Quantum spaces with no group structure

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Quantum spaces

- \mathfrak{C} category of C^* -algebras
 - Objects: unital C*-algebras
 - Morphisms: unital *-homomorphisms

Definition

A compact quantum space is an object of the category dual to \mathfrak{C} .

• Compact Hausdorff spaces are quantum spaces

$$X \iff \mathsf{C}(X)$$

(we call such quantum spaces classical)

 A quantum space corresponding to A ∈ Ob(𝔅) is classical ⇔ the C*-algebra A is commutative.

Compact quantum semigroups

Definition

A compact quantum semigroup is a pair

•
$$(A, \Delta)$$

unital C^* -algebra unital *-homomorphism $A \to A \otimes A$

•
$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta.$$

Example A = C(S) (S — compact semigroup), $\Delta(f) \in A \otimes A = C(S \times S),$ $\Delta(f)(s, t) = f(st).$

Compact quantum groups

Definition

A compact quantum group is a compact quantum semigroup (A, Δ) such that

$$\operatorname{span}\left\{ (a \otimes \mathbf{1})\Delta(b) \, \middle| \, a, b \in A \right\} \subset_{\operatorname{dense}} A \otimes A,$$

 $\operatorname{span}\left\{ \Delta(a)(\mathbf{1} \otimes b) \, \middle| \, a, b \in A \right\} \subset_{\operatorname{dense}} A \otimes A.$

• In case A = C(S) density conditions correspond to

$$(s \cdot t = s \cdot t') \implies (t = t'),$$

 $(s \cdot t = s' \cdot t) \implies (s = s').$

Example

- $A = C^*(\Gamma)$ (Γ discrete group),
- $\Delta(\gamma) = \gamma \otimes \gamma$ $(\gamma \in \Gamma).$

Haar measure

Theorem (S.L. Woronowicz)

Let (A, Δ) be a compact quantum group. Then there exists a unique state h on A such that

$$(\mathrm{id}\otimes h)\Delta(a)=(h\otimes\mathrm{id})\Delta(a)=h(a)\mathbf{1}$$

for all $a \in A$.

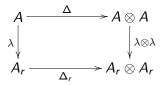
• For A = C(G) (*G* — compact group) $h(f) = \int_{G} f(t) dt.$ • For $A = C^{*}(\Gamma)$ (Γ — discrete group)

$$h(x) = \left(\delta_e | \lambda(x) \delta_e\right),\,$$

where λ is the regular representation $C^*(\Gamma) \to C^*_r(\Gamma)$.

Reduced quantum group

- (A, Δ) compact quantum group, h it's Haar measure.
- Let $J = \{a \in A \mid h(a^*a) = 0\}$, $A_r = A/J$, $\lambda : A \twoheadrightarrow A_r$.
- There is a unique $\Delta_r : A_r \to A_r \otimes A_r$ such that



(A_r, Δ_r) is a compact quantum group — reduced (A, Δ).
For A = C*(Γ) we have A_r = C^{*}_r(Γ).

Hopf algebra

- (A, Δ) compact quantum group.
- There exists a unique dense unital *-subalgebra $\mathscr{A} \subset A$ such that

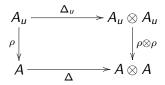
$$\Delta(\mathscr{A})\subset \mathscr{A}\otimes_{\mathrm{alg}}\mathscr{A}$$

and $(\mathscr{A}, \Delta|_{\mathscr{A}})$ is a **Hopf** *-algebra (counit *e*, antipode κ).

- For $A = C^*(\Gamma)$ we have $\mathscr{A} = \mathbb{C}[\Gamma]$.
- If A = C(G) then \mathscr{A} is the span of matrix elements of irreps.

Universal quantum group

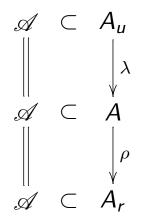
- (A, Δ) compact quantum group, \mathscr{A} it's Hopf algebra.
- The enveloping C*-algebra A_u of A carries a unique comultiplication Δ_u : A_u → A_u ⊗ A_u such that



where $\rho : A_u \rightarrow A$ is the quotient map.

- (A_u, Δ_u) is a compact quantum group universal (A, Δ) .
- The Hopf algebra associated with (A_u, Δ_u) is \mathscr{A} .
- Also the Hopf algebra associated with (A_r, Δ_r) is \mathscr{A} .

Completions of \mathscr{A}



Woronowicz characters

- (A, Δ) compact quantum group, \mathscr{A} it's Hopf algebra.
- ∃! family (f_z)_{z∈C} of non-zero multiplicative functionals on A such that
 - for an $a \in \mathscr{A}$ the function $z \mapsto f_z(a)$ is entire,
 - $f_0 = e$, $f_{z_1} * f_{z_2} = f_{z_1+z_2}$, $\left(\psi * \varphi = (\psi \otimes \varphi) \circ \Delta\right)$

•
$$f_{\overline{z}}(a^*) = \overline{f_{-z}(a)}$$
 for all $a \in \mathscr{A}$, $z \in \mathbb{C}$,

- $f_z(\kappa(a)) = f_{-z}(a)$ for all $a \in \mathscr{A}$, $z \in \mathbb{C}$,
- $\kappa^2(a) = f_{-1} * a * f_1$ for all $a \in \mathscr{A}$. $\left(\psi * a = (\mathrm{id} \otimes \psi)\Delta(a)\right)$

• $(f_{it})_{t \in \mathbb{R}}$ are *-characters of \mathscr{A} \implies they extend to characters of A_u .

- The family (f_z)_{z∈C} is related to the modular function on the dual of (A, Δ).
- We have $f_z = e$ for all z iff the Haar measure is a trace.

Quantum two-torus

•
$$\theta \in]0,1[, A_{\theta} = C^{*}(u,v)$$

 $u^{*}u = \mathbf{1} = uu^{*}, v^{*}v = \mathbf{1} = vv^{*}, uv = e^{2\pi i\theta}vu.$

- A_θ admits a faithful trace.
- If there is $\Delta : A_{\theta} \to A_{\theta} \otimes A_{\theta}$ such that (A_{θ}, Δ) is a c.q.g. then
 - the Haar measure of $(A_{ heta}, \Delta)$ is a trace,
 - $\kappa^2 = \mathrm{id}$ (i.e. (A_{θ}, Δ) is a Kac algebra). (P.M.S.)
- A_{θ} is nuclear. Therefore
 - $A_{\theta_r} = A_{\theta_u}$, (This property is called *co-amenability.*)
 - the counit of \mathscr{A} is continuous on A_{θ} . (Bedos, Murphy & Tuset)
- This means that $A_{ heta}$ must admit a character, but it does not.
- The quantum two-torus is not a quantum group (for $\theta \neq 0$).
- Neither is any higher dimensional quantum torus.

Bratteli-Elliott-Evans-Kishimoto quantum two-spheres

• $C_{\theta} = C(S_{\theta}^2)$ is defined as $C_{\theta} = A_{\theta}{}^{\alpha}$, where $\alpha \in Aut(A_{\theta})$

$$\alpha(u) = u^*, \quad \alpha(v) = v^*.$$

- C_{θ} admits a faithful trace,
- C_{θ} is nuclear,
- C_{θ} does not admit a character.

Standard Podleś quantum two-spheres

- $q\in [-1,1]\setminus \{0\}$, $\mathsf{C}(S^2_{q,0})=\mathscr{K}^+.$
- Assume that there is $\Delta:\mathscr{K}^+\to\mathscr{K}^+\otimes\mathscr{K}^+$ such that (\mathscr{K}^+,Δ) is a c.q.g.
- One can show that it's Haar measure must be faithful.
- \mathscr{K}^+ admits a character, and so (\mathscr{K}^+, Δ) is co-amenable. Thus
 - all Woronowicz characters are continuous,
 - but there is only one character on \mathscr{K}^+ ,
 - so $f_{it} = e$ for all $t \in \mathbb{R}$,
 - so $f_z = e$ for all $z \in \mathbb{C}$,
 - so Haar measure of (\mathscr{K}^+, Δ) is a trace.
- There are no faithful traces on \mathscr{K}^+ .

Natsume-Olsen quantum two-spheres

•
$$t \in \left[0, \frac{1}{2}\right[, \quad B_t = C(S_t^2), \quad B_t = C^*(\zeta, z)$$

 $\zeta^* \zeta + z^2 = \mathbf{1} = \zeta \zeta^* + (t\zeta \zeta^* + z)^2,$
 $\zeta z - z\zeta = t\zeta(\mathbf{1} - z^2).$

• For
$$t = 0$$
 we get $C(S^2)$ and S^2 is not a group.

- We can show that if there is $\Delta : B_t \to B_t \otimes B_t$ such that (B_t, Δ) is a c.q.g. then
 - The Haar measure of G cannot be a trace,
 - *B_{tr}* possesses a character.
- Thus (B_t, Δ) must be co-amenable, so B_{tr} = B_{tu} = B_t ⇒ all Woronowicz characters are continuous on B_t.
- But B_t has only two characters (not enough).