULAR K-OPERAD "GRAVITY OPERAD"

THM. $F_k(\mathrm{Grav}(g,n)) \simeq C_*(\overline{\mathcal{M}}_{g,n}; \mathbb{C})$

" OPEN MODULI "
SPACE
(IE COMPUTE THE H_* OF)

$\check{\epsilon} = \mathbb{1}$ $\xrightarrow{\text{ANALY}}$ $F_k \underbrace{C_*(\overline{\mathcal{M}}_{g,n}; \mathbb{C})}_{\substack{\text{FORMAL} \\ \text{AS AN OPERAD} \\ \Rightarrow \text{JUST } H_*}} \simeq \mathrm{Grav}(g,n)$