I.1. PLAN I. the Connes Embedding Conjecture & Its Equivalent

II. Kirchberg's Conj & Tsirelson's Problem

III. Algebraic Reformulations , etc

CEC

$\boxed{\forall \text{ sep } (= \text{countably generated}) \text{ finite vNa is embeddable into } R^\omega}$

<u>Ultralimit</u>

$\ell_\infty \mathbb{N} \overset{\pi}{\simeq} C(\beta\mathbb{N})$ $\quad \beta\mathbb{N}$ Stone–Čech Compactification

Choose $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$.

$\lim_{n\to\omega} : \ell_\infty \mathbb{N} \to \mathbb{C}$ the character corr. to $\omega$

$\lim_{n\to\omega} f(n) := \pi(f)(\omega)$

- positive linear functional
- $\lim_\omega f(n) \in$ limit pts of $(f(n))_{n=1}^\infty$
- $\lim_\omega f(n)g(n) = (\lim_\omega f(n))(\lim_\omega g(n))$

$\rightsquigarrow \forall E \subseteq \mathbb{N}$ $\quad \lim_\omega \chi_E(n) = \begin{cases} 1 & \text{if } E \text{ is "big"} \\ 0 & \text{if } E \text{ is "small"} \end{cases}$

$\omega \in \beta\mathbb{N}\setminus\mathbb{N}$ is identified with the non-principal ultrafilter

$\{E \subseteq \mathbb{N} : \lim_\omega \chi_E(n) = 1\}$

Filter Axiom
- $\phi \notin E$
- $E \subseteq F$, $E \in \omega \Rightarrow F \in \omega$
- $E, F \in \omega \Rightarrow E \cap F \in \omega$

ultrafilter
- $\forall E \subseteq \mathbb{N}$ either $E \in \omega$ or $E^c \in \omega$

$X$ top sp $\quad (x(n))_{n=1}^\infty$, $x \in X$

$x = \lim_{n\to\omega} x(n) \overset{\text{def}}{\iff} \forall U \in \mathcal{N}_x \ \{n : x(n) \in U\} \in \omega$

Thm $X$ cpt Hausdorff $\omega$ ultrafilter

$\Rightarrow \forall (x(n))_{n=1}^\infty \ \exists! \lim_{n\to\omega} x(n) \in X$

## I.2.) Hahn-Banach-Eidelheit-Kakutani Separation Thm

$V$ $\mathbb{R}$-vector sp

$C \subseteq V$ convex, algebraically solid $\left( \exists c_0 \in C \text{ algebraic interior pt: } \forall x \in V \; \exists \varepsilon > 0 \text{ s.t. } c_0 + \lambda x \in C \text{ for } |\lambda| < \varepsilon \right.$

$v \in V \setminus C$

$\Rightarrow \exists \varphi : V \to \mathbb{R}$ linear, non-zero

s.t. $\varphi(v) \leq \inf_{c \in C} \varphi(c) \quad (< \varphi(c_0))$

## Operator Algebra Preliminary

All algebras are assumed to be unital alg $/\mathbb{C}$

$\mathcal{H}$ Hilbert sp

$\mathbb{B}(\mathcal{H})$ $*$-alg $\langle x\eta, \xi \rangle = \langle \eta, x^* \xi \rangle \qquad x \in \mathbb{B}(\mathcal{H}),\ \xi, \eta \in \mathcal{H}$

When $\mathcal{H} = \ell_2^n$ $\quad \mathbb{B}(\mathcal{H}) = \mathbb{M}_n \qquad x^* = \overline{x^T}$

$*$ conjugate linear involution $(xy)^* = y^* x^*$

$A$ $*$-subalg $A \subseteq \mathbb{B}(\mathcal{H})$ is a $C^*$-alg if closed under the norm top

a von Neumann alg if SOT

$A_+ = \{ a \in A : \langle a\xi, \xi \rangle \geq 0 \text{ for } \forall \xi \in \mathcal{H} \}$

$= \{ x^*x : x \in A \}$

$\mathcal{U}(A) :=$ the unitary elements of $A = \{ u \in A : u^*u = 1 = uu^* \}$

## Algebraic Treatment

$\mathcal{A}$ $*$-alg $/\mathbb{C}$ $\quad 1 \in \mathcal{A}$

$\mathcal{A}_{\mathbb{R}} := \{ a \in \mathcal{A} : a^* = a \}$ $\mathbb{R}$-vector sp $\mathcal{A} = \mathcal{A}_{\mathbb{R}} + i\mathcal{A}_{\mathbb{R}}$

$\mathcal{A}_+ \subseteq \mathcal{A}_{\mathbb{R}}$ $*$-positive cone
- $1 \in \mathcal{A}$
- $a \in \mathcal{A}_+,\ x \in \mathcal{A} \Rightarrow x^* a x \in \mathcal{A}_+$

$a \leq b \overset{\text{def}}{\Leftrightarrow} b - a \in \mathcal{A}_+$

$A^{\text{bdd}} := \{ a \in \mathcal{A} : \exists R > 0 \text{ s.t. } a^*a \leq R1 \}$

Exercise: Prove $\mathcal{A}^{\text{bdd}}$ is a $*$-subalg of $\mathcal{A}$

(Tricky: $a^*a \leq R1 \Rightarrow aa^* \leq R1$)

$\mathcal{A}$ is a "pre-semi-$C^*$-alg" if $\mathcal{A} = \mathcal{A}^{\text{bdd}}$

$\xrightarrow[\text{Exercise}]{}$ $1$ alg interior pt in $\mathcal{A}_+$

I.3. $C^*_u(\mathcal{A})$ the universal enveloping $C^*$-alg of $\mathcal{A}$

$\forall\ \pi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ positive $*$-repn

extends to a $*$-repn on $C^*_u(\mathcal{A})$

Exercise : $\mathrm{Ker}(\mathcal{A} \to C^*_u(\mathcal{A})) = \{ a \in \mathcal{A} : a^*a \le \varepsilon 1 \text{ for } \forall \varepsilon > 0 \}$

DAY III

"infinitesimal elements"

GNS construction

$\varphi : \mathcal{A} \to \mathbb{C}$ state $\overset{\text{def}}{\iff} \varphi(\mathcal{A}_+) \subseteq \mathbb{C}_+$ & $\varphi(1) = 1$

$\varphi \in S(\mathcal{A})$ faithful $\overset{\text{def}}{\iff} \varphi(a^*a) > 0$ for $\forall\ a \ne 0$

Given $\varphi \in S(\mathcal{A})$

$\langle y, x \rangle := \varphi(x^*y)$ semi inner prod $\quad \|x\|_2 = \varphi(x^*x)^{1/2}$

$\longrightarrow$ Hilbert space $L^2(\mathcal{A}, \varphi)$

$\mathcal{A} \ni x \mapsto \hat{x} \in L^2(\mathcal{A}, \varphi)$

$a : \hat{x} \mapsto \widehat{ax}$ gives $\pi_\varphi(a) \in \mathbb{B}(L^2(\mathcal{A},\varphi))$

$( a^*a \le R1 \Rightarrow \|\pi(a)\|^2 \le R )$

$\pi_\varphi : \mathcal{A} \to \mathbb{B}(L^2(\mathcal{A},\varphi))$ positive $*$-repn

$\langle \pi_\varphi(a)\hat{1}, \hat{1} \rangle = \varphi(a)$ extends to a normal state on

$\pi_\varphi(\mathcal{A})'' := \overline{\pi_\varphi(\mathcal{A})}^{SOT} \subseteq \mathbb{B}(L^2(\mathcal{A},\varphi))$ vN alg

Fact : If $\mathcal{A}$ is a vNa w/ normal faithful state $\varphi$,

then $\pi_\varphi(\mathcal{A})'' \cong \mathcal{A}$

$\varphi \in S(\mathcal{A})$ tracial $\overset{\text{def}}{\iff} \varphi(ab) = \varphi(ba)$ for $\forall a, b \in \mathcal{A}$

Examples · $(\mathbb{M}_n, \mathrm{tr})$ $\quad \mathrm{tr} = \frac{1}{n}\mathrm{Tr}$

· $\mathbb{M}_{n_k} \hookrightarrow \mathbb{M}_{n_{k+1}} \hookrightarrow \cdots = \varinjlim \mathbb{M}_{n_k} =: \mathcal{A}$

$n_k \mid n_{k+1} \quad \mathbb{M}_{n_k} \ni X \mapsto X \otimes 1 \in \mathbb{M}_{n_{k+1}}$

$R := \pi_\tau(\mathcal{A})''$ the hyperfinite $\mathrm{II}_1$ factor

· $\Gamma$ discrete grp $\quad \mathbb{C}\Gamma \quad f^*(t) = \overline{f(t^{-1})}$

$\tau(f) := f(e)$ tracial state on $\mathbb{C}\Gamma$

$L^2(\mathbb{C}\Gamma, \tau) = \ell_2\Gamma \qquad \pi_\tau(\mathbb{C}\Gamma)'' =: L\Gamma$ grp vNa

Fact : $\mathcal{A} \subseteq \mathcal{B}$, $\tau_\mathcal{A} = \tau_\mathcal{B}|_\mathcal{A} \Rightarrow \pi_{\tau_\mathcal{A}}(\mathcal{A})'' \subseteq \pi_{\tau_\mathcal{B}}(\mathcal{B})''$ naturally

I.4.] <u>Ultraproduct</u> of finite vNa $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ fixed

$(M_n, \tau)$ finite vNa (i.e. $\tau$ faithful normal tracial state)

$\prod M_n = \{(x_n)_{n=1}^{\infty} : x_n \in M_n, \ \sup \|x_n\| < +\infty\}$

$\tau_\omega((x_n)_{n=1}^{\infty}) := \lim_\omega \tau(x_n)$ tracial state

$\prod M_n/\omega := \pi_{\tau_\omega}(\prod M_n)'' \underset{\text{fact}}{=} \prod M_n / \{x : \tau_\omega(x^*x) = 0\}$

$R^\omega = \prod R/\omega$ ultrapower of $R$

<u>Thm</u> For a sep finite vNa $(M, \tau)$, TFAE

(1) $M \hookrightarrow R^\omega$ ($\tau$-preserving) embedding

(2) $M \hookrightarrow \prod M_n/\omega$

(3) $\forall d \ \forall x_1, \dots, x_d \in M_{sa} \quad \forall m \quad \forall \varepsilon > 0$

$\exists N \ \exists X_1, \dots, X_d \in (M_N)_{sa}$ (with $\|X_i\| \le \|x_i\|$)

s.t. $|\tau(p(x_1, \dots, x_d)) - tr(p(X_1, \dots, X_d))| < \varepsilon$

for all $p$ NC polyn of degree $\le m$ in d-variable

(4) $\forall d \ \forall u_1, \dots, u_d \in \cancel{M_u}\, \mathcal{U}(M)$ (unitary elements) $\forall m \ \forall \varepsilon > 0$

$\exists N \ \exists U_1, \dots, U_d \in \cancel{(M_N)_u}\, \mathcal{U}(M_N)$

s.t. same as above, but use $p(u_1, \dots, u_d, u_1^*, \dots, u_d^*)$

(5) Kirchberg $\forall d \ \forall u_1, \dots, u_d \in \mathcal{U}(M) \quad \forall \varepsilon > 0$

$\exists N \ \exists U_1, \dots, U_d \in \mathcal{U}(M_N)$

s.t. $|\tau(u_i^* u_j) - tr(U_i^* U_j)| < \varepsilon$

Note: $u \in A$ is a unitary element $\Leftrightarrow u^*u = 1 = uu^*$

<u>Fact</u>: $A$ $C^*$-alg $\Rightarrow \overline{\mathrm{conv}}\, \mathcal{U}(A) = \overline{\mathrm{Ball}}(A)$.

<u>Fact</u>: $\pi : A \to B$ linear $\pi(\mathcal{U}(A)) \subseteq \mathcal{U}(B)$

unital $C^*$-alg's

$\Rightarrow \pi(1)^* \pi(\cdot)$ is a Jordan hom

<u>Fact</u>: $M \hookrightarrow R^\omega \Leftrightarrow M * R \hookrightarrow R^\omega$

Dykema

$II_1$ factor

I.5.1 Full group $C^*$-alg

$\Gamma$ discrete group

$C^*\Gamma = C^*_u(\mathbb{C}\Gamma)$ : $\forall \pi : \Gamma \to \mathbb{B}(\mathcal{H})$ unitary repn extends to a $*$-hom $\pi : C^*\Gamma \to \mathbb{B}(\mathcal{H})$

$\rightsquigarrow$ A sep unital $C^*$-alg $\Rightarrow \exists\, C^*\mathbb{F}_\infty \twoheadrightarrow A$

Thm The following conjectures are equivalent

(CEC) $\forall$ sep finite vNa $\hookrightarrow R^\omega$

Kirchberg DAY II (KC) $C^*\Gamma \otimes C^*\Gamma$ has unique $C^*$-norm

for $\Gamma$ one/all of $\mathbb{F}_2, \cdots, \mathbb{F}_\infty$, $\underbrace{\mathbb{Z}/m\mathbb{Z} * \cdots * \mathbb{Z}/m\mathbb{Z}}_{d}$ $(m,d) \neq (2,2)$

DAY II $(TP_\infty)$ Tsirelson's Problem on Equivalence of Two Bell Scinario in Quantum Information Theory.

Matrix valued

Alice state Bob

DAY III (Alg) Something purely algebraic

(Radulescu, Klep - Schweighofer, Juschenko-Popovych)

(EM) Something on Effros - Maréchal Topology

( Haagerup - Winsløw )

⋮

I.6.

# Tensor Product of Operator Algebras

$A, B$ $C^*$-algs $\quad A \odot B$ algebraic tensor prod $/\mathbb{C}$

**min tensor**

Take $A \subseteq \mathbb{B}(\mathcal{H})$, $B \subseteq \mathbb{B}(\mathcal{K})$

$\to A \odot B \subseteq \mathbb{B}(\mathcal{H} \otimes_2 \mathcal{K})$

$\rightsquigarrow A \otimes_{\min} B$ minimal tensor prod

indep of choices of faithful repns $A \subseteq \mathbb{B}(\mathcal{H})$, $B \subseteq \mathbb{B}(\mathcal{K})$

**max tensor** $\quad (A \odot B)_+^{\max} = \{ x^* x : x \in A \odot B \}$

$A \otimes_{\max} B = C^*_u(A \odot B)$

- $\forall \pi : A \odot B \to \mathbb{B}(\mathcal{H})$ $*$-repn

  $\leftrightsquigarrow$ $\pi|A$ $\pi|B$ $*$-hom with commuting ranges

  extends to a $*$-hom on $A \otimes_{\max} B$

- $A \otimes_{\max} B \twoheadrightarrow A \otimes_{\min} B$ canonically
- $C^*\Gamma \otimes_{\max} C^*\Gamma = C^*(\Gamma \times \Gamma)$ $\overset{??}{\neq}$ $C^*\Gamma \otimes_{\min} C^*\Gamma$
- $C^*_\lambda \Gamma \otimes_{\max} C^*_\lambda \Gamma = \otimes_{\min}$ iff $\Gamma$ amenable

**Completely positive maps**

$\varphi : A \to B$ cp $\overset{\text{def}}{\iff}$ $\forall n$ $\mathrm{id}_{\mathbb{M}_n} \otimes \varphi : \mathbb{M}_n(A) \to \mathbb{M}_n(B)$ positive

$\iff \varphi \otimes \mathrm{id}_C : A \odot C \to B \odot C$ is positive wrt $\otimes_{\min}$ & $\otimes_{\max}$ for $\forall C$

Thm / Def $\quad A$ nuclear (amenable) $\iff A \otimes_{\max} B = A \otimes_{\min} B$ for $\forall B$

II.1.

## Tensor Product of $C^*$-algebras

$A \odot B$ alg tensor prod $\quad (A\odot B)_+ := \text{cone}\{x^*x : x \in A\odot B\}$

$A \otimes_{\min} B \subseteq \mathbb{B}(\mathcal{H}\bar{\otimes}K)$, $\quad (A\odot B)_+^{\min} := A\odot B \cap (A\otimes_{\min} B)_+$

$A \otimes_{\max} B = C^*_u(A\odot B)$, $\quad (A\odot B)_+^{\max} := A\odot B \cap (A\otimes_{\max} B)_+$

Exercise: $\mathcal{A}$ pre-semi-$C^*$-alg (e.g. $\mathcal{A} = A\odot B$)

$\Rightarrow \mathcal{A} \cap C^*_u(\mathcal{A})_+ = \{a \in \mathcal{A} : a + \varepsilon \geq 1 \text{ for } \forall \varepsilon > 0\}$

Exercise: $\varphi \in S(A\odot B)$ or $\pi : A\odot B \to \mathbb{B}(\mathcal{H})$

is $\otimes_{\min}$-conti (extends on $A\otimes_{\min} B$ by continuity)

iff positive w.r.t. $(A\odot B)_+^{\min}$.

## GNS construction for tracial states

$A$ $C^*$-alg $\quad \tau$ tracial state on $A$

$\longrightarrow L^2(A,\tau) \qquad \langle \hat{y}, \hat{x}\rangle = \tau(x^*y)$

$\pi_\tau : A \to \mathbb{B}(L^2(A,\tau))$, $\quad \pi_\tau(a)\hat{x} = \widehat{ax}$

Since $\tau$ is tracial, $\hat{x} \mapsto \widehat{xa}$ is also continuous. (Check)

Opposite $C^*$-alg $\left[\begin{array}{l} A^{op} := \{a^{op} : a \in A\}, \quad a^{op} b^{op} := (ba)^{op} \\ \text{E.g. } \mathbb{M}_n^{op} \simeq \mathbb{M}_n \quad x^{op} \leftrightarrow x^T, \quad (C^*\Gamma)^{op} \simeq C^*\Gamma \quad g^{op} \leftrightarrow g^{-1}, \end{array}\right.$

$\pi_\tau^{op} : A^{op} \to \mathbb{B}(L^2(A,\tau))$, $\quad \pi_\tau^{op}(a^{op})\hat{x} = \widehat{xa}$

is a $*$-hom whose range commutes with $\pi_\tau(A)$.

$\rightsquigarrow \pi_\tau \times \pi_\tau^{op} : A \odot A^{op} \to \mathbb{B}(L^2(A,\tau))$ $\quad$ $*$-repn

$\mu_\tau \in S(A\odot A^{op})$

$\mu_\tau(a\otimes b^{op}) := \langle \pi_\tau \times \pi_\tau^{op}(a\otimes b^{op})\hat{1}, \hat{1}\rangle = \tau(ab)$

$A$ sep

**Thm** (Connes '76, Kirchberg '93) For $(A,\tau)$ TFAE

(1)
$$\begin{tikzcd} & \prod \mathbb{M}_n \arrow[d] \\ A \arrow[ur, "\exists\varphi"] \arrow[r, "\exists\pi"'] & \prod \mathbb{M}_n/\omega \end{tikzcd}$$
$\pi$ $*$-hom s.t. $\tau_\omega \circ \pi = \tau$

$\varphi$ ucp lifting of $\pi$.

(2) $\pi_\tau \times \pi_\tau^{op}$ $\otimes_{\min}$-conti on $A\odot A^{op}$.

(3) $\mu_\tau$ ______________ .

ucp = "unital completely positive"

II.2.

Proof (1) ⇒ (3): $\varphi_n : A \to \mathbb{M}_n$ ucp $\|\cdot\|_2$-approx multiplicative

$\mu_n \in S(\mathbb{M}_n \otimes \mathbb{M}_n^{op})$ $a \otimes b^{op} \longmapsto \mathrm{tr}(ab)$

$\mu_n \circ (\varphi_n \otimes \varphi_n^{op}) \in S(A \otimes_{\min} A^{op})$ since $\mathbb{M}_n \otimes \mathbb{M}_n^{op} = \mathbb{M}_n \otimes_{\min} \mathbb{M}_n^{op}$

$$\mu_n \circ (\varphi_n \otimes \varphi_n^{op})(a \otimes b^{op}) = \mathrm{tr}(\varphi_n(a)\varphi_n(b)) \approx \mathrm{tr}(\varphi_n(ab)) \to \tau(ab)$$

$\mu_n \circ (\varphi_n \otimes \varphi_n^{op}) \xrightarrow[\omega]{} \tau$ and $\tau \in S(A \otimes_{\min} A^{op})$.

(3) ⇒ (2): $\hat{1}$ is cyclic for $\pi_\tau \times \pi_\tau^{op}$.

(2) ⇒ (1): Look at my book.

Def $\tau$ amenable $\overset{\text{def}}{\Longleftrightarrow}$ $\tau$ satisfies the above conditions

- $\tau$ on $L\Gamma$ (or $C^*_\lambda\Gamma$) amenable $\Longleftrightarrow$ $\Gamma$ amenable
- If $\Gamma$ is RF, then $C^*\Gamma \twoheadrightarrow C^*_\lambda\Gamma \xrightarrow{\tau} \mathbb{C}$ is amenable.

M sep

Thm For $(M, \tau)$, TFAE

(1) $M \hookrightarrow R^\omega$ $\tau$-preserving

(2) $\exists/\forall$ $C^*\mathbb{F}_\infty \xrightarrow{\sigma} M$ $\|\cdot\|_2$-dense range

$\tau \circ \sigma$ is amenable on $C^*\mathbb{F}_\infty$

Proof

(1) ⇒ (2∀) $C^*\mathbb{F}_\infty \xrightarrow{\sigma} M \hookrightarrow \prod \mathbb{M}_n/\omega$ $\exists\varphi \longrightarrow \prod \mathbb{M}_n \Downarrow$ $\exists\varphi$ $*$-hom lifting by universality

(2∃) ⇒ (1) $C^*\mathbb{F}_\infty \xrightarrow{\pi} \prod \mathbb{M}_n/\omega$, $\tau_\omega \circ \pi = \tau \circ \sigma$.

$\rightsquigarrow \overline{\pi(C^*\mathbb{F}_\infty)}^{SOT} \simeq M$. ▨

Consequence (CEC) $\Longleftrightarrow$ $\forall \tau$ on $C^*\mathbb{F}_\infty$ is amenable

$\Uparrow$ By the previous Thm $\Downarrow$ modular theory etc

(KC): $C^*\mathbb{F}_\infty \otimes C^*\mathbb{F}_\infty$ max = min

Case for $M = L\Gamma$

$Q : \mathbb{F}_r \twoheadrightarrow \Gamma$

$\mu_\tau : C^*\mathbb{F}_r \otimes C^*\mathbb{F}_r \to \mathbb{C}$

$$\mu_\tau(g \otimes h) = \begin{cases} 1 & \text{if } Q(g) = Q(h) \\ 0 & \text{otherwise} \end{cases}$$

$L\Gamma \hookrightarrow R^\omega \Longleftrightarrow \mu_\tau$ $\otimes_{\min}$-conti

II.3.

## Operator Systems

Op sys : $S \subseteq A$ unital $*$-subspace (not nec norm closed)

Op sys str : $\mathbb{M}_n(S)_+ := \mathbb{M}_n(S) \cap \mathbb{M}_n(A)_+$

$\varphi : S \to B$ cp $\overset{\text{def}}{\iff}$ $\mathrm{id} \otimes \varphi : \mathbb{M}_n(S) \to \mathbb{M}_n(B)$ positive for $\forall n$

$S \cong T$ if $\exists \varphi : S \to T$ $\varphi$ & $\varphi^{-1}$ ucp

Exercise : $\varphi : S \to \mathbb{C}$ positive $\Rightarrow \|\varphi\| = \varphi(1)$.

Axiom of Op Sys (Choi-Effros)
$S$, $\mathbb{M}_n(S)_+$
- $a \in \mathbb{M}_n(S)_+$, $X \in \mathbb{M}_{n,m}(\mathbb{C}) \Rightarrow X^* a X \in \mathbb{M}_m(S)_+$
- $1_n = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \in \mathbb{M}_n(S)$ is an Archimedean order unit

Thm (Arveson) $\mathbb{B}(\mathcal{H})$ is injective

$\forall S \subseteq A$ $\forall \varphi : S \to \mathbb{B}(\mathcal{H})$ cp

$\Rightarrow \exists \bar{\varphi} : A \to \mathbb{B}(\mathcal{H})$ cp

Combined with multiplicative domain,

Pisier's Trick

$S \subseteq A$ op sys

$\mathcal{S} \subseteq S$ a family of unitary elements which generates $A$ as a C*-alg

$\varphi : S \to B$ ucp s.t. $\varphi(u) \in \mathcal{U}(B)$ for $\forall u \in \mathcal{S}$

$\Rightarrow \varphi$ extends to a $*$-hom $\bar{\varphi} : A \to B$.

Exercise $\varphi : \mathbb{M}_n(S) \to \mathbb{C}$, $\tilde{\varphi} : S \to \mathbb{M}_n$

$\tilde{\varphi}(a) = [\varphi(e_{ij} \otimes a)]_{ij}$

Then $\varphi$ is positive $\iff$ $\tilde{\varphi}$ is cp

II.4.

<u>Tsirelson's Problem</u> $\qquad m \geq 2,\ d \geq 2,\ (m,d) \neq (2,2)$ fixed

POVM with $m$ outputs $(A_i)_{i=1}^m$, $A_i \geq 0$ & $\Sigma A_i = 1$

Measurement by $\xi \in \mathcal{H}$, $\|\xi\| = 1$ : $(\langle A_i \xi, \xi \rangle)_{i=1}^m$

Alice $\quad d$ POVMs $\quad (A_i^k)_{i=1}^m \quad k = 1, \cdots, d$

Bob $\quad (B_j^l)_{j=1}^m \quad l = 1, \cdots, d$

Shared state

Independence $\quad A_i^k \in \mathbb{B}(\mathcal{H})$, $B_j^l \in \mathbb{B}(K)$, $\xi \in \mathcal{H} \bar{\otimes} K$

or

$A_i^k, B_j^l \in \mathbb{B}(\mathcal{H})$ s.t. $[A_i^k, B_j^l] = 0$, $\xi \in \mathcal{H}$

Quantum corelation matrices

$$Q_s := \text{closure}\left\{ [\langle A_i^k \otimes B_j^l \xi, \xi \rangle]_{i,j}^{k,l} : \begin{array}{l} A \text{ POVMs on } \mathcal{H} \\ B \qquad\quad K \\ \xi \in \mathcal{H}\bar{\otimes}K \quad \|\xi\|=1 \end{array} \right\}$$

closed convex $\quad \cap$

$$Q_c := \left\{ [\langle A_i^k B_j^l \xi, \xi \rangle]_{i,j}^{k,l} : \begin{array}{l} A, B \text{ POVMs on } \mathcal{H} \\ [A, B] = 0 \\ \xi \in \mathcal{H} \quad \|\xi\| = 1 \end{array} \right\}$$

In general

$$Q_s^{(n)} := \text{closure}\left\{ \underbrace{[V^*(A_i^k \otimes B_j^l) V]_{i,j}^{k,l}}_{mdn \text{ size matrices}} : \underset{\text{isometry}}{V : \ell_2^n \to \mathcal{H}\bar{\otimes}K} \right\}$$

$$Q_c^{(n)} := \cdots$$

~~[illegible]~~.

(TP): $Q_s = Q_c$ $\qquad$ True if $\dim \mathcal{H} < +\infty$

$(\mathrm{TP}_\infty)$: $Q_s^{(n)} = Q_c^{(n)}$ for all $n$

Proof of $(\mathrm{TP}_\infty) \Leftrightarrow (\mathrm{KC})$ $\qquad \Gamma = \overbrace{\mathbb{Z}/m\mathbb{Z} * \cdots * \mathbb{Z}/m\mathbb{Z}}^{d}$

$$C^*_u \Gamma = C^*(\mathbb{Z}/m\mathbb{Z}) * \cdots * C^*(\mathbb{Z}/m\mathbb{Z}) = \ell_\infty^m * \cdots * \ell_\infty^m$$

$$S := (\ell_\infty^m)^{(1)} + \cdots + (\ell_\infty^m)^{(d)} = \text{span}\{ e_i^k : \begin{array}{l} i = 1, \cdots, m \\ k = 1, \cdots, d \end{array} \}$$

- <u>Exercise</u>: $\varphi : \ell_\infty^m \to \mathbb{B}(\mathcal{H})$, $e_i \mapsto A_i$ $\quad$ ucp $\Leftrightarrow$ $A_i \geq 0$ & $\Sigma A_i = 1$ i.e. POVM

II.5.

**Fact (Boca)** $\quad \varphi_i : C_i \to \mathbb{B}(\mathcal{H})$ ucp

$C_i$ $C^*$-alg's $\quad C = C_1 * C_2$ full free product $/\mathbb{C}1$

$\Longrightarrow \exists\, \varphi : C \to \mathbb{B}(\mathcal{H})$ ucp s.t. $\varphi|_{C_i} = \varphi_i$.

Cor: $\quad Q_s = \{ \varphi(e_i^k \otimes e_j^l) : \varphi \in S(C^*\Gamma \underset{min}{\otimes} C^*\Gamma) \}$

$\quad Q_c = \{ \varphi(e_i^k \otimes e_j^l) : \varphi \in S(C^*\Gamma \underset{max}{\otimes} C^*\Gamma) \}$

$Q_s = Q_c \iff \begin{array}{c} S \otimes S \\ \cap\!| \\ C^*\Gamma \underset{min}{\otimes} C^*\Gamma \end{array} \xrightarrow{id} C^*\Gamma \underset{max}{\otimes} C^*\Gamma$ is positive

$\begin{array}{c} Q_s^{(n)} = Q_c^{(n)} \\ \text{for all } n \end{array} \iff$ ucp

$\Longrightarrow$ min = max on $C^*\Gamma \otimes C^*\Gamma$ i.e. (KC).

Pisier's Trick

## Tensor Product of Op Sys (Paulsen, Kavruk, ...)

$S, T$ op sys $\quad S \otimes T$ alg tensor prod $/\mathbb{C}$

$\mathbb{M}_n(S\otimes T)_+^{min} := \{ x \in \mathbb{M}_n(S\otimes T)_{h} :$
$\varphi \otimes \psi\,(x) \geq 0$ in $\mathbb{M}_{nkl}$
for $\forall \varphi : S \to \mathbb{M}_k$ $\psi : T \to \mathbb{M}_l$ cp $\}$

"Obviously" positive elements

$D_n^+ := \{ X^*(a \otimes b) X : \begin{array}{l} a \in \mathbb{M}_k(S)_+ \\ b \in \mathbb{M}_l(T)_+ \\ X \in \mathbb{M}_{kl,n}(\mathbb{C}) \end{array} \}$

$\mathbb{M}_n(S\otimes T)_+^{max} := \{ x \in \mathbb{M}_n(S\otimes T)_{h} : x + \varepsilon 1 \in D_n^+ \text{ for } \forall \varepsilon > 0 \}$

**Thm** If $A, B$ $C^*$-alg's, then min and max are compatible with the previous definitions.

Proof for max

$D_n^+ \subseteq (A\otimes B)_+^{C^*\text{-}max}$ $\quad$ Easy

If $(A\otimes B)_+^{C^*\text{-}max} \not\subseteq (A\otimes B)_+^{OS\ max}$, then

$\exists\, \varphi$ state wrt OS max which is non-positive wrt $C^*$-max.

But $(\Sigma a_i \otimes b_i)^*(\Sigma a_i \otimes b_i) = \Sigma a_i^* a_j \otimes b_i^* b_j \in (A\otimes B)_+^{OS\ max}$.

$\varphi$ extends to a state on $A \underset{max}{\otimes} B$. A contradiction $\quad \blacksquare$

II. 6.1

Thm (Kirchberg 93) $A_i$: $C^*$-alg $A = A_1 * A_2$

If max = min on $A_i \otimes \mathbb{B}(\ell_2)$,

then max = min on $A \otimes \mathbb{B}(\ell_2)$.

In particular $C^*\mathbb{F}_\infty \otimes \mathbb{B}(\ell_2)$ max = min.

Proof

$$S := A_1 + A_2 \cong A_1 \oplus A_2 / \mathbb{C}(1,-1)$$

By Pisier's Trick, it suffices to show

$S \otimes \mathbb{B}(\ell_2) \longrightarrow A \otimes_{max} \mathbb{B}(\ell_2)$ is min completely positive.

Lem $x \in (S \otimes \mathbb{B}(\ell_2))_+^{min}$

$\Rightarrow \exists\, a_i \in (A_i \otimes \mathbb{B}(\ell_2))_+^{min}$

s.t. $x = a_1 + a_2$.

Pf. First $\ell_2^n$

$\{a_1 + a_2 : a_i \in (A_i \otimes \mathbb{B}(\ell_2^n))_+\} \subseteq (S \otimes \mathbb{B}(\ell_2^n))_+$

positive cone, alg solid

By HBEK & Boca, they are equal.

$P_n : \mathbb{B}(\ell_2) \to \mathbb{B}(\ell_2^n)$ cut off

$x \in (S \otimes \mathbb{B}(\ell_2))_+^{min}$ given

$\rightsquigarrow \exists\, E_i \subseteq A_i$ finite dim s.t. $x \in (E_1 + E_2) \otimes \mathbb{B}(\ell_2)$

$(id \otimes P_n)(x) = a_1^{(n)} + a_2^{(n)}$

$a_i^{(n)} \in (S \otimes \mathbb{B}(\ell_2^n))_+$ ; $a_i^{(n)} \in E_i \otimes \mathbb{B}(\ell_2^n)$

$0 \leq a_i^{(n)} \leq \|x\|$

$\rightsquigarrow$ Take a limit pt ( $\lim_{n \to \omega} a_i^{(n)} \in E_i \otimes \mathbb{B}(\ell_2)$ )

$a_i \in E_i \otimes \mathbb{B}(\ell_2)$, $a_i \geq 0$

$x = a_1 + a_2$ $\square$

Since $(A_i \otimes \mathbb{B}(\ell_2))_+^{min} = (A_i \otimes \mathbb{B}(\ell_2))_+^{max}$,

$a_i$'s are "obviously" positive.

$\rightsquigarrow$ $x \geq 0$ in $A \otimes_{max} \mathbb{B}(\ell_2)$ $\square$

III. 1.

Recall: $\mathcal{A}$ unital $*$-alg $/\mathbb{C}$

$\mathcal{A}_+ \subseteq \mathcal{A}_h$ $*$-positive cone
- $1 \in \mathcal{A}_+$
- $a \in \mathcal{A}_+, x \in \mathcal{A} \Rightarrow x^* a x \in \mathcal{A}$

$\mathcal{A}$ is a semi-pre-$C^*$-alg

if $\mathcal{A} = \mathcal{A}^{\mathrm{bdd}} := \{ x \in \mathcal{A} : \exists R > 0 \text{ s.t. } x^* x \le R 1 \}$

$\rightsquigarrow$ 1 is an alg interior pt of $\mathcal{A}_+$ $\left( \because h \le \frac{h^2+1}{2} \right)$

$C^*_u(\mathcal{A})$ the univ enveloping $C^*$-alg of $\mathcal{A}$

$\forall \pi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ positive $*$-repn extends to a $*$-repn of $C^*_u(\mathcal{A})$

Example
- $\mathbb{C}\Gamma$, $(\mathbb{C}\Gamma)_+ := \operatorname{cone}\{ x^* x : x \in \mathbb{C}\Gamma \}$

$\to C^*_u(\mathbb{C}\Gamma) = C^*\Gamma$

- $\mathbb{C}\langle x_1, \cdots, x_d \rangle$ NC polyn $x_i^* = x_i$

$\mathbb{C}\langle x_1, \cdots, x_d \rangle_+ :=$ $*$-positive cone generated by $1 - x_i^2$

$\to C^*_u(\mathbb{C}\langle x_1, \cdots, x_d \rangle) = C([-1,1]) * \cdots * C([-1,1])$

$\mathbb{C}[x_1, \cdots, x_d]$ commutative

$C^*_u(\quad) = C([-1,1]^d)$

Thm
- $\operatorname{Ker}(\mathcal{A} \to C^*_u(\mathcal{A})) = \{ x \in \mathcal{A} : x^* x \le \varepsilon 1 \text{ for } \forall \varepsilon > 0 \}$
- $\mathcal{A} \cap C^*_u(\mathcal{A})_+ = \{ a \in \mathcal{A}_h : a + \varepsilon 1 \in \mathcal{A}_+ \text{ for } \forall \varepsilon > 0 \}$

Pf of the second: $\supseteq$ clear

$\subseteq$: Let $a \in \mathcal{A}_h$ be s.t. $a + \varepsilon 1 \notin \mathcal{A}_+$ for some $\varepsilon > 0$.

By HBEK, $\exists \varphi \in S(\mathcal{A})$ s.t. $\varphi(a + \varepsilon 1) \le 0$.

$\xrightarrow{\text{GNS}}$ $\pi_\varphi(a) \not\ge 0$. ▨

Thm (Choi) $C^*\mathbb{F}_r$, $C([-1,1])^{*d}$ are RFD

i.e. $\bigoplus_{\pi \text{ fd repn}} \pi : A \to \prod \mathbb{M}_{\dim \pi}$ is faithful.

Lubotzky - Shalom

Cor $f \in \mathbb{C}\mathbb{F}_r$ or $\mathbb{C}\langle x_1, \cdots, x_d \rangle$

If $\pi(f) \ge 0$ for $\forall$ fd repn $\pi$,

then $f + \varepsilon 1 \in (\mathbb{C}\mathbb{F}_r)_+$ or $(\mathbb{C}\langle x_1, \cdots, x_d \rangle)_+$ for $\forall \varepsilon > 0$.

Cor KC $\iff$ $\left( \begin{array}{l} f \in \mathbb{C}(\mathbb{F}_r \times \mathbb{F}_r) \\ \pi(f) \ge 0 \text{ for } \forall \text{ fd repn } \pi \\ \Rightarrow f + \varepsilon 1 \in \mathbb{C}(\mathbb{F}_r \times \mathbb{F}_r)_+ \end{array} \right.$

☺ $\pi(f) \ge 0$ for $\forall$ fd repn $\pi$ $\iff$ $f \ge 0$ wrt min

$f + \varepsilon 1 \in \mathbb{C}(\mathbb{F}_r \times \mathbb{F}_r)_+$ $\iff$ $f \ge 0$ wrt max ▨

III.2.

Radulescu,
Klep-Schweighofer 2008, Juschenko-Popovych 2011

Tracial Setting $K = \mathrm{span}\{[a,b] : a,b \in \mathcal{A}\}$

$\tau \in S(\mathcal{A})$ tracial $\Leftrightarrow \tau = 0$ on $K$

Thm For $a \in \mathcal{A}_{\mathbb{R}}$, TFAE

(1) $\tau(a) \geq 0$ for $\forall \tau \in S(\mathcal{A})$ tracial

(2) $a + \varepsilon 1 \in \mathcal{A}_+ + K$ for all $\varepsilon > 0$.

Pf. (2) $\Rightarrow$ (1) Obvious.

(1) $\Rightarrow$ (2) Suppose $a + \varepsilon 1 \notin \mathcal{A}_+ + K$.

$\mathcal{A}_+ + K_{\mathbb{R}}$ alg solid (because $\mathcal{A}_+$ is)

By HBEK $\exists \varphi$ non-zero

$\varphi(\mathcal{A}_+ + K_{\mathbb{R}}) \subseteq \mathbb{R}_{\geq 0}$

$\varphi(a + \varepsilon 1) \leq 0$

WMA $\varphi(1) = 1$ i.e. $\varphi \in S(\mathcal{A})$, $\rightsquigarrow \varphi(a) < 0$

$\varphi|_{K_{\mathbb{R}}} \geq 0 \rightsquigarrow \varphi|_{K_{\mathbb{R}}} = 0 \rightsquigarrow \varphi$ tracial

$(K = K_{\mathbb{R}} + iK_{\mathbb{R}})$ $\blacksquare$

Klep-Schweighofer 08

$x_1, \cdots, x_d$ hermitian

$\mathcal{A} := \mathbb{C}\langle x_1, \cdots, x_d \rangle$ NC polyn

$\mathcal{A}_+ := \mathrm{cone}(\{a^*a : a \in \mathcal{A}\} \cup \{a^*(1-x_i^2)a : \begin{smallmatrix} a \in \mathcal{A} \\ i=1,\cdots,d \end{smallmatrix}\})$

$(KS_d)$ $\left\{ \begin{array}{l} f \in \mathbb{C}\langle x_1, \cdots, x_d \rangle \text{ is s.t.} \\ \mathrm{tr}(f(X_1, \cdots, X_d)) \geq 0 \\ \text{for } \forall n \ \forall X_i \in \mathbb{M}_n(\mathbb{C})_{\mathbb{R}} \text{ contractions} \\ \Rightarrow f + \varepsilon 1 \in \mathcal{A}_+ + K \end{array} \right.$

Rem. ~~[illegible]~~ True in non tracial setting.

Thm (KS) $KS_\infty \Leftrightarrow$ CEC

$\Leftrightarrow KS_2$

Fact $M \bar{\otimes} R$ is generated by two hermitian elements.

III. 3.

Juschenko – Popovych 2011

$\mathcal{A} = \mathbb{C}\mathbb{F}_d = \mathbb{C}\langle u_1, \cdots, u_d\rangle$, $u_i^* u_i = 1 = u_i u_i^*$

$\mathcal{A}_+ = \text{cone}\{x^* x : x \in \mathbb{C}\mathbb{F}_d\}$

$(JP_d)$ $\left\{ \begin{array}{l} \alpha \in \mathbb{M}_d(\mathbb{C})_{\mathbb{R}} \text{ is s.t.} \\ tr(\sum \alpha_{ij} U_i^* U_j) \geq 0 \text{ for } \forall n\ \forall U_i \in \mathcal{U}(\mathbb{M}_n) \\ \Rightarrow \sum \alpha_{ij} u_i^* u_j + \varepsilon 1 \in \mathcal{A}_+ + K \text{ for } \forall \varepsilon > 0 \end{array}\right.$

Thm (JP) $\quad JP_\infty \Leftrightarrow CEC$

We'll prove this.

Recall Kirchberg's Thm

$F_d :=$ closure $\{[tr(U_i^* U_j)]_{i,j} : n \in \mathbb{N}, U_1, \cdots, U_d \in \mathcal{U}(\mathbb{M}_n)\}$

$= \{[\tau_\omega(U_i^* U_j)]_{i,j} : U_i \in \mathcal{U}(R^\omega)\}$

$(M, \tau)$ $\quad G_d^M := \{[\tau(V_i^* V_j)]_{i,j} : V_1, \cdots, V_d \in \mathcal{U}(M)\}$

$G_d := \bigcup_M G_d^M$

$\Theta_d :=$ positive semi definite matrices with 1's on diagonals

$= \{[\langle \xi_i, \xi_j\rangle] : \xi_1, \cdots, \xi_d \in \mathcal{H}$ unit vectors$\}$

Thm (Kirchberg) $(M, \tau) \hookrightarrow (R^\omega, \tau_\omega) \Leftrightarrow G_d^M \subseteq F_d$ for $\forall d$ //

$\rightsquigarrow JP_\infty \Rightarrow CEC$.

Cor For $(M, \tau)$, TFAE

(1) $(M, \tau) \hookrightarrow (R^\omega, \tau_\omega)$

(2) $\forall d\ \forall \alpha \in \mathbb{M}_d(\mathbb{C})_{\mathbb{R}}$

If $tr(\sum \alpha_{ij} U_i^* U_j) \geq 0$ for $\forall n\ \forall U_i \in \mathcal{U}(\mathbb{M}_n)$

then $\tau(\sum \alpha_{ij} V_i^* V_j) \geq 0$ for $\forall V_i \in \mathcal{U}(M)$.

Pf (1) ⇒ (2) : Obvious.

(2) ⇒ (1) Suppose $(M, \tau) \not\hookrightarrow (R^\omega, \tau_\omega)$ and

$\exists Z \in G_d^M \setminus F_d$.

By HB, $\exists \alpha \in \mathbb{M}_d(\mathbb{C})$

s.t. $\operatorname{Re} \sum \alpha_{ij} Z_{ij} < \inf_{W \in F_d} \operatorname{Re} \sum \alpha_{ij} W_{ij}$.

III.4.

Since $Z$ and $W$ are hermitian,

We can replace $\alpha$ with $\frac{\alpha+\alpha^*}{2} \in \mathbb{M}_d(\mathbb{C})_h$

and $\sum \alpha_{ij} Z_{ij} < \inf_{W \in F_d} \sum \alpha_{ij} W_{ij}$

Arrange $\alpha_{11} \in \mathbb{R}$ so that

$$\sum \alpha_{ij} Z_{ij} < 0 \le \inf_{W \in F_d} \sum \alpha_{ij} W_{ij} \quad . \quad \blacksquare$$

Combined with previous results,

we obtain JP's Thm $JP_\infty \Leftrightarrow CEC$.

<u>Some known facts about $F_d \subseteq G_d \subseteq \Theta_d$</u> Dykema – Juschenko

Thm (Grothendieck)

$$\Theta_d \subseteq K_G^{\mathbb{C}} \operatorname{conv}\{ [\bar{\alpha}_i \beta_j]_{ij} : \alpha_i, \beta_j \in \mathbb{C}, |\alpha_i|, |\beta_j| = 1 \}$$

Thm $\Theta_d \cap \mathbb{M}_d(\mathbb{R}) = F_d \cap \mathbb{M}_d(\mathbb{R})$

<u>Pf.</u>

$\Theta_d \cap \mathbb{M}_d(\mathbb{R}) = \{ [\langle \xi_i, \xi_j \rangle]_{ij} : \xi_i \in \mathcal{H}_{\mathbb{R}}, \|\xi_i\| = 1 \}$

Clifford alg (CAR algebra)

$a(\xi) \in (\mathbb{M}_{2^{\dim \mathcal{H}_{\mathbb{R}}}})_h$ $\quad a(\xi)$ unitary

$a(\xi)^* a(\xi) = \|\xi\|^2$ $\quad$ if $\|\xi\| = 1$

$a(\xi) a(\eta) + a(\eta) a(\xi) = 2 \langle \xi, \eta \rangle$

$\rightsquigarrow \operatorname{tr}( a(\eta)^* a(\xi)) = \langle \xi, \eta \rangle$ $\quad \blacksquare$

Thm (Dykema – Juschenko)

- $F_3 = \Theta_3$
- $G_4 \subsetneq \Theta_4$

OPEN: Does $F_4 = G_4$ imply CEC ??

$\quad JP_4$

III. 5.

Approx Commuting Unitary Matrices ( arXiv 1211.2712 )

$E = \mathrm{span}\{U_1, U_2, U_3\} \subseteq C^*\mathbb{F}_3$ op space

$\simeq \ell_1^3$

KC $\iff \| \ \|_{max} = \| \ \|_{min}$ completely on $E \otimes E$

Pisier's Trick

$\iff \forall n \ \forall \alpha \in \mathbb{M}_3(\mathbb{M}_n(\mathbb{C}))$

$\|\alpha\|_{max} := \sup\{ \|\sum \alpha_{ij} U_i V_j\| : U_i, V_j \in \mathcal{U}(\mathbb{B}(\mathcal{H})), \ [U_i, V_j] = 0 \}$

coincides with

$\|\alpha\|_{min} := \sup\{ \text{———} : \text{———}, \dim \mathcal{H} < +\infty \}$.

Lem $\forall n \ \forall \alpha \in \mathbb{M}_3(\mathbb{M}_n(\mathbb{C}))$

$$\|\alpha\|_{max} = \inf_{\varepsilon > 0} \sup\{ \|\sum \alpha_{ij} U_i V_j\| : U_i, V_j \in \mathcal{U}(\mathbb{B}(\mathcal{H})), \ \dim \mathcal{H} < +\infty, \ \|[U_i, V_j]\| < \varepsilon \}$$

Pf.

$\geq$ Contradiction argument involving norm ultraproduct.

$\leq$ Follows from the fact that $C^*(\mathbb{F}_\infty \times \mathbb{F}_\infty)$ is QD. $\boxed{/\!/}$

Fact max = min for $\alpha \in \mathbb{M}_d(\mathbb{M}_n(\mathbb{R}))$ $\alpha_{ij} \in \mathbb{R}$

max = min for $\alpha \in \mathbb{M}_2(\mathbb{M}_n(\mathbb{C}))$ ∵ WMA $U_1 = 1$, $V_1 = 1$

Fact $U, V \in \mathcal{U}(\mathbb{M}_n)$ $\|[U, V]\| < \varepsilon$

( There need not $\exists U', V' \in \mathcal{U}(\mathbb{M}_n)$ s.t.

$[U', V'] = 0$ & $U \approx U'$, $V \approx V'$ [Voiculescu 83]

$\exists m \geq n$ $\exists \tilde{U}, \tilde{V} \in \mathcal{U}(\mathbb{M}_m)$ $[\tilde{U}, \tilde{V}] = 0$

$U \underset{\sqrt{\varepsilon}}{\approx} P_n(\tilde{U})$, $V \underset{\sqrt{\varepsilon}}{\approx} P_n(\tilde{V})$ compression

Strong KC : $\forall \kappa > 0 \ \exists \varepsilon > 0$

If $U_i, V_j \in \mathcal{U}(\mathbb{M}_n)$, $\|[U_i, V_j]\| < \varepsilon$

$\Rightarrow \exists m \geq n$ $\exists \tilde{U}_i, \tilde{V}_j \in \mathcal{U}(\mathbb{M}_m)$, $[\tilde{U}_i, \tilde{V}_j] = 0$

$\|U_i - P_n(\tilde{U}_i)\| < \kappa$,

$\|V_j - P_n(\tilde{V}_j)\| < \kappa$ for all $i, j$

$\Leftarrow$? $C^*\mathbb{F}_2 \otimes C^*\mathbb{F}_2 \otimes \mathbb{B}(\ell_2)$ max = min