Spectra of C* algebras, classification.

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Conventions and Notations

- Spaces $P, X, Y, \cdots$ are second countable, algebras $A, B, \ldots$ are separable, ...
- ... except corona spaces $\beta(P) \setminus P$, multiplier algebras $\mathcal{M}(B)$, and ideals of corona algebras $\mathrm{Q}(B) := \mathcal{M}(B)/B$, ...
  as e.g., $\mathrm{Q}(\mathbb{R}_+, B) := \mathrm{C_b}(\mathbb{R}_+, B)/\,\mathrm{C_0}(\mathbb{R}_+, B) \subset \mathrm{Q}(SB)$.
- we use the naturally isomorphism $\mathcal{I}(A) \cong \mathbb{O}(\mathrm{Prim}(A))$.
- $\mathbb{Q}$ denotes the Hilbert cube (with its coordinate-wise order).

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$(\bigcap_n \pi^{-1}(U_n))^\circ = \pi^{-1}((\bigcap_n U_n)^\circ)$ *for each sequence $U_1, U_2, \ldots \in \mathbb{O}(X)$*

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The algebra $A \otimes \mathcal{O}_2 \otimes \mathbb{K}$ is uniquely determined by X up to (unitarily homotopic) isomorphisms.

Recall (2):

Notice: A continuous epimorphism $\pi\colon P \to X$ is **is not pseudo-open.**

There is **no pseudo-open** continuous epimorphism from the Cantor space $\{0,1\}^\infty$ onto the Hausdorff space $[0,1]$.

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We call a map $\Psi\colon \mathbb{O}(X) \to \mathbb{O}(Y)$ "*lower semi-continuous*" if $(\bigcap_n \Psi(U_n))^\circ = \Psi((\bigcap_n U_n)^\circ)$ for each sequence $U_1, U_2, \ldots \in \mathbb{O}(X)$.

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If one works with *closed sets*, then one has to replace intersections by unions and interiors by closures.

Recall (3):

A subset C of X is "saturated" if $C = \mathrm{Sat}(C)$, where $\mathrm{Sat}(C)$ means the intersection of all $U \in \mathbb{O}(X)$ with $U \supset C$.

Definition

A sober T_0 space X is called "**coherent**" if the intersection $C_1 \cap C_2$ of two *saturated* quasi-compact subsets $C_1, C_2 \subset X$ is again quasi-compact.

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Next we give some partial results concerning **Question 4:**
Is every (second-countable) *coherent* locally quasi-compact sober T_0 space X homeomorphic to the primitive ideal spaces Prim(A) of some *amenable* A?

Let X a locally quasi-compact sober T_0 space, $\mathcal{F}(X)$ the lattice of closed subsets $F \subset X$.

Definition

The topological space $\mathcal{F}(X)_{\mathrm{lsc}}$ is the set $\mathcal{F}(X)$ with the Scott topology T_0 topology (or: **order topology**) that is **generated** by the complements

$$\mathcal{F}(X) \setminus [\emptyset, F] = \{G \in \mathcal{F}(X)\, ;\;\; G \cap U \neq \emptyset\} =: \mu_U$$

(where $U = X \setminus F$) of the intervals $[\emptyset, F]$ for all $F \in \mathcal{F}(X)$.

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(where $U = X \setminus F$) of the intervals $[\emptyset, F]$ for all $F \in \mathcal{F}(X)$.

The **Fell-Vietoris topology** is the topology, that is *generated* by the sets μ_U ($U \in \mathbb{O}(X)$) and the sets $\mu_C := \{G \in \mathcal{F}(X)\,;\; G \cap C = \emptyset\}$ for all *quasi-compact* $C \subset X$.

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Definition

A map $f\colon X \to [0, \infty)$ is a **Dini function** if it is lower semi-continuous and $\sup f(F) = \inf_{\{} \sup f(F_n)\}$ for every decreasing sequence $F_1 \supset F_2 \supset \cdots$ of closed subsets and $F = \bigcap_n F_n$.

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For sober spaces X one has also that a function $f\colon X \to [0, 1]$ is Dini, if and only if, $f\colon X \to [0, 1]_{\text{lsc}}$ is continuous and the restriction $f\colon X \setminus f^{-1}(0) \to (0, 1]_{\text{lsc}}$ is **proper**.

The ordered Hilbert cube $\mathbb{Q}$ is nothing else $\mathcal{F}(Y)$ for $Y := X_0 \uplus X_0 \uplus \cdots$ where $X_0 := (0, 1]_{\mathrm{lsc}}$. The Fell-Vietoris topology becomes just the ordinary Hausdorff topology on $\mathbb{Q}$.

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If X is locally quasi-compact sober T_0 space, then a dense sequence $g_1, g_2, \ldots$ in the Dini functions g on X with $\sup g(X) = 1$ defines an order isomorphism $\iota\colon \mathcal{F} \to \mathbb{Q}$ onto a max-closed subset $\iota(\mathcal{F})$ of $\mathbb{Q}$. Indeed, $\iota(F) := (\sup g_1(F), \sup g_2(F), \ldots) \in \mathbb{Q}$ does the job, and $\iota(\emptyset) = 0$, $\iota(X) = 1$.

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The image $\iota(\mathcal{F}(X))$ is closed in $\mathbb{Q}$ (with Hausdorff topology) and ι defines an isomorphism from $\mathcal{F}(X)$ (with Fell-Vietoris topology) onto $\iota(\mathcal{F}(X))$.

In a T_0 space X (e.g. $X = [0,1]_{\mathrm{lsc}}$) one has usually that quasi-G_δ subsets $Z \subset X$, — i.e., intersections of a sequence $Z_1, Z_2, \ldots$ with $Z_n = U_n \cup F_n$ (U_n open, F_n closed) — are not G_δ subsets of X. But, for continuous map $\pi \colon P \to X$, one has that $\pi^{-1}(Z)$ is G_δ, hence is a Polish space.

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The Scott-topology on $\mathbb{Q}$ induces the Scott-topology on $\mathcal{F}(X)$, in which X becomes an quasi-G_δ of $\mathcal{F}(X)$ and $\mathbb{Q}$. Since $\mathbb{Q}$ is a primitive ideal space, we get that *there is a (not necessarily l.c.) Polish space P and an open and continuous surjection $\pi \colon P \to X$, such that the fibers $\pi^{-1}(x)$ are disjoint unions of infinite-dimensional projective spaces.*

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In a more direct way one sees, that X has a Polish topology (induced from $\mathbb{Q}$ with Hausdorff topology) and a continuous partial order on it with the property that the corresponding Scott topology is just the T_0 topology of X.

In this way $X \subset \overline{X}^H \setminus \{0\} \subset \mathcal{F}(X) \subset \mathbb{Q}$ as Polish spaces.

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Below, we denote by $Y = \overline{X}^H \setminus \{\emptyset\} \subset \mathcal{F}(X) \setminus \{\emptyset\}$ the closure of X in $\mathbb{Q} \setminus \{0\}$.

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Proposition

*The image $\eta(X) \cong X$ in $\mathcal{F}(X) \setminus \{\emptyset\}$ of a l.q-c. (second countable) sober T_0 space X is **closed** in $\mathcal{F}(X) \setminus \{\emptyset\}$ with respect to the Fell-Vietoris topology on $\mathcal{F}(X)$,*
if and only if,
*X is **coherent**, if and only if,*
*the set $\mathcal{D}(X)$ of Dini functions on X is **convex**, if and only if,*
$\mathcal{D}(X)$ is min***-closed**, if and only if,*
*$\mathcal{D}(X)$ is **multiplicatively closed**.*

Lemma

Each closed subset $F \subset \mathbb{Q}_H$ is a coherent sober subspace F_{lsc} of $\mathbb{Q}_{\mathrm{lsc}}$. If $F = \bigcap_n F_n$ for sequence $F_1 \supset F_2 \supset \cdots$ in $\mathcal{F}(\mathbb{Q}_H)$, and if each $(F_n)_{\mathrm{lsc}}$ is the primitive ideal space of an amenable C-algebra, then F_{lsc} is the primitive ideal space of an amenable C*-algebra.*

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Corollary

If there is a coherent sober l.c. space X *that is not homeomorphic to the primitive ideal space of an amenable C*-algebra, then there is* $n \in \mathbb{N}$ *and a finite union* Y *of (Hausdorff-closed) cubes in* $[0, 1]^n$ *such that* Y *with induced order-topology is not the primitive ideal space of any amenable C*-algebra.*

I do not know if the following (Hausdorff) closed subset F of $[0,1]^2$ (with the coherent topology on F that is induced from $([0,1]_{\mathrm{lsc}})^2$) is the primitive ideal space of an amenable C^*-algebra:
F is the union of the segments $\overline{(0,0)\,(1,0)}$, $\overline{(1,0)\,(1,1)}$, $\overline{(1/2,1)\,(1,1)}$, and $\overline{(1/2,1/2)\,(1/2,1)}$.

A subspace $Z \subset [0,1]_{\mathrm{lsc}}$ The *sober* subspaces Z of $[0,1]_{\mathrm{lsc}}$ are all coherent and are primitive ideal spaces of amenable C^*-algebras, because the subsets $Z \cup \{\inf Z\}$ are order isomorphic to closed subsets of $[0,1]$.

The saturated quasi-compact subsets of the cartesian product $([0,1]_{\mathrm{lsc}})^n$ are the upward directed closed sets.

Examples of ***non-coherent*** *and of* ***coherent*** *Prim(A)*:

Let $X := \text{Prim}(A)$ for the C^*-algebra $A \subset C([0,1], M_2)$ consisting of the continuous maps $h\colon [0,1] \to M_2$ with $h(1) \in \Delta :=$ diagonal matrices.

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Then $X = [0,1] \cup_\pi \{2,3\}$ with $\pi(2) := \pi(3) := 1$ (The point 1 is replaced by two points 2 and 3). We have $Y = [0,1] \cup \{2,3\} \subset \mathbb{R}$ with its ordinary Hausdorff topology for $Y \cong$ closure of X in $\mathcal{F}_H$ ($= \mathcal{F}$ with Fell-Vietoris topology). The Dini functions on X are given by the set of non-negative continuous functions $g \in C(Y)$ with $g(1) = \max(g(2), g(3))$. The closed subset F_1 of X that correponds to 1 is $F_1 = \{2,3\}$.

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The natural embedding of X into Y maps X onto $Y \setminus \{1\}$. Thus, the condition $g(1) = \max(g(2), g(3))$ reads as $\lim_{t \nearrow 1} g(t) = \max(g(2), g(3))$.

The topology Y_{lsc} on Y generated by the supports of the Dini function (on X but naturally extended to Y as continuous functions) is given by the lattice of those *open* subsets V of Y that satisfy $1 \in V$ if $V \not\subset [0, 1)$, i.e., $V \cap \{1, 2, 3\} \in \{ \emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\} \}$.

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With this topology, the space Y is the primitive ideal space $Y \cong \mathrm{Prim}(B)$ of a unital separable nuclear C^*-algebra B, as follows:

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With this topology, the space Y is the primitive ideal space $Y \cong \mathrm{Prim}(B)$ of a unital separable nuclear C^*-algebra B, as follows: Let $D := \mathbb{K} + (\mathbb{C}1 \oplus \mathbb{C}1) \subset \mathcal{L}(\ell_2 \oplus \ell_2)$. Then D is unital, and has three primitive ideals $1 \cong \{0\}$, $2 \cong \mathbb{K} + (\mathbb{C}1 \oplus 0)$ and $3 \cong \mathbb{K} + (0 \oplus \mathbb{C})$ with topology as induced on $\{1,2,3\} \subset Y$ by $Y_{\rm lsc}$.

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The Dini functions on Y_{lsc} are given by the continuous functions $g \in C(Y)_+$ with $g(1) \geq \max(g(2), g(3))$. It follows that $\mathcal{D}(Y_{\mathrm{lsc}})$ is invariant under min (i.e., Y_{lsc} is *coherent*). Thus, $C(Y) = C^*(\mathcal{D}(Y_{\mathrm{lsc}})) = C^*(\mathcal{D}(X)) \subset \ell_\infty(X)$.

The Dini functions on $Y_{\rm lsc}$ are given by the continuous functions $g \in C(Y)_+$ with $g(1) \geq \max(g(2), g(3))$. It follows that $\mathcal{D}(Y_{\rm lsc})$ is invariant under min (i.e., $Y_{\rm lsc}$ is *coherent*). Thus, $C(Y) = C^*(\mathcal{D}(Y_{\rm lsc})) = C^*(\mathcal{D}(X)) \subset \ell_\infty(X)$.

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(Notice that $Y_{\mathrm{lsc}} \setminus \{1\} = X$ as topological spaces.)

The natural continuous epimorphisms from Y onto Y_{lsc}, and from $Y \setminus \{1\}$ onto X are *not* pseudo-open. Indeed, the closure of $[0, 1) = \bigcap_n [0, 1 - 1/n]$ in Y (respectively in $Y \setminus \{1\}$) is not closed in Y_{lsc}, (respectively in X), but $[0, 1 - 1/n]$ is closed in X and Y_{lsc} for each $n \in \mathbb{N}$.

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It follows, that $\mathcal{F}(Y_{\mathrm{lsc}})_H \to \mathcal{F}(Y_{\mathrm{lsc}})_{\mathrm{lsc}}$ and $\mathcal{F}(X)_H \to \mathcal{F}(X)_{\mathrm{lsc}}$ are not pseudo-open (even if we remove $\emptyset$).

The map $\psi\colon [0,1] \cup [4,5] \to X$ with $\psi(t) := \psi(4+t) := t$ for $t \in [0,1)$ and $\psi(1) := 2$, $\psi(5) := 3$ defines a continuous map from $[0,1] \cup [4,5]$ onto X that is pseudo-open.

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The compression map $g\in C([0,1]\cup[4,5])_+ \to \widehat{g}\in\mathcal{D}(X)\subset C(Y)$ is given by $\widehat{g}(t) := \max(g(t), g(4+t))$ for $t\in[0,1]$ and $\widehat{g}(2) := g(1)$, $\widehat{g}(3) := g(5)$.

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(We have no explicite construction for Z, but it seems likely, that one can take a suitable subset Z of $[0,1]\times\{1,2,3\}$ or of $[0,1]\times\{0,1,1/n,1-1/n\,;\,n\in\mathbb{N}\}$.)

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Example of quasi-open and quasi-epimorphic continuous map that is not surjective (an is not open):

Take $\pi\colon (0,1)\to(0,1]_{\mathrm{lsc}}$ with $\pi(t):=t$.

Let $\mathbb{O}(X) :=$ the lattice of open sets of a T_0 space X.

Definition (Actions of T0 spaces)

An increasing map $\Psi \colon \mathbb{O}(X) \to \mathcal{I}(A)$ is called an **action** of X on A.

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The action Ψ is *lower semi-continuous* if the functions

$x \in X \mapsto \|a|\overline{\{x\}}\|$ are l.s.c. on X for all $a \in A$.

(Equivalently: $\Psi((\bigcap_n U_n)^\circ) = \bigcap_n \Psi(U_n)$ for $U_n \in \mathbb{O}(X)$.)

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In case of l.s.c. actions one can take $X' := (\bigcap_{U \in \Psi^{-1}(A)} U)^\circ$ and $A' := A/\,\Psi(\emptyset)$ to get non-degenerate actions.

Example 1: X locally compact Hausdorff and A a $C_0(X)$-algebra. Then $\Psi(U) := C_0(U)A$ defines an upper s.c. action of X on A. This action is also *lower* s.c., iff, A is the algebra of continuous sections (zero at ∞) of a continuous field over X.

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Example 2: Let $X = \mathrm{Prim}(B)$ then $\Psi_B(U) := \bigcap_{J \notin U} J$ is a lattice isomorphism from $\mathbb{O}(X)$ onto $\mathcal{I}(B)$. This Ψ_B is the *natural* action of $\mathrm{Prim}(B)$ on B.

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Example 3: If $\mathcal{S} \subset \mathrm{CP}(A, B)$ and $X := \mathrm{Prim}(B)$, then, using the inverse of the natural action Ψ_B, we can define closed ideas $\Psi(U)$ of A by

$$\Psi_{\mathcal{S}}(U)_+ := \{a \in A\,;\ T(c^* ac) \in \Psi_B(U),\ \text{for all } T \in \mathcal{S},\ c \in A\}\,.$$

I.e., for $J \lhd B$, $\Psi_{\mathcal{S}}(J)$ is the maximal closed ideal I of A with $T(I) \subset J$ for all $T \in \mathcal{S}$.

The action $\Psi_{\mathcal{S}}$ is lower s.c. $\Psi_{\mathcal{S}}$ is non-degenerate, iff, $\mathcal{S}$ is non-degenerate in sense of following Definition.

Definition (Non-degenerate sets of c.p. maps)

We call a subset $\mathcal{S} \subset \mathrm{CP}(A, B)$ *non-degenerate*, if the ideal generated by $\{ T(a) ;\ a \in A,\ T \in \mathcal{S} \}$ is dense in B, and $a \in A_+$ and $T(a) = 0 \, \forall T \in \mathcal{S}$ implies $a = 0$.

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Example 4: If $\mathcal{H}$ is a Hilbert B-module and $d \colon A \to \mathcal{L}(\mathcal{H})$ is a *-representation, then one can consider the set $\mathcal{S}$ of B-valued coefficients $T \colon a \mapsto \langle d(a)y, y \rangle \in B$ for $y \in \mathcal{H}$. The action $\Psi_{\mathcal{S}} \colon \mathcal{I}(B) \to \mathcal{I}(A)$ of Example 3 has the property, that $\Psi_{\mathcal{S}}(J)$ is the kernel of the (induced) representation $[d] \colon A \to \mathcal{L}(\mathcal{H} / \mathcal{H}J)$.

Definition (M.o.c. Cones)

A subset $\mathcal{C} \subset \mathrm{CP}(A, B)$ is a *matricially operator-convex cone* (m.o.c.c.), if

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Can define the m.o.c.c. $\mathcal{C}(\mathcal{S})$ generated by a subset $\mathcal{S} \subset \mathrm{CP}(A, B)$, because intersections $\mathcal{C} := \bigcap_\alpha \mathcal{C}_\alpha$ of families of m.o.c. cones $\mathcal{C}_\alpha \subset \mathrm{CP}(A, B)$ are again a m.o.c. cones.

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Recall that (non-degenerate) m.o.c. cones $\mathcal{C} \subset$ define (non-degenerate) lower s.c. actions $\Psi_{\mathcal{C}} \colon \mathcal{I}(B) \cong \mathbb{O}(\mathrm{Prim}(B)) \to \mathcal{I}(A)$.

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Let F_∞ free group, $E := C^*(F_\infty)$, and let $\mathcal{C}' := \mathcal{C} \otimes^{\max} \mathcal{C}(\mathrm{id}) \subset$ denote the m.o.c. cone in $\mathrm{CP}(A \otimes^{\max} E, B \otimes^{\max} E)$ that is generated by $\mathcal{S} = \{V \otimes \mathrm{id}\,;\ V \in \mathcal{C}\}$. Let $\Psi' \colon \mathcal{I}(B \otimes^{\max} E) \to \mathcal{I}(A \otimes^{\max} E)$ the action corresponding to $\mathcal{C}'$.

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Theorem (Separation)

If $\mathcal{C} \subset \mathrm{CP}(A, B)$ is given, and the action Ψ' is defined as above, then $V \in \mathrm{CP}(A, B)$ is in $\mathcal{C}$, if and only if, $(V \otimes \mathrm{id})(\Psi'(J)) \subset J$ for all $J \in \mathcal{I}(B \otimes^{\max} E)$.

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Corollary (Cones determined by its action, see Example 3)

If B is nuclear, or if A is exact and $\mathcal{C} \subset \mathrm{CP}_{\mathrm{nuc}}(A, B)$, then, for $V \in \mathrm{CP}_{\mathrm{nuc}}(A, B)$ holds: $V \in \mathcal{C}$ iff $V(\Psi_{\mathcal{C}}(J)) \subset J$ for all $J \lhd B$.

Hilbert A-B-modules versus m.o.c. cones.

We say that a Hilbert A–B-module (given by $\mathcal{H}_B$ and *-morphism $d\colon A \to \mathcal{L}(\mathcal{H}_B)$) is $\mathcal{C}$-*compatible* if the B-valued coefficient maps $a \mapsto \langle d(a)y, y\rangle$ are in $\mathcal{C}$ for all $y \in \mathcal{H}_B$.

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(i) Isometric A-B-module morphisms,

(ii) (infinite) Hilbert A-B-module sums, and

(iii) passage to Hilbert A-B-submodules.

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Proposition (Modules versus Cones, see Example 4)

The relation between m.o.c. cones $\mathcal{C} \subset \mathrm{CP}(A, B)$ and the family of $\mathcal{C}$-compatible Hilbert A–B-modules, is a bijection between m.o.c. cones and all families of Hilbert A–B-modules that are invariant under the operations (i)–(iii) above.

Theorem (Existence of h_0)

Suppose that A and B are stable, A exact and B strongly purely infinite, and that $\Psi\colon \mathbb{O}(\mathrm{Prim}(B)) \to \mathcal{I}(A)$ is a non-degenerate action of $\mathrm{Prim}(B)$ on A lower s.c. and monotone upper s.c.
Then there is a non-degenerate nuclear monomorphism $h_0\colon A \to B$ such that $h_0 \oplus h_0$ is unitarily equivalent to h_0, and $\mathcal{C}(h_0) = \mathrm{CP}_{\mathrm{rn}}(\mathrm{Prim}(B); A, B)$.

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Corollary (lifting of G-actions on Prim(A))

If A is nuclear, stable and $A \cong A \otimes \mathcal{O}_2$, then every continuous action $\widehat{\alpha}\colon G \to \mathrm{Homeo}(\mathrm{Prim}(A))$ lifts to an continuous action $\alpha\colon G \to \mathrm{Aut}(A)$ on A.

The proof needs two “reconstruction theorems”:

Theorem (Reconstruction, H.H.,E.K.)

Suppose that A is a nuclear and stable, that Ω *is a* sup–inf *closed sub-lattice of* $\mathcal{I}(A) \cong \mathbb{O}(\mathrm{Prim}(A))$ *with* $\mathrm{Prim}(A), \emptyset \in \Omega$ *. Then there is a non-degenerate *-monomorphism* $H_0 \colon A \to \mathcal{M}(A)$ *with following properties:*

(i) *The infinite repeat* $\delta_\infty \circ H_0$ *is unitarily equivalent to* H_0*.*

(ii) *For every* $U \in \mathbb{O}(\mathrm{Prim}(A))$ *holds* $H_0(J(V)) = H_0(A) \cap \mathcal{M}(A, J(U))$ *where* $V \in \Omega$ *is given by* $V = \bigcup\{W \in \Omega\, ;\ W \subset U\}$*.*

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(ii) *For every* $U \in \mathbb{O}(\mathrm{Prim}(A))$ *holds* $H_0(J(V)) = H_0(A) \cap \mathcal{M}(A, J(U))$ *where* $V \in \Omega$ *is given by* $V = \bigcup\{W \in \Omega\,;\; W \subset U\}$.

The H_0 *is uniquely determined up to unitary homotopy.*
The Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}}$ *of the Hilbert A-A-module* $\mathcal{H} := (A, H_0)$ *is stable and strongly purely infinite; and it is the same as the C*-Fock algebra* $\mathcal{F}(\mathcal{H})$ *of* $\mathcal{H}$.
The natural embedding of A into $\mathcal{O}_{\mathcal{H}}$ *defines a lattice isomorphism from* Ω *onto* $\mathbb{O}(\mathrm{Prim}(\mathcal{O}_{\mathcal{H}}))$ *and a* $\mathrm{KK}(\Omega; \cdot, \cdot)$*-equivalence.*

Theorem (G-equivariant reconstruction)

If a locally compact group G acts on A by $\alpha\colon G \to \mathrm{Aut}(A)$ with $\alpha(g)(J) \in \Omega$ for all $J \in \Omega$, then H_0 (in the Reconstruction theorem) can be found such that H_0 is G-equivariant,
i.e., there is an action $\gamma\colon G \to \mathrm{Aut}(A)$ of G on A that is outer conjugate to α, such that

$$\gamma(g)\,(H_0(a)b) = H_0\,(\gamma(g)(a))\,\gamma(g)(b)\,.$$

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Then G acts on $\mathcal{O}_{\mathcal{H}}$ such that that $\iota\colon A \hookrightarrow \mathcal{O}_{\mathcal{H}}$ defines a $\mathrm{KK}^G(\Omega;\cdot,\cdot)$-equivalence from A into $\mathcal{O}_{\mathcal{H}}$ (w.r.t. γ on A).

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If A is of type I, then $\mathcal{O}_{\mathcal{H}}$ is a $\mathbb{Z}$-crossed product of an inductive limit of type I C^*-algebras by an automorphism.