18/11/2009

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Regularity properties and classification of nuclear C^*-algebras

1.1 Nuclearity

Theorem (Elliott) AF algebras are classified by their scaled ordered K_0-groups

1.2 Conjecture (Elliott) Separable nuclear C^*-alg are classified by K-theoretic data

Why nuclear C^*-algebras?

1.3 Theorem (Choi-Effros; Kirchberg)
A is nuclear iff A has the completely positive approximation property (CPAP)

$$\begin{array}{ccc} A & & A \\ & {}_{\psi_\lambda}\searrow \quad \nearrow^{\varphi_\lambda} & \\ & F_\lambda & \end{array} \qquad \varphi_\lambda \circ \psi_\lambda \to id_A$$

F_λ finite-dimensional $\quad \varphi_\lambda, \psi_\lambda$ cpc maps

1.4 Remarks
- Finite-dimensional approximations seem promising but cp approximations are not a natural framework to study K-theoretic data
- Nuclearity is a flexible concept; it can be characterized in many different ways which make contact with many areas of operator algebras

1.2 Closeness

1.5 Definition (Kadison - Kastler; Christensen)
$A, B \subset \mathcal{B}(H)$ C^*-algebras acting on the same Hilbert space. Write $d(A,B) < \gamma$ if, for each $a \in A^1$, $\exists b \in B^1$ with $\|a-b\| < \gamma$ and vice versa.
Write $A \subset_\gamma B$ if $\exists\, 0 < \gamma' < \gamma$ such that for each $a \in A^1$ $\exists b \in B^1$ with $\|a - b\| < \gamma'$.

1.6 Conjecture (Kadison - Kastler) If $A, B \subset B(H)$ are separable C^*-algebras and $d(A,B) < \gamma$ for some small enough γ, then A, B are unitarily isomorphic

ie. $\exists u \in B(\mathcal{H})$ such that $A = uBu^*$?

1.7 Theorem (Christensen - Sinclair - Smith - White - Winter)
Let $A, B \subset \mathcal{B}(\mathcal{H})$ be C^*-algebras with A separable and nuclear and $d(A,B) < 10^{-11}$.
Then there is a unitary $u \in B(\mathcal{H})$ such that $A = uBu^*$

1.8 Theorem (Christensen - Sinclair - Smith - White - Winter)
For $n \in \mathbb{N}$ $\exists \gamma > 0$ such that the following holds
Let $A, B \subset \mathcal{B}(\mathcal{H})$ be C^*-algebras with A separable and $\dim_{nuc} A \leq n$ and with $A \subset_\gamma B$.
Then there exists an embedding $A \hookrightarrow B$.

Remark: K-theoretic invariants tend to be preserved under closeness.

1.3 The purely infinite case

1.9 Definition (Cuntz): A simple C^*-algebra is called purely infinite if for any $0 \neq a, b \in A_+$ $\exists x \in A$ such that $a = x^* b x$

Being purely infinite means that positive elements can be compared

1.10 Theorem (Kirchberg) Let A be separable, nuclear and simple C^*-algebra. Then A is purely infinite if and only if $A \cong A \otimes \mathcal{O}_\infty$

$$\mathcal{O}_\infty = C^*\left(s_1, s_2, \dots \mid \sum_{i=1}^{n} s_i s_i^* \leq 1 \quad \forall n, \quad s_i^* s_i = 1\right)$$

1.11 Theorem (Kirchberg) Let A be separable and exact. Then A embeds into $\mathcal{O}_2$, $A \hookrightarrow \mathcal{O}_2$

$$\mathcal{O}_2 = C^*\left(s_1, s_2 \mid s_i^* s_i = 1, \ s_1 s_1^* + s_2 s_2^* = 1\right)$$

If moreover A is nuclear then there are an embedding

1.12 Theorem (Kirchberg, Phillips) Kirchberg algebras (sep., simple, nuclear, purely infinite) with UCT are classified by their K-theory (K-theory isomorphic ~~with~~ isomorphism of algs. inducing the K-theoretic isomorphism)

1.4 The Stably finite case

There are many partial results, for example:

1.13 Theorem (Elliott - Gong - Li): Simple AH algebras of bounded topological dimension are classified by their Elliott invariants.

B homogeneous: $B = p(M_k \otimes C(X))p$

B semihomogeneous: $B = \bigoplus_i p_i(M_{k_i} \otimes C(X))p_i$

B AH (approximately homogeneous): direct limit of semihomogeneous C^*-algebras.

In fact, E-G-L show that very slow dimension growth is enough.

(bounded topological dimension means that the covering dimension of the X_i's are bounded)

Task: Find stably finite versions of $\mathcal{O}_\infty$ (maybe also $\mathcal{O}_2$) to shed new light on existing classification results in the stably finite case and also to (hopefully) unify the purely infinite and stably finite case

1.5 Towards a structural conjecture

1.14 Question: In what way, and under what conditions, are finite topological dimension, Murray von-Neumann comparison of positive elements, $\mathcal{D}$-stability (for $\mathcal{D} = \mathcal{O}_\infty, \mathcal{O}_2, \dots$) and classifiability related?

Topological Dimension

2.1 Order zero maps

2.1 Definition A cpc map $\varphi : A \to B$ has order zero if it respects orthogonality ie

$$(e \perp f \in A_+ \Rightarrow \varphi(e) \perp \varphi(f) \in B_+)$$

$$CPC_\perp(A,B) := \{\text{c.p.c. order zero maps } A \to B\}$$

For general $e, f \in A$,

$$e \perp f \ :\Leftrightarrow\ ef = ef^* = ef^* = e^*f = e^*f^* = 0$$

2.2 Theorem (Winter - Zacharias; using results of Wolff)
Let $\varphi : A \to B$ be a CPC order zero map
Then there is a $*$-homomorphism

$$\pi_\varphi : A \longrightarrow \mathcal{M}(C^*(\varphi(A))) \subset B^{**}$$

and an element

$$0 \leq h_\varphi \in \mathcal{M}(C^*(\varphi(A))) \cap \pi_\varphi(A)'$$

such that

$$\varphi(a) = h_\varphi \pi_\varphi(a) \quad \text{for } a \in A.$$

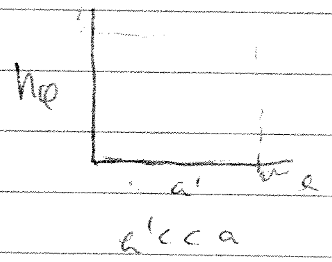
Note that $\|\varphi\| = \|h_\varphi\|$

The universal C^*-alg generated by a contraction h:

$$C^*(h \mid 0 \leq h \ \ \|h\| \leq 1) = C_0((0,1])$$

2.3 Corollary Let $\varphi: A \to B$ be a c.p.c. order zero map. Then

- $\varphi^{(n)}: M_n(A) \to M_n(B)$ has order zero for all $n \in \mathbb{N}$
- there is an induced map $W(\varphi): W(A) \to W(B)$
- for $0 \le f \in C_0((0,1])$ we may define a c.p. order zero map
$$f(\varphi)(\cdot) := f(h_\varphi)\pi_\varphi(\cdot) : A \to B$$
(functional calculus for order zero maps)



$$a' << a \Rightarrow \varphi(a') << \varphi(a)$$
$$\Rightarrow f_\varepsilon(\varphi(a')) << \varphi(a)$$

Moreover, there is a 1-1 correspondence
$$CPC_\perp(A,B) \longleftrightarrow Hom(C_0((0,1]) \otimes A, B)$$

2.4 Theorem (Loring): cpc order zero maps with finite-dimensional domains are given by weakly stable relations

More precisely: Let F be a finite-dimensional C^*-algebra and let $\varepsilon > 0$. Then there is $\delta > 0$ such that the following holds:

If $\varphi: F \to A$ is c.p.c. δ-order zero then there is a c.p.c. order zero map $\bar{\varphi}: F \to A$ such that $\|\varphi - \bar{\varphi}\| < \varepsilon$.

2.2 Decomposition rank

2.5 Definition (Kirchberg - Winter): Let A be a C^*-algebra, $n \in \mathbb{N}$. We say A has decomposition rank at most n, $\mathrm{dr}\, A \leq n$, if the following holds: For any $\mathcal{F} \subset A$ finite and any $\varepsilon > 0$ there is a finite-dimensional c.p.c approximation

$$A \xrightarrow{\psi} F \xrightarrow{\varphi} A$$

with

$$\varphi \circ \psi =_{\mathcal{F}, \varepsilon} \mathrm{id}_A$$

and such that F can be written as

$$F = F^{(0)} \oplus \ldots \oplus F^{(n)}$$

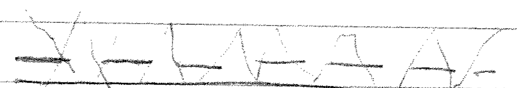
with

$$\varphi^{(i)} := \varphi|_{F^{(i)}}$$

having order zero.

2.6 **Proposition**: Let X be a locally compact metrizable space, $\mathrm{dr}\, C_0(X) = \dim X$

Proof: Uses partitions of unity and barycentric subdivision.

$X = [0,1]$  partition of unity

partition of unity: $X \longrightarrow \Delta^r$

2.7 Proposition: (A separable?) $\mathrm{dr}\, A = 0$ if and only if A is AF

Proof: If A is unital, consider $A \to F \to A$ and note that almost unital order zero maps are almost $*$-homomorphisms ($h_\varphi \approx 1$)
In the nonunital case, use an idempotent approximate unit for A ($e_{n+1}e_n = e_n$) ∎

2.8 Proposition: Decomp. rank behaves well with respect to quotients, limits, tensor products, hereditary subalgebras, morita equivalence.

$$dr(\varinjlim A_i) \leq \liminf_{i\to\infty} (dr\, A_i)$$

$$dr(A \otimes B) \leq (dr(A)+1)(dr(B)+1) - 1$$

$$B \subseteq_{her} A \qquad dr(B) \leq dr(A)$$

2.9 Corollary: If A has continuous trace then $dr\, A = \dim \hat{A}$.

2.10 Theorem: A separable. If A is l-subhomogeneous then $dr\, A = \max_{k=1,\dots,l} \dim(\mathrm{Prim}_k A)$

2.11 Corollary: $dr\, A \leq \dim_{ASH} A \leq \dim_{AH} A$

(if A is not ASH $\dim_{ASH} A = \infty$)

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2.8 Proposition: dr behaves well wrt quotients limits, tensor products, hereditary subalgebras, Morita equivalence

2.9 Corollary: If A has continuous trace then $dr\, A = \dim A$

2.12 Definition A has locally finite decomposition rank if for any finite $\mathcal{F} \subset A$, $\varepsilon > 0$ $\exists B \subset A$ s.t. $dr\, B < \infty$ and $\mathcal{F} \subset_\varepsilon B$.

2.3 Quasidiagonality

2.13 Lemma: $\varphi : B \to A$ cpc between C^*-algs A, B. Then for any $x, y \in B$ we have

$$\|\varphi(xy) - \varphi(x)\varphi(y)\| \leq \|\varphi(xx^*) - \varphi(x)\varphi(x^*)\|^{\frac{1}{2}} \|y\|$$

Proof: Uses Stinespring's Theorem

2.14 Lemma A, B C^*-algebras, $a \in A_+$, $\|a\| \le 1$ and $\eta > 0$. If

$$A \xrightarrow{\psi} B \xrightarrow{\varphi} A$$

are c.p.c maps satisfying

$$\|\varphi\psi(a) - a\|,\ \|\varphi\psi(a^2) - a^2\| \le \eta$$

then, for all $b \in B$,

$$\|\varphi(\psi(a)b) - \varphi\psi(a)\varphi(b)\| \le 3^{1/2}\eta^{1/2}\|b\|$$

Proof: We have

$$\|\varphi(\psi(a)^2) - (\varphi\psi(a))^2\| \le \|\varphi(\psi(a)^2) - (\varphi\psi(a^2))\| \le 3\eta$$

so

$$\|\varphi(\psi(a)b) - \varphi\psi(a)\varphi(b)\| \le (3\eta)^{1/2}\|b\|$$

for all $b \in B$ by Lemma 2.13. ■

~~If b is a central projection in~~ $\cancel{B = F^{(0)} \oplus \cdots \oplus F^{(n)}}$

$\cancel{\varphi/\psi}$

$$B = F^{(0)} \oplus \cdots \oplus F^{(n)}$$

$$\begin{array}{l} \varphi(\psi(a)1_{F^{(i)}}) \approx \varphi\psi(a)\varphi(1_{F_i}) \\ \quad \| \\ \varphi^{(i)}\psi^{(i)}(a) \qquad \varphi^{(i)}, \psi^{(i)} \text{ cpc order zero maps} \end{array}$$

2.15 Prop (Kirchberg-Winter): If $\mathrm{dr}\, A \le n < \infty$ then $\exists$ a system

$$\left(A \xrightarrow{\psi_\lambda} F_\lambda \xrightarrow{\varphi_\lambda} A\right)_{\lambda \in \Lambda}$$

of c.p.c. approximations for A with finite-dim F_λ, n-decomposable c.p.c maps φ_λ and approximately multiplicative cpc maps ψ_λ. In particular, A embeds into $\prod_\lambda F_\lambda \big/ \bigoplus_\lambda F_\lambda$

$A \xrightarrow{\psi_\lambda} F_\lambda$

$A \longrightarrow \prod F_\lambda / \bigoplus F_\lambda$

2.16 Corollary: If $\mathrm{dr}\, A < \infty$ then A is quasidiagonal (and hence stably finite)

Proof of 2.15 (Sketch): Assume A unital. $\mathcal{F} \subset\subset A_+$, $\varepsilon > 0$
Take $A \xrightarrow{\psi} F \xrightarrow{\varphi} A$

Since n-decomposable, can write $F = F^{(0)} \oplus \ldots \oplus F^{(n)}$
Find a projection $p = p^{(0)} \oplus \ldots \oplus p^{(n)} \in F$ such that
$\|\varphi(1_F - p)\| < \eta$, $\|\varphi^{(i)}(e)\| \geq \eta \|e\|$ for all $e \in p^{(i)} F^{(i)} p^{(i)}$

Set
$$\begin{array}{ccccc} A & \longrightarrow & F & \longrightarrow & A \\ & \bar{\psi}\searrow & \downarrow\uparrow & \nearrow \varphi & \\ & & pFp & & \end{array}$$

Check that $\bar{\psi}$ is $\mathcal{F}$-$\eta^{1/2}$-multiplicative ∎

2.17 Examples The Toeplitz algebra $\mathcal{T}$ and Cuntz algebra $\mathcal{O}_\infty$ have infinite dr.

2.4 Nuclear dimension

What to do when A is not necessarily finite?

~~2.18 D~~

2.18 Definition (Winter-Zacharias)
nuclear dimension, $\dim_{nuc} A \leq n$.
defined as decomposition rank, but in

$$\left(F = F^{(0)} \oplus \dots \oplus F^{(n)}, \psi, \varphi = \varphi^{(0)} + \dots + \varphi^{(n)}\right)$$

only asking the $\varphi^{(i)}$ be contractions (instead of φ)

2.19 Prop

Nuclear dimension agrees with decomposition rank in the commutative and in zero-dim

2.20 Prop (Winter-Zacharias) If $\dim_{nuc} A \leq n < \infty$ then there is a system

$$\left(A \xrightarrow{\psi_\lambda} F_\lambda \xrightarrow{\varphi_\lambda} A\right)_{\lambda \in \Lambda}$$

of c.p. approximations for A with finite-dimensional F_λ, n-decomposable cp maps φ_λ and approximately order zero ~~embedding~~ cpc maps ψ_λ

In particular, there is a cpc order zero embedding of A into $\prod_\Lambda F_\lambda / \bigoplus_\Lambda F_\lambda$

2.12 Corollary: If A is unital

2.5 Kirchberg algebras

2.22 Theorem (Winter–Zacharias) For $n = 2, 3, \ldots$, we have $\dim_{nuc} \mathcal{O}_n \leq 2$

2.23 Corollary: Let A be a UCT Kirchberg algebra. Then $\dim_{nuc} A \leq 5$.

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~~Str~~

Strongly self-absorbing algebras

3.1 Being strongly self-absorbing

3.1 Definition (Toms–Winter) A unital separable C^*-algebra D is strongly self-absorbing if $D \neq \mathbb{C}$ and there is a ~~$*$-homomorphism~~ $*$-isomorphism

$$\varphi : D \xrightarrow{\cong} D \otimes D$$

such that

$$\varphi \approx_{au} id_D \otimes 1_D$$

A C^*-algebra is D-stable if $A \cong A \otimes D$

3.2 Theorem (Effros–Rosenberg; Kirchberg) If D is strongly self-absorbing then D is simple and nuclear and D either has a unique tracial state or is purely infinite.

3.3 Examples

(i) UHF algebras of infinite type M_{2^∞}, M_{3^∞}, ...

(ii) $\mathcal{O}_2 = C^*(s_1, s_2 \mid s_i^* s_i = 1, \sum_{i=1,2} s_i s_i^*)$

(iii) $\mathcal{O}_\infty$

(iv) $\mathcal{O}_\infty \otimes M_{2^\infty}$

(v) $\mathcal{Z}$ the Jiang-Su algebra, a finite analogue of $\mathcal{O}_\infty$, $\mathcal{Z}$ can be written as a stationary inductive limit

$$\varinjlim (\mathcal{Z}_{2^\infty, 3^\infty}, \alpha)$$

where

$$\mathcal{Z}_{2^\infty,3^\infty} = \left\{ f \in C([0,1], M_{2^\infty} \otimes M_{3^\infty}) \;\middle|\; \begin{array}{l} f(0) \in M_{2^\infty} \otimes 1 \\ f(1) \in 1 \otimes M_{3^\infty} \end{array} \right\}$$

and α is trace-collapsing endomorphism of $\mathcal{Z}_{2^\infty, 3^\infty}$ (Rørdam-Winter)

i.e.

$$\alpha : \mathcal{Z}_{2^\infty, 3^\infty} \longrightarrow \mathcal{Z}_{2^\infty, 3^\infty}$$

$$T(\quad) \longleftarrow T(\quad) : \alpha_T$$

all traces are sent by α_T to a single trace

3.4 **Theorem** (Dadarlat - Rørdam; Winter)
If D is strongly self-absorbing then $D \cong D \otimes \mathcal{Z}$

Remarks

▷ Any strongly self-absorbing C^*-alg is K_1-injective (using Gong - Jiang - Su); respective hypotheses in earlier papers are obsolete

▷ The known strongly self-absorbing examples form a hierarchy with $\mathcal{O}_2$ ~~at~~ the top and $\mathcal{Z}$ at the bottom (everything embeds into $\mathcal{O}_2$, $\mathcal{Z}$ embeds into everything)

3.2 D-stability

3.5 **Theorem** (Rørdam; Toms - Winter) Let A be separable and D strongly self-absorbing. Then A is D-stable if and only if $\exists$ $*$-homomorphism

$$\varphi : A \otimes D \longrightarrow \prod_{\mathbb{N}} A \Big/ \bigoplus_{\mathbb{N}} A$$

Such that $\varphi \circ id_A \otimes 1_D = \iota_A$

If A is unital and $D = \varinjlim D_i$ then A is D stable if and only if for each i there is a unital $*$-homomorphism

$$\varrho_i : D_i \longrightarrow \left(\prod_{\mathbb{N}} A \Big/ \bigoplus_{\mathbb{N}} A \right) \cap A'$$

3.6 **Theorem** (Toms - Winter) For any strongly self-absorbing D, D-stability passes to limits, quotients, hereditary subalgebras and ~~exten~~ extensions

Extensions:

σ a $*$-homomorphism

$$0 \to J \to A \underset{\sigma}{\overset{\longleftarrow}{\longrightarrow}} B \to 0$$

J: $\mathcal{D}$-stable $\qquad$ B: $\mathcal{D}$-stable

Assume that A is unital

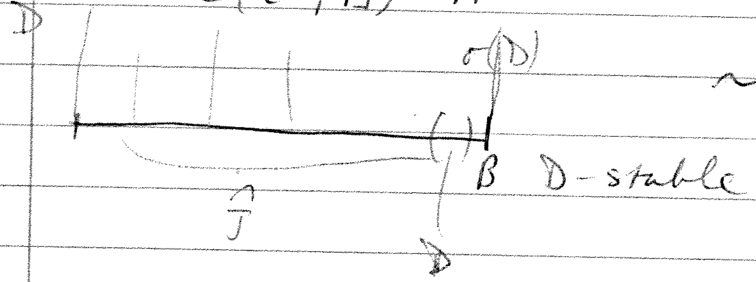
$$\sigma : B \to A \quad \rightsquigarrow \quad \sigma : B \otimes \mathcal{D}^{\otimes\infty} \to A$$

$$\rightsquigarrow \sigma|_{\mathcal{D}} : \mathcal{D} \to A$$

$(h_n)^{\subset J}$ a quasicentral approximate unit

$\rightsquigarrow 0 \le h \le 1, \ h \in A' \cap A$

$\rightsquigarrow C([0,1]) \subset A$



$$\rightsquigarrow C([0,1]) \otimes \mathcal{D} \subset \prod A / \bigoplus A \cap A'$$

<u>Remark</u>: There are results on the structure of $C(X)$-algebras with strongly self-absorbing fibres or $\mathcal{D}$-stable fibres

<u>3.2 $\mathcal{Z}$-stability</u>

<u>3.7 Proposition (Toms-Winter)</u>

Approximate divisibility $\Rightarrow$ $\mathcal{Z}$-stability

<u>3.8 Theorem (Toms-Winter)</u>

All C^*-algebras classified to date are $\mathcal{Z}$-stable

Pure finiteness and $\mathcal{Z}$-stability

4.1 Strict comparison

4.1 Definition: A has strict comparison of positive elements if

$$d_\tau(a) < d_\tau(b) \ \forall \tau \in T(A)$$
$$\Rightarrow \langle a \rangle \leq \langle b \rangle$$

for all $0 \neq a, b \in A_+$.

4.2 Theorem (Rørdam) If A is exact, unital and $\mathcal{Z}$-stable then A has strict comparison (A also simple).

4.3 Lemma (Toms-Winter): Suppose A is a unital C^*-algebra with $\operatorname{dr} A \leq n$. Given $a, d^{(0)}, \ldots, d^{(n)} \in A_+$ such that

$$d_\tau(a) < d_\tau(d^{(i)}) \text{ for } i = 0, \ldots, n \text{ and } \forall \tau \in T(A)$$

Then

$$\langle a \rangle \leq \langle d^{(0)} \rangle + \cdots + \langle d^{(n)} \rangle \text{ in } W(A).$$

4.4 Remark One can show directly that if $\operatorname{dr} A < \infty$, then quasitraces are traces

4.2 The conjecture:

4.5 Conjecture (Toms-Winter) For a nuclear, separable simple finite unital and nonelementary C^*-alg A TFAE

(i) A has finite decomp rank

(ii) A is $\mathcal{Z}$-stable

(iii) A has strict comparison of positive elements

Remarks:

(a) (ii) $\Rightarrow$ (iii) Rørdam

(b) (ii) $\Rightarrow$ (i) known in many cases, using classification results

(c) conjecture can also be formulated in the nonunital and nonsimple case

(d) in the simple but not necessarily finite case

4.3 Finite decomposition rank and $\mathcal{Z}$-stability

4.6 Theorem (Winter): A separable simple unital with $dr(A) < \infty$, then A is $\mathcal{Z}$-stable

Ingredients of Proof

1. An approx. central sequence of unital $*$-homoms
$$Z_{p,p+1} = \{ f \in C_0([0,1], M_p \otimes M_{p+1}) \mid f(0) \in M_p \otimes 1, f(1) \in 1 \otimes M_{p+1} \} \to A$$
for any $p \in \mathbb{N}$ will do

2. By Rørdam-Winter, need to find $x \in W(A)$ such that
$$px \leq \langle 1_A \rangle \leq (p+1)x$$
(in an approximately central way)

3. Finite decomposition rank implies "enough" comparison (by Lemma 4.3) to find x as above

4. n-decomposable approxs allow to do things in an approximately central way.

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4.4 Pure finiteness

4.7 Definition A separable simple nuclear C^*-algebra is purely finite if it has strict comparison and if $W(A)$ is almost divisible

$W(A)$ almost divisible.

$\forall p \in \mathbb{N} \quad x \in W(A) \quad \exists y \in W(A) : \quad py \leq x \leq (p+1)y$

Question: Is almost divisibility implied by strict comparison?

Conjecture: 4.8 will probably work for locally finite nuclear dimension

4.8 Theorem (Winter, in progress)
Let A be simple, separable unital with locally finite decomposition rank.
If A is purely infinite then A is $\mathcal{Z}$-stable.

Partial confirmation of the question
strict comparison
$\overset{?}{\Rightarrow}$ $\mathcal{Z}$-stability

Ingredients of proof: As in pf of Thm 4.6, an approx. central sequence of unital $*$-homomorphisms

$$Z_{p,p+1} \longrightarrow A$$

for any $p \in \mathbb{N}$ will do $(p \cdot x \leq \langle 1_A \rangle \leq (p+1)x)$

By Rørdam - Winter, need to find a c.p.c. order zero map

$$\Phi : M_p \longrightarrow A$$

and $v \in A$ such that

$$vv^* = 1_A - \Phi(1_{M_p}) \text{ and } v^*v \leq \Phi(e_{11})$$

and such that $\Phi(M_p)$ and v are approximately central.

Key result for constructing Φ and v:

Lemma: For $m \in \mathbb{N}$ $\exists \alpha_m > 0$ s.t.:
Let A be sep, simple, unital, purely finite
Let $1_A \in B \subset A$ be a C^*-subalgebra with $\mathrm{dr}\, B \leq m$ and let $k, l \in \mathbb{N}$.
If

$$\varphi : M_l \longrightarrow A_\infty \cap B'$$

is a c.p.c. order zero then there is a c.p.c order zero map

$$\psi : M_k \longrightarrow A_\infty \cap B' \cap \varphi(M_l)'$$

such that

$$\tau(\psi(1_k)\varphi(1_l)b) \geq \alpha_m \cdot \tau(\varphi(1_l)b)$$

$\forall b \in B_+$ and $\tau \in T_\infty(A)$ ($\tau \in T_\infty(A)$ if $\lim\limits_{\omega} \tau_n(\cdot) = \tau$ for some $(\tau_n)_{\mathbb{N}} \subset T(A)$ so $T_\infty(A) \subset T(A_\infty)$; ω free ultrafilter on $\mathbb{N}$)

use a geometric series argument after repeated application of lemma $\alpha_m \cdot \sum\limits_k (1-\alpha_m)^k = 1$

TAS algebras

5.1 Definition Let $\mathcal{S}$ be a class of separable unital C^*-algebras. Let A be simple, separable and unital. We say A is TAS if the following holds: Given $0 \neq e \in A_+$, $F \subset A$ finite, $\varepsilon > 0$ $\exists B \subset A$ with $B \in \mathcal{S}$ and

(i) $\|[1_B, a]\| < \varepsilon$ $\forall a \in F$

(ii) $1_B F 1_B \subset_\varepsilon B$

(iii) $(1_A - 1_B) \precsim e$

If $\mathcal{S}$ is the class of finite-dimensional C^*-algebras (or tensor products of such with closed intervals) we write TAF (or TAI, respectively)

<u>5.2 Theorem (Lin)</u>: The class of UCT TAI algebras satisfies the Elliott conjecture.

5.2 The lfdr, rr0, Z-stable case

<u>5.3 Theorem</u>: A separable, simple, unital and Z-stable with locally finite decomposition rank and real rank zero.
Then A is TAF

Assume A has finite decomposition rank, and that $T(A) = \{\tau\}$ ie. is just a point
Take c.p. approximation

$$\begin{array}{c} A \searrow \quad \nearrow A \\ F \qquad \varphi = \varphi^{(0)} + \dots + \varphi^{(n)} \\ \parallel \\ F^{(0)} \oplus \dots \oplus F^{(n)} \\ \cup \\ 1_{F^{(i)}} \end{array}$$

order zero maps

$\varphi(1_{F^{(i)}})$

approx by elems of finite spec using RR0

Can assume $\varphi^{(i)} = \bigoplus_j \lambda_j \cdot \pi_j \qquad \pi_j$ a $*$-homomorphism

These generate finite-dimensional subalgs.
At least one has to be at least $\frac{1}{n}$ in trace.
Repeat

End up with geometric series argument...

5.3 Localizing at $\mathcal{Z}$

5.4 Theorem (Winter): Let $\mathcal{A}$ be a class of separable, simple, nuclear, unital C^*-algebras such that, for any $A, B \in \mathcal{A}$ and any isomorph. of invariants

$$\Lambda : \mathrm{Inv}(A) \to \mathrm{Inv}(B)$$

there are prime integers p, q such that Λ can be lifted along $\mathcal{Z}_{p^\infty, q^\infty}$

i.e. $\Phi : A \otimes \mathcal{Z}_{p^\infty, q^\infty} \xrightarrow{\cong} B \otimes \mathcal{Z}_{p^\infty, q^\infty}$

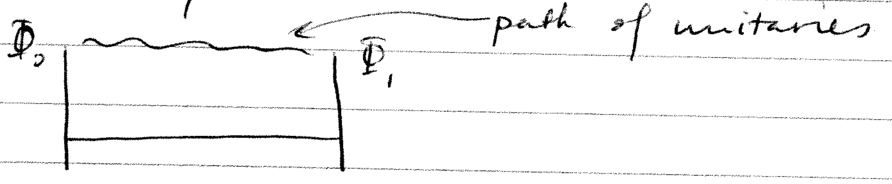
$\uparrow$

as $C([0,1])$-algebras

such that Φ_0 induces $\Lambda \otimes \mathrm{id}_{M_{p^\infty}}$

Φ_1

Φ_0, Φ_1 unitarily intertwined



Then

$$\mathcal{A}^{\mathcal{Z}} := \{A \otimes \mathcal{Z} \mid A \in \mathcal{A}\}$$

satisfies the Elliott conjecture.

5.5 Theorem (Lin–Niu): Let $\mathcal{B}$ denote the class of separable, simple nuclear unital C^*-algebras with UCT and such that tensor products with UHF algebras are TAI. Then $\mathcal{B}^{\mathcal{Z}} = \{B \otimes \mathcal{Z} \mid B \in \mathcal{B}\}$ satisfies the Elliott conjecture.

5.6 Corollary (Using Q. Lin and Phillips)
C^*-algebras associated to minimal uniquely ergodic, smooth dynamical systems are classified

5.7 Corollary (Using Gong and Theorems 4.8 and 5.5)
Simple, unital AH algebras with slow dimension growth are classified

Remark: The elements of $\mathcal{B}$ have rationally Riesz K_0-groups.

Minimal dynamical systems

6.1 The set up
X a compact metrizable space (infinite)
$\alpha: X \to X$ a homeomorphism
The crossed product is given by
$$C(X) \rtimes_\alpha \mathbb{Z} := C^*(C(X), u \mid u f(\cdot) u^* = f(\alpha^{-1}(\cdot)))$$

6.1 Proposition: If α is minimal then $C(X) \rtimes_\alpha \mathbb{Z}$ is simple unital nuclear with tracial state. If α is uniquely ergodic then the tracial state is unique.

6.2 Problem
(i) Determine the structure of such crossed products
(ii) Classify them
(iii) Draw conclusions about the underlying dynamical systems

To do: If $A_{\{y\}} \otimes \mathcal{U}$ is TAS for one $y \in X$ then TAS $\forall y \in X$?

- will focus on (i) and (ii) with minimal actions
- will solve (ii) in the finite dimensional minimal uniquely ergodic case

(there is little hope for a complete solution by K-theory in the infinite dimensional case)

- joint A Toms and K Strung and Danish furniture

6.2 Classification up to $\mathcal{Z}$-stability

6.3 Theorem: (Strung - W based on Lin - Phillips)
X compact, metrizable, infinite. Let $\alpha: X \to X$ be a minimal homeomorphism. Let $\mathcal{S}$ be a class of separable unital C^*-algebras which is closed under taking hereditary unital C^*-subalg
Let $\mathcal{U}$ be a UHF algebra and $y \in X$
Set

$$A_{\{y\}} = C^*(C(X), C_0(X \setminus \{y\})u)$$

If $A_{\{y\}} \otimes \mathcal{U}$ is TAS then $(C(X) \rtimes_\alpha \mathbb{Z}) \otimes \mathcal{U}$ is TAS.

Remarks

- In the above situation, $A_{\{y\}}$ is simple, ASH and $\mathrm{dr}\, A_{\{y\}} \leq \dim X$

(A_Y subhomogeneous if $Y \neq \emptyset$ and is closed if $Y_1 \supset Y_2 \supset Y_3 \supset \ldots$ with $\bigcap Y = \{y\}$ then $\varinjlim A_{Y_i} = A_{\{y\}}$)

- $A_{\{y\}} \otimes \mathcal{U}$ is TAF if projections separate traces (by theorem 5.3)

6.4 Corollary X compact, metrizable infinite. Let $\alpha: X \to X$ be minimal uniquely ergodic homeomorphism. Then

$$(C(X) \rtimes_\alpha \mathbb{Z}) \otimes \mathcal{U} \text{ is TAF}$$

for any UHF algebra $\mathcal{U}$

6.5 Corollary The result yields classification up to $\mathcal{Z}$-stability without any dimension restriction on X.

6.3 $\mathcal{Z}$-stability

6.6 Theorem (Toms-Winter) Let X be compact metrizable, infinite with finite topological dimension. Let $\alpha: X \to X$ be a minimal homeomorphism. Then $C(X) \rtimes_\alpha \mathbb{Z}$ is $\mathcal{Z}$-stable

6.7 Corollary

$$\mathcal{E} = \{ C(X) \rtimes_\alpha \mathbb{Z} \mid X \text{ cpct, metrzble, fin-dim, } \alpha \text{ uniquely ergodic, minimal homeomorphism} \}$$

is classified by ordered K-theory.

~~Theorem~~

Proof of Theorem 6.6

Remark: one also proves that

$$\dim_{nuc}(C(X) \rtimes_\alpha \mathbb{Z}) \leq 2 \dim X + 1$$

in a similar fashion

Proof of 6.6 (Sketch) Need $\mathcal{Z} \to A_\infty \cap A'$

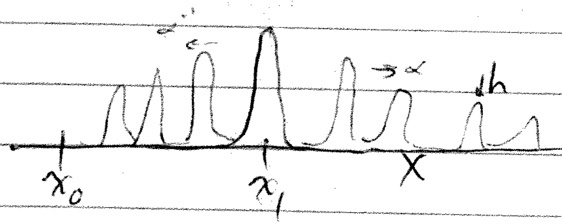
For $Y \subseteq_{\text{closed}} X$, set $A_Y := C^*(C(X), C_0(X \setminus Y)u) \subseteq C(X) \rtimes_\alpha \mathbb{Z} =: A$

Proposition (Putnam; Lin-Phillips; Toms-Winter)
Y finite then A_Y is simple, ASH with finite dr

$\Rightarrow A_Y$ is $\mathcal{Z}$-stable

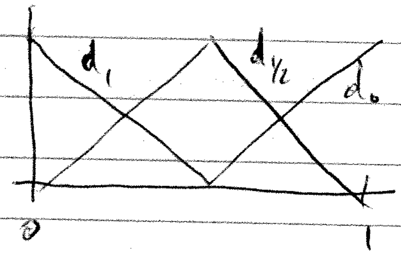
Choose $x_0, x_1 \in X$ with disjoint orbits. Set $A_0 := A_{\{x_0\}}$
$A_{1/2} := A_{\{x_0, x_1\}}$, $A_1 := A_{\{x_1\}}$. Then $A_{1/2} \subset A_1$ and $A_{1/2} \subset A_0$

Let $\mathcal{F} \subset\subset A$, $\varepsilon > 0$, $\mathcal{G} \subset\subset \mathcal{Z}$
Choose $0 \le h \le 1$ in $C(X) \subset A$ such that
$h(x_0) = 0$, $h(x_1) = 1$, $\|[h, a]\| < \varepsilon'$ $\forall a \in \mathcal{F}$



(this uses minimality of α.
h is almost invariant under α
hence almost commutes with
$\forall a \in \mathcal{F}$)

For $i = 0, 1/2, 1$ define



$\bar{d}_i := d_i(h)$

Then $\bar{d}_i a \in_{\varepsilon''} A_i$