Regularity properties and classification of nuclear C^*-algebras

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1.1 Nuclearity

1.1 THEOREM (Elliott)
AF algebras are classified by their scaled ordered K_0-groups.

1.2 CONJECTURE (Elliott)
Separable nuclear C^*-algebras are classified by K-theoretic data.

But why *nuclear* C^*-algebras?

1.3 THEOREM (Choi–Effros; Kirchberg)
A is nuclear iff A has the CPAP.

1.4 REMARKS

- Finite-dimensional approximations seem promising, but c.p. approximations are not a natural framework to study K-theoretic data.
- Nuclearity is a flexible concept; it can be characterized in many different ways, which make contact with many areas of operator algebras.

1.2 Closeness

1.5 DEFINITION (Kadison–Kastler; Christensen)
Let $A, B \subset \mathcal{B}(\mathcal{H})$ be C^*-algebras acting on the same Hilbert space.
We write $d(A, B) < \gamma$, if for each $a \in A^1$ there is $b \in B^1$ with $\|a - b\| < \gamma$ and vice versa.
We write $A \subset_\gamma B$, if there is $0 < \gamma' < \gamma$ such that for each $a \in A^1$ there is $b \in B^1$ with $\|a - b\| < \gamma'$.

1.6 CONJECTURE (Kadison–Kastler)
If $A, B \subset \mathcal{B}(\mathcal{H})$ are separable C^*-algebras and $d(A, B) < \gamma$ for some small enough γ, then A and B are unitarily isomorphic.

1.7 THEOREM (Christensen–Sinclair–Smith–White–W)
Let $A, B \subset \mathcal{B}(\mathcal{H})$ be C^*-algebras, with A separable and nuclear and $d(A, B) < 10^{-11}$.
Then, there is a unitary $u \in \mathcal{B}(\mathcal{H})$ such that $A = uBu^*$.

1.8 THEOREM (Christensen–Sinclair–Smith–White–W)
For $n \in \mathbb{N}$ there is $\gamma > 0$ such that the following holds:
Let $A, B \subset \mathcal{B}(\mathcal{H})$ be C^*-algebras, with A separable and $\dim_{\mathrm{nuc}} A \leq n$, and with $A \subset_\gamma B$.
Then, there exists an embedding $A \hookrightarrow B$.

REMARK K-theoretic invariants tend to be preserved under closeness.

1.3 The purely infinite case

1.9 DEFINITION (Cuntz)
A simple C^*-algebra A is called purely infinite, if for any $0 \neq a, b \in A_+$ there is $x \in A$ such that $a = x^* b x$.

Being purely infinite means that positive elements can be compared (in the sense of Murray–von Neumann subequivalence) in a strong way.

1.10 THEOREM (Kirchberg)
Let A be a separable, nuclear and simple C^*-algebra.
Then, A is purely infinite iff $A \cong A \otimes \mathcal{O}_\infty$.

1.11 THEOREM (Kirchberg)
Let A be separable and exact.
Then, A embeds into $\mathcal{O}_2$, $A \hookrightarrow \mathcal{O}_2$.
If, moreover, A is nuclear, then there are an embedding $\iota : A \hookrightarrow \mathcal{O}_2$ and a conditional expectation $\Phi : \mathcal{O}_2 \to \iota(A)$.

1.12 THEOREM (Kirchberg; Phillips)
Kirchberg algebras with UCT are classified by their K-theory.

1.4 The stably finite case

There are many partial results, for example:

1.13 THEOREM (Elliott–Gong–Li)
Simple AH algebras of bounded topological dimension are classified by their Elliott invariants.
(In fact, E–G–L show that *very* slow dimension growth is enough.)

TASK Find stably finite versions of $\mathcal{O}_\infty$ (and maybe also of $\mathcal{O}_2$) to shed new light on existing classification results in the stably finite case, and also to (hopefully) unify the purely infinite and the stably finite cases.

1.5 Towards a structural conjecture

1.14 QUESTION In what way, and under which conditions, are finite topological dimension, Murray–von Neumann comparison of positive elements, $\mathcal{D}$-stability (for $\mathcal{D} = \mathcal{O}_\infty, \mathcal{O}_2, \ldots$) and classifiability related?

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2.1 Order zero maps

2.1 DEFINITION A c.p.c. map $\varphi : A \to B$ has order zero, if it respects orthogonality, i.e.

$$(e \perp f \in A_+ \Rightarrow \varphi(e) \perp \varphi(f) \in B_+).$$

$\mathrm{CPC}_\perp(A,B) := \{$c.p.c. order zero maps $A \to B\}$.

2.2 THEOREM (W–Zacharias; using results of Wolff)
Let $\varphi : A \to B$ be a c.p.c. order zero map.
Then, there are a $*$-homomorphism

$$\pi_\varphi : A \to \mathcal{M}(C^*(\varphi(A))) \subset B^{**}$$

and

$$0 \leq h_\varphi \in \mathcal{M}(C^*(\varphi(A))) \cap \pi_\varphi(A)'$$

such that

$$\varphi(a) = h_\varphi \pi_\varphi(a) \text{ for } a \in A.$$

2.3 COROLLARY Let $\varphi : A \to B$ be a c.p.c. order zero map. Then,

- $\varphi^{(n)}$ has order zero for all n
- there is an induced map $W(\varphi) : W(A) \to W(B)$
- for $0 \leq f \in \mathcal{C}_0((0,1])$, we may define a c.p. order zero map

$$f(\varphi)(\,.\,) := f(h_\varphi)\pi_\varphi(\,.\,) : A \to B$$

(functional calculus for order zero maps).

Moreover, there is a 1-1 correspondence

$$\mathrm{CPC}_\perp(A,B) \longleftrightarrow \mathrm{Hom}(\mathcal{C}_0((0,1]) \otimes A, B).$$

2.4 THEOREM (Loring)
C.p.c. order zero maps with finite-dimensional domains are given by weakly stable relations.

More precisely: Let F be a finite-dimensional C^*-algebra and let $\epsilon > 0$.
Then, there is $\delta > 0$ such that the following holds:
If $\varphi : F \to A$ is c.p.c. δ-order zero, then there is a c.p.c. order zero map $\bar{\varphi} : F \to A$ such that $\|\varphi - \bar{\varphi}\| < \epsilon$.

2.2 Decomposition rank

2.5 DEFINITION (Kirchberg–W)
Let A be a C^*-algebra, $n \in \mathbb{N}$. We say A has decomposition rank at most n, $\mathrm{dr}\, A \leq n$, if the following holds:

For any $\mathcal{F} \subset A$ finite and any $\varepsilon > 0$ there is a finite-dimensional c.p.c. approximation

$$A \xrightarrow{\psi} F \xrightarrow{\varphi} A$$

with

$$\varphi \circ \psi =_{\mathcal{F},\varepsilon} \mathrm{id}_A$$

and such that F can be written as

$$F = F^{(0)} \oplus \ldots \oplus F^{(n)}$$

with

$$\varphi^{(i)} := \varphi|_{F^{(i)}}$$

having order zero.

2.6 PROPOSITION For X a locally compact metrizable space, $\mathrm{dr}\,\mathcal{C}_0(X) = \dim X$.

PROOF Use partitions of unity and barycentric subdivision. ∎

2.7 PROPOSITION $\mathrm{dr}\,A = 0$ iff A is AF.

PROOF If A is unital, consider $A \to F \to A$ and note that almost unital order zero maps are almost $*$-homomorphisms.

In the nonunital case, use an idempotent approximate unit for A. ∎

2.8 PROPOSITION dr behaves well w.r.t. quotients, limits, tensor products, hereditary subalgebras, Morita equivalence.

2.9 COROLLARY If A has continuous trace, then $\mathrm{dr}\, A = \dim \hat{A}$.

2.10 THEOREM If A is l-subhomogeneous, then
$\mathrm{dr}\, A = \max_{k=1,\ldots,l} \dim(\mathrm{Prim}_k A)$.

2.11 COROLLARY $\mathrm{dr}\, A \leq \dim_{\mathrm{ASH}} A \leq \dim_{\mathrm{AH}} A$.

2.12 DEFINITION A has locally finite decomposition rank, if for any finite $\mathcal{F} \subset A$ and $\varepsilon > 0$ there is $B \subset A$ such that $\mathrm{dr}\, B < \infty$ and $\mathcal{F} \subset_{\varepsilon} B$.

2.3 Quasidiagonality

2.13 LEMMA Let $\varphi : B \to A$ be a c.p.c. map between C^*-algebras. Then, for any $x, y \in B$ we have

$$\|\varphi(xy) - \varphi(x)\varphi(y)\| \leq \|\varphi(xx^*) - \varphi(x)\varphi(x^*)\|^{\frac{1}{2}} \|y\|.$$

PROOF Use Stinespring's Theorem. ∎

2.14 LEMMA Let A, B be C^*-algebras, $a \in A_+$ with $\|a\| \leq 1$ and $\eta > 0$. If

$$A \xrightarrow{\psi} B \xrightarrow{\varphi} A$$

are c.p.c. maps satisfying

$$\|\varphi\psi(a) - a\|, \ \|\varphi\psi(a^2) - a^2\| \leq \eta,$$

then, for all $b \in B$,

$$\|\varphi(\psi(a)b) - \varphi\psi(a)\varphi(b)\| \leq 3^{\frac{1}{2}}\eta^{\frac{1}{2}}\|b\|.$$

PROOF We have

$$\|\varphi(\psi(a)^2) - (\varphi\psi(a))^2\| \leq \|\varphi(\psi(a)^2) - (\varphi\psi(a^2))\| \leq 3\eta,$$

so

$$\|\varphi(\psi(a)b) - \varphi\psi(a)\varphi(b)\| \leq (3\eta)^{\frac{1}{2}}\|b\|$$

for all $b \in B$ by Lemma 2.13. ∎

2.15 PROPOSITION (Kirchberg–W)
If $\operatorname{dr} A \leq n < \infty$, then there is a system

$$(A \xrightarrow{\psi_\lambda} F_\lambda \xrightarrow{\varphi_\lambda} A)_{\lambda \in \Lambda}$$

of c.p.c. approximations for A with finite-dimensional F_λ, n-decomposable c.p.c. maps φ_λ and approximately multiplicative c.p.c. maps ψ_λ.

In particular, A embeds into $\prod_\Lambda F_\lambda / \bigoplus_\Lambda F_\lambda$.

2.16 COROLLARY If $\operatorname{dr} A < \infty$, then A is quasidiagonal (and hence stably finite).

2.17 EXAMPLES The Toeplitz algebra $\mathcal{T}$ and the Cuntz algebras $\mathcal{O}_n$ have infinite decomposition rank.

2.4 Nuclear dimension

What do we do when A is not necessarily finite?

2.18 DEFINITION (W–Zacharias)
nuclear dimension, $\dim_{\mathrm{nuc}} A \leq n$:
defined as decomposition rank, but in

$$(F = F^{(0)} \oplus \ldots \oplus F^{(n)}, \psi, \varphi = \varphi^{(0)} + \ldots + \varphi^{(n)})$$

only asking the $\varphi^{(i)}$ to be contractions (instead of φ).

2.19 PROPOSITION (W–Zacharias)
Nuclear dimension agrees with decomposition rank in the commutative and in the zero-dimensional case; it behaves well w.r.t. quotients, limits, tensor products, hereditary subalgebras, Morita equivalence, and even extensions.

2.20 PROPOSITION (W–Zacharias)
If $\dim_{\text{nuc}} A \leq n < \infty$, then there is a system

$$(A \xrightarrow{\psi_\lambda} F_\lambda \xrightarrow{\varphi_\lambda} A)_{\lambda \in \Lambda}$$

of c.p. approximations for A with finite-dimensional F_λ, n-decomposable c.p. maps φ_λ and approximately order zero c.p.c. maps ψ_λ.

In particular, there is a c.p.c. order zero embedding of A into $\prod_\Lambda F_\lambda / \bigoplus_\Lambda F_\lambda$.

2.21 COROLLARY If A is unital with finite nuclear dimension and no nontrivial trace, then A is (weakly) purely infinite.

(Using the previous proposition in connection with results of Kirchberg.)

2.5 Kirchberg algebras

2.22 THEOREM (W–Zacharias)
For $n = 2, 3, \ldots$, we have $\dim_{\mathrm{nuc}} \mathcal{O}_n \leq 2$.

2.23 COROLLARY Let A be a UCT Kirchberg algebra.
Then, $\dim_{\mathrm{nuc}} A \leq 5$.

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3.1 Being strongly self-absorbing

3.1 DEFINITION (Toms–W)

A unital C^*-algebra $\mathcal{D}$ is strongly self-absorbing, if $\mathcal{D} \neq \mathbb{C}$ and there is a $*$-isomorphism

$$\varphi : \mathcal{D} \xrightarrow{\cong} \mathcal{D} \otimes \mathcal{D}$$

such that

$$\varphi \approx_{\mathrm{a.u.}} \mathrm{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}}.$$

A C^*-algebra A is $\mathcal{D}$-stable, if $A \cong A \otimes \mathcal{D}$.

3.2 THEOREM (Effros–Rosenberg; Kirchberg)
If $\mathcal{D}$ is strongly self-absorbing, then $\mathcal{D}$ is simple and nuclear, and $\mathcal{D}$ either has a unique tracial state or is purely infinite.

3.3 EXAMPLES

(i) UHF algebras of infinite type; $M_{2^\infty} = M_2^{\otimes\infty}, M_{3^\infty}, \ldots$

(ii) $\mathcal{O}_2 = \mathrm{C}^*(s_1, s_2 \mid s_i^* s_i = \mathbf{1} = \sum_{i=1,2} s_i s_i^*)$

(iii) $\mathcal{O}_\infty = \mathrm{C}^*(s_1, s_2, \ldots \mid s_i^* s_i = \mathbf{1} \geq \sum_{i\in\mathbb{N}} s_i s_i^*)$

(iv) $\mathcal{O}_\infty \otimes M_{2^\infty}, \ldots$

(v) $\mathcal{Z}$ the Jiang–Su algebra, a finite analogue of $\mathcal{O}_\infty$. $\mathcal{Z}$ can be written as a stationary inductive limit

$$\varinjlim(Z_{2^\infty,3^\infty}, \alpha),$$

where

$$Z_{2^\infty,3^\infty} = \{f \in \mathcal{C}([0,1], M_{2^\infty} \otimes M_{3^\infty}) \mid f(0) \in M_{2^\infty} \otimes \mathbf{1},\ f(1) \in \mathbf{1} \otimes M_{3^\infty}\}$$

and α is a trace-collapsing endomorphism of $Z_{2^\infty,3^\infty}$ (Rørdam–W).

3.4 THEOREM (Dadarlat–Rørdam; W)
If $\mathcal{D}$ is strongly self-absorbing, then $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{Z}$.

REMARKS

- Any strongly self-absorbing C^*-algebra is K_1-injective (using Gong–Jiang–Su); respective hypotheses in earlier papers are obsolete.
- The known strongly self-absorbing examples form a hierarchy, with $\mathcal{O}_2$ at the top and $\mathcal{Z}$ at the bottom.

3.2 $\mathcal{D}$-stability

3.5 THEOREM (Rørdam; Toms–W)
Let A be separable and $\mathcal{D}$ strongly self-absorbing. Then, A is $\mathcal{D}$-stable iff there is a $*$-homomorphism

$$\varphi : A \otimes \mathcal{D} \to \prod_{\mathbb{N}} A / \bigoplus_{\mathbb{N}} A$$

such that $\varphi \circ \mathrm{id}_A \otimes \mathbf{1}_{\mathcal{D}} = \iota_A$.

If A is unital and $\mathcal{D} = \lim_{\to} D_i$, then A is $\mathcal{D}$-stable iff for each i there is a unital $*$-homomorphism

$$\varrho_i : D_i \to \left(\prod_{\mathbb{N}} A / \bigoplus_{\mathbb{N}} A\right) \cap A'.$$

3.6 THEOREM (Toms–W)
For any strongly self-absorbing $\mathcal{D}$, $\mathcal{D}$-stability passes to limits, quotients, hereditary subalgebras, and extensions.

REMARK There are results on the structure of $\mathcal{C}(X)$-algebras with strongly self-absorbing fibres or $\mathcal{D}$-stable fibres.

3.3 $\mathcal{Z}$-stability

3.7 PROPOSITION (Toms–W)
Approximate divisibility implies $\mathcal{Z}$-stability.

3.8 THEOREM (Toms–W)
– as far as we know –
All C^*-algebras classified to date are $\mathcal{Z}$-stable.

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4.1 Strict comparison

4.1 DEFINITION A simple C^*-algebra A has strict comparison of positive elements, if

$$d_\tau(a) < d_\tau(b)\ \forall\, \tau \in T(A) \Longrightarrow \langle a\rangle \leq \langle b\rangle$$

for all $0 \neq a, b \in A_+$.

4.2 THEOREM (Rørdam)
If A is exact, unital and $\mathcal{Z}$-stable, then A has strict comparison.

4.3 LEMMA (Toms–W)
Suppose A is a unital C^*-algebra with $\mathrm{dr}\, A \leq n$.
Given $a, d^{(0)}, \ldots, d^{(n)} \in A_+$ such that

$$d_\tau(a) < d_\tau(d^{(i)}) \text{ for } i = 0, \ldots, n \text{ and all } \tau \in T(A),$$

then, $\langle a \rangle \leq \langle d^{(0)} \rangle + \ldots + \langle d^{(n)} \rangle$ in $W(A)$.

4.4 REMARK One can show directly that if $\mathrm{dr}\, A < \infty$, then quasitraces are traces.

4.2 The conjecture

4.5 CONJECTURE (Toms–W)
For a nuclear, separable, simple, finite, unital and nonelementary C^*-algebra A, t.f.a.e.:

(i) A has finite decomposition rank

(ii) A is $\mathcal{Z}$-stable

(iii) A has strict comparison of positive elements

REMARKS

(a) (ii) $\Longrightarrow$ (iii) has been shown by Rørdam

(b) (ii) $\Longrightarrow$ (i) is known in many cases, using classification results

(c) the conjecture can also be formulated in the nonunital and nonsimple case

d) in the simple, but not necessarily finite case, replace 'decomposition rank' by 'nuclear dimension' in (i).

4.3 Finite decomposition rank and $\mathcal{Z}$-stability

4.6 THEOREM (W)
Let A be separable, simple, unital with finite decomposition rank.
Then, A is $\mathcal{Z}$-stable.

Ingredients of the proof:

1. An approximately central sequence of unital $*$-homomorphisms

   $$Z_{p,p+1} = \{f \in \mathcal{C}_0([0,1], M_p \otimes M_{p+1} \mid f(0) \in M_p \otimes \mathbf{1},\ f(1) \in \mathbf{1} \otimes M_{p+1}) \to A$$

   for any $p \in \mathbb{N}$ will do.
2. By Rørdam–W, we need to find $x \in W(A)$ such that

   $$px \leq \langle \mathbf{1}_A \rangle \leq (p+1)x$$

   (in an approximately central way).
3. Finite decomposition rank implies ‘enough’ comparison (by Lemma 4.3) to find x as above.
4. n-decomposable approximations allow to do things in an approximately central way.

4.4 Pure finiteness

4.7 DEFINITION We say a separable, simple, nuclear C^*-algebra A is purely finite if it is finite, if it has strict comparison and if $W(A)$ is almost divisible.

QUESTION Is almost divisibility implied by strict comparison?

4.8 THEOREM (W; in progress)
Let A be simple, separable, unital, with locally finite decomposition rank.
If A is purely finite, then A is $\mathcal{Z}$-stable.

Ingredients of the proof:

As in the proof of Theorem 4.6, an approximately central sequence of unital $*$-homomorphisms

$$Z_{p,p+1} \to A$$

for any $p \in \mathbb{N}$ will do.

By Rørdam–W, we need to find a c.p.c. order zero map

$$\Phi : M_p \to A$$

and $v \in A$ such that

$$vv^* = \mathbf{1}_A - \Phi(\mathbf{1}_{M_p}) \text{ and } v^*v \leq \Phi(e_{11})$$

and such that $\Phi(M_p)$ and v are approximately central.

The following is the key result for constructing both Φ and ν.

LEMMA For $m \in \mathbb{N}$, there is $\alpha_m > 0$ such that the following holds:
Let A be separable, simple, unital, purely finite.
Let $\mathbf{1}_A \in B \subset A$ be a C^*-subalgebra with $\mathrm{dr}\, B \leq m$, and let $k, l \in \mathbb{N}$.
If

$$\varphi : M_l \to A_\infty \cap B'$$

is c.p.c. order zero, then there is a c.p.c. order zero map

$$\psi : M_k \to A_\infty \cap B' \cap \varphi(M_l)'$$

such that

$$\tau(\psi(\mathbf{1}_k)\varphi(\mathbf{1}_l)b) \geq \alpha_m \cdot \tau(\varphi(\mathbf{1}_l)b)$$

for all $b \in B_+$ and $\tau \in T_\infty(A)$.

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5.1 TA$\mathcal{S}$-algebras

5.1 DEFINITION Let $\mathcal{S}$ be a class of separable unital C^*-algebras.
Let A be simple, separable and unital.
We say A is TA$\mathcal{S}$, if the following holds:
Given $0 \neq e \in A_+$, $\mathcal{F} \subset A$ finite and $\epsilon > 0$, there is $B \subset A$ with $B \in \mathcal{S}$ and

(i) $\|[\mathbf{1}_B, a]\| < \epsilon$ for $a \in \mathcal{F}$

(ii) $\mathbf{1}_B \mathcal{F} \mathbf{1}_B \subset_\epsilon B$

(iii) $(\mathbf{1}_A - \mathbf{1}_B) \prec e$.

If $\mathcal{S}$ is the class of finite-dimensional C^*-algebras (or tensor products of such with closed intervals), we write TAF (or TAI, respectively).

5.2 THEOREM (Lin)
The class of UCT TAI algebras satisfies the Elliott conjecture.

5.2 The lfdr, rr0, $\mathcal{Z}$-stable case

5.3 THEOREM Let A be separable, simple, unital and $\mathcal{Z}$-stable, with locally finite decomposition rank and real rank zero.
Then, A is TAF.

5.3 Localizing at $\mathcal{Z}$

5.4 THEOREM (W)
Let $\mathcal{A}$ be a class of separable, simple, nuclear, unital C^*-algebras such that, for any $A, B \in \mathcal{A}$ and any isomorphism of invariants

$$\Lambda : \mathrm{Inv}(A) \to \mathrm{Inv}(B),$$

there are prime integers p, q such that Λ can be lifted along Z_{p^∞, q^∞}.

Then,

$$\mathcal{A}^{\mathcal{Z}} := \{A \otimes \mathcal{Z} \mid A \in \mathcal{A}\}$$

satisfies the Elliott conjecture.

5.5 THEOREM (Lin–Niu)
Let $\mathcal{B}$ denote the class of separable, simple, nuclear, unital C^*-algebras with UCT, and such that tensor products with UHF algebras are TAI.

Then,

$$\mathcal{B}^{\mathcal{Z}} := \{B \otimes \mathcal{Z} \mid B \in \mathcal{B}\}$$

satisfies the Elliott conjecture.

5.6 COROLLARY (Using Q.Lin–Phillips)
C^*-algebras associated to minimal, uniquely ergodic, smooth dynamical systems are classified by their ordered K-theory.

5.7 COROLLARY (Using Gong and Theorems 4.8 and 5.5)
Simple, unital AH algebras with slow dimension growth are classified.

REMARK The elements of $\mathcal{B}$ have rationally Riesz K_0-groups.

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4.3 Finite decomposition rank and $\mathcal{Z}$-stability
4.4 Pure finiteness

Classification up to $\mathcal{Z}$-stability
5.1 TA$\mathcal{S}$ algebras
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Minimal dynamical systems
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6.3 $\mathcal{Z}$-stability

6.1 The setup

Let X be a compact, metrizable, infinite space and $\alpha : X \to X$ a homeomorphism.

The crossed product is given by

$$\mathcal{C}(X) \rtimes_\alpha \mathbb{Z} := \mathrm{C}^*(\mathcal{C}(X), u \mid uf(\,.\,)u^* = f(\alpha^{-1}(\,.\,))).$$

6.1 PROPOSITION If α is minimal, then $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$ is simple, unital, nuclear, with a tracial state.
If α is uniquely ergodic, then the tracial state is unique.

6.2 PROBLEM

(i) Determine the structure of such crossed products.

(ii) Classify them.

(iii) Draw conclusions about the underlying dynamical systems.

- Will focus on (i) and (ii) with minimal actions.
- Will solve (ii) in the finite dimensional, minimal, uniquely ergodic case.

  (There is little hope for a complete solution in the infinite dimensional case.)
- This is joint work with A. Toms, and with K. Strung.

6.2 Classification up to $\mathcal{Z}$-stability

6.3 THEOREM (Strung–W; based on work of Lin and Phillips)
Let X be compact, metrizable, infinite. Let $\alpha : X \to X$ be a minimal homeomorphism.
Let $\mathcal{S}$ be a class of separable, unital C^*-algebras which is closed under taking hereditary unital C^*-subalgebras.
Let $\mathcal{U}$ be a UHF algebra and $y \in X$.
Set

$$A_{\{y\}} := \mathrm{C}^*(\mathcal{C}(X), \mathcal{C}_0(X \setminus \{y\})u).$$

If $A_{\{y\}} \otimes \mathcal{U}$ is TA$\mathcal{S}$, then $(\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}) \otimes \mathcal{U}$ is TA$\mathcal{S}$.

REMARKS

- In the above situation, $A_{\{y\}}$ is simple, ASH, with $\mathrm{dr}\, A_{\{y\}} \leq \dim X$.
- $A_{\{y\}} \otimes \mathcal{U}$ is TAF if projections separate traces (by Theorem 5.3).
- $A_{\{y\}}$ has the same ordered K_0-group and the same trace space as the crossed product. (Lin–Phillips)

6.4 COROLLARY Let X be compact, metrizable, infinite. Let $\alpha : X \to X$ be a minimal, uniquely ergodic homeomorphism.
Then, $(\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}) \otimes \mathcal{U}$ is TAF for any UHF algebra $\mathcal{U}$.

6.5 COROLLARY The result yields classification up to $\mathcal{Z}$-stability, without any dimension restriction on X.

6.3 $\mathcal{Z}$-stability

6.6 THEOREM (Toms–W)
Let X be compact, metrizable, infinite, with finite topological dimension.
Let $\alpha : X \to X$ be a minimal homeomorphism.
Then, $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$ is $\mathcal{Z}$-stable.

REMARK Note that α is not asked to be uniquely ergodic.

6.7 COROLLARY The class

$$\mathcal{E} = \{\mathcal{C}(X) \rtimes_\alpha \mathbb{Z} \mid X \text{ compact, metrizable, infinite, finite dimensional, } \alpha \text{ a uniquely ergodic, minimal homeomorphism} \}$$

is classified by ordered K-theory.

In the remainder, let us focus on the proof of the

THEOREM Let X be compact, metrizable, infinite, with finite topological dimension. Let $\alpha : X \to X$ be minimal. Then, $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$ is $\mathcal{Z}$-stable.

REMARK This is doubly instructive, since one proves

$$\dim_{\mathrm{nuc}}(\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}) \leq 2 \dim X + 1$$

in a similar fashion.