Andrew Toms - Cuntz Semigroup

Classification of C^*-algebras

- consider C^*-algebras A which are:
  - separable
  - unital
  - nuclear / amenable
  - usually simple

A is nuclear if for any other C^*-algebra B there is only one way to complete $A \otimes B$ to get a C^*-algebra

Examples (1) $C(X) \rtimes_\alpha \mathbb{Z}$ X compact, Hausdorff

$\alpha : X \to X$ a homeomorphism

$$C(X) \rtimes_\alpha \mathbb{Z} = C^*(C(X), u) \qquad ufu = f \circ \alpha^{-1}$$

Outer auto $\rightsquigarrow$ Inner auto

(2) Recursive subhomogeneous C^*-algebra

$X_1, \dots, X_l$ compact metric spaces

$X_i^{(0)} \subseteq X_i$ closed subsets

$n_1, \dots, n_l \in \mathbb{N}$

$$A_1 = M_{n_1}(C(X_1)) \qquad A_2 = A_1 \oplus_{(M_{n_2}(C(X_2^{(0)})), \varphi_2)} M_{n_2}(C(X_2))$$

$\varphi_2 = A_1 \to M_{n_2}(C(X_2^{(0)}))$ unital $*$-homomorp

$$A_2 = \{(a,b),\ a \in A,\ b \in M_{n_2}(C(X_2)) \mid \varphi(a) = b|_{X_2^{(0)}}\}$$

$$A_3 = A_2 \oplus_{(M_{n_3}(C(X_3)), \varphi_3)} M_{n_3}(C(X_3))$$

$$A = \left[\quad \left[A_1 \underset{(C_2,\varphi_2)}{\oplus} M_{n_2}(C(X_2))\right] \oplus \cdots \quad \right]$$ (Type I algebra)

What kind of theorem do we want?

"Theorem" Let A, B be simple unital separable amenable C^*-algebras of some class $\mathcal{C}$.
$\exists$ a functor $F : \mathcal{C} \to \mathcal{C}'$ such that if $F(A) \overset{\varphi}{\cong} F(B)$ is an isomorphism, then there exists a $*$-isomorphism $\Phi : A \to B$ such that $F(\Phi) = \varphi$

What is F typically? K-theory and traces

K_0-group A unital, $\mathcal{K}$ compact operators on separable infinite dimensional Hilbert space
p, q projections in $A \otimes \mathcal{K}$. Say $p \sim q \iff \exists v \in A \otimes \mathcal{K}$ such that $v^* v = p$, $vv^* = q$

$$V(A) = \{\text{projections in } A \otimes \mathcal{K}\}/\sim \qquad p \longmapsto [p]$$

define an addition $[p] + [q] = \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\right]$

$\rightsquigarrow V(A)$ is a semigroup

$$V(A) \xmapsto{\text{Grothendieck}} K_0(A)$$
$$\Gamma$$

$$K_0(A)^+ = \Gamma(V(A)), \quad [1_A]$$

$(K_0(A), K_0(A)^+, [1_A])$ is a preordered, pointed abelian group

A projection p is infinite if $p \sim q \lneq p$, finite otherwise.
A stably finite if all projections in $M_n(A)$ are finite. $\forall n$.
In this case $K_0(A)$ is ordered

~~The~~ K_1-group $\mathcal{U}(A)$ unitaries in A. $\mathcal{U}_0(A)$ connected component of 1_A. The map $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1_A \end{pmatrix}$, $a \in M$

induces a homomorphism $\varphi_n : \frac{\mathcal{U}(M_n(A))}{\mathcal{U}_0(M_n(A))} \to \frac{\mathcal{U}(M_{n+1}(A))}{\mathcal{U}_0(M_{n+1}(A))}$

$$\underset{G_n}{} \qquad \qquad \underset{G_{n+1}}{\shortparallel}$$

$$K_1(A) = \varinjlim G_n$$

$$[u]_1 + [v]_1 = [uv]_1$$

A tracial state on A is a linear functional, $\tau \geq 0$
$\tau : A \to \mathbb{C}$ such that $\tau(1_A) = 1$ and $\tau(xy) = \tau(yx)$ $\forall x, y \in A$. The set $T(A)$ of these is a metrizable Choquet simplex
A trace defines a state on $(K_0(A), [1_A])$ via $\tau([p])$

So we get a map $\rho_A : T(A) \to S(K_0(A)_{+}^{?}, [1_A])$

A unital: The Elliott Invariant of A is
$$\left((K_0(A), K_0(A)^+, [1_A]), K_1(A), T(A), \rho_A \right)$$

$$K_*(A) = K_0(A) \oplus K_1(A)$$

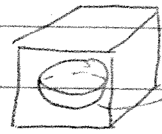
If projections separate traces, one would expect not to need $T(A)$, ρ_A

On a good day, $(K_0(A), K_0(A)^+, [1_A], T(A), \rho_A)$ is equivalent to the Cuntz semigroup of A

The Cuntz Semigroup

A unital, $a, b \in (A \otimes \mathcal{K})_+$. Say a is Cuntz dominated by B if $\exists (v_n) \in A \otimes \mathcal{K}$ such that $v_n b v_n^* \xrightarrow{\|\cdot\|} a$.
Write $a \precsim b$.
We say $a \sim b$, a is Cuntz equivalent to b, if $a \precsim b$ and $b \precsim a$.

Example: $a \sim \lambda a$, $\lambda > 0$
Ex. $A = M_n(\mathbb{C})$; $a \precsim b$ iff $\operatorname{rank}(a) \leq \operatorname{rank}(b)$
Ex. $A = M_n(C([0,1]))$: $a \precsim b$ iff $\operatorname{rank}(a(t)) \leq \operatorname{rank}(b(t))$ $\forall t \in [0,1]$
Why? Because a, b can be approximately unitarily diagonalized
Ex. X a CW-complex, $\dim X \geq 3$, $n \geq 2$. Then $\exists a, b \in M_n(C(X))_+$ such that $\operatorname{rank}(a(x)) = \operatorname{rank}(b(x))$ $\forall x \in X$ yet $a \not\sim b$



copy of S_2

Ex $f, g \in C(X)$, $f, g \geq 0$ then
$$f \precsim g \iff \operatorname{supp}(f) \subseteq \operatorname{supp}(g)$$

Define $Cu(A) = \{\text{positive elements in } A \otimes \mathcal{K}\}/\sim$
$$a \mapsto \langle a \rangle$$
As before define $\langle a \rangle + \langle b \rangle = \left\langle \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\rangle$
and $\langle a \rangle \leq \langle b \rangle \iff a \precsim b$
We get an ordered abelian semigroup called the Cuntz semigroup

Ex $A = M_n(\mathbb{C})$: $Cu(A) = \mathbb{N} \cup \{\infty\}$, $x + \infty = \infty$
$\langle 1_A \rangle = n$

Ex $A = M_n(C[0,1])$
$Cu(A) = f : [0,1] \to \mathbb{N} \cup \{\infty\} \mid f$ is supremum of an increasing sequence (f_n) of functions ~~f~~ $[0,1]$
$f_n : [0,1] \to \{0, \dots, n\}$

Definition: $T(A)$ trace simplex, $\mathrm{Aff}\, T(A) = \{$cts. affine $\mathbb{R}$-valued functions on $T(A)\}$

$L(T(A)) = \{$sups of increasing sequences (f_n) in $\mathrm{Aff}(T(A))$, $f_n \geq 0\}$

Why $Cu(A)$?

(i) If $Cu(A)$ is "nice" you can prove classification theorems for such A
(ii) $Cu(A)$ is more sensitive as an invariant than K-theory and traces.

A unital, exact $T(A) \neq \emptyset$.
$\tau \in T(A)$ extends to an unbounded trace on $A \otimes K$. If $a \in A \otimes K$, define

gives a notion of rank!

$$d_\tau(a) = \lim_{n \to \infty} \tau(a^{1/n})$$

This is an example of a dimension function on A, ie an additive order-preserving map: $\varphi : Cu(A) \to [0, \infty]$ such that $\varphi(\langle 1_A \rangle) = 1$

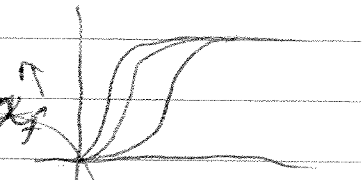
Ex. $a \in M_n(\mathbb{C})_+$, $d_\tau(a) = \mathrm{rank}(a)/n$

For $\langle a \rangle \in Cu(A)$ we define $\iota(\langle a \rangle): T(A) \to [0, \infty]$ by $\iota(\langle a \rangle)(\tau) = d_\tau(a)$

Facts: (i) $\iota(\langle a \rangle)$ is in $L(T(A))$ since $\tau \mapsto \tau(a^{1/n})$ is continuous and $\tau(a^{1/n}) \leq \tau(a^{\frac{1}{n+1}})$ (since we can choose to represent a by a contraction)

(ii) If $f \in C^*(a)$, $a \geq 0$, $f \geq 0$ then
$d_\tau(f(a)) = \mu_\tau(\mathrm{supp}(f) \cap \sigma(a))$

where μ_τ is the spectral measure induced by τ.

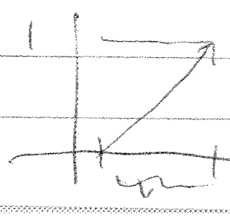


Fact: $a \precsim b \iff \forall \varepsilon > 0\ \exists \delta > 0$ st $(a-\varepsilon)_+ \leq (b-\delta)_+$
where $(a-\varepsilon)_+ = f(a)$ where $f = \begin{cases} 0 & [0, \varepsilon] \\ t-\varepsilon, & [\varepsilon, \infty) \end{cases}$

Question: When is $\langle a \rangle = \langle p \rangle$ for some projection?

Lemma: If A is unital, simple, $T(A) \neq \emptyset$ then $\langle a \rangle = \langle p \rangle$ ~~iff~~ p a projection $\iff$ 0 is not a limit point of $\sigma(a)$

Proof: $(\Leftarrow)$ ~~0 $\notin$~~ 0 a limit point of $\sigma(a)$



$a \sim 1_X(a)$

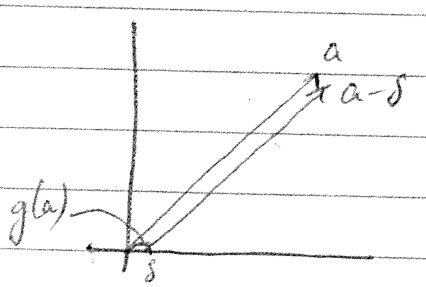
Assume 0 a limit pt

($\Rightarrow$) Suppose $\langle p \rangle = \langle a \rangle$. Pick $0 < \varepsilon < 1$. Find $\delta > 0$ such that $(p-\varepsilon)_+ \precsim (a-\delta)_+ \precsim a \sim p$

But also $p \sim (p-\varepsilon)_+$

$$\Rightarrow (a-\delta)_+ \sim p \quad \forall \delta \text{ sufficiently small}$$

$$\Rightarrow d_\tau((a-\delta)_+) = d_\tau(p) \quad \forall \delta \text{ small}$$



$$(a-\delta)_+ \leq g(a) + (a-\delta)_+ \leq a$$

$$\rightsquigarrow d_\tau((a-\delta)_+) \leq d_\tau(g(a)) + d_\tau((a-\delta)_+)$$

$$\leq d_\tau(a)$$

$$\Rightarrow d_\tau(g(a)) = 0$$

Note: A stably finite $\Rightarrow$ $p \sim_{\mathrm{MvN}} q \Leftrightarrow p \sim_{\mathrm{Cuntz}} q$

Now for A unital simple $T(A) \neq \emptyset$ we have $Cu(A) = V(A) \sqcup Cu(A)_+$ where

$$Cu(A)_+ = \{ \langle a \rangle \mid 0 \text{ is a limit point of } \sigma(a) \}$$

$Cu(A)_+$ is absorbing in that $x + y \in Cu(A)_+$ if $y \in Cu(A)_+$

Definition: A unital. A has strict comparison of positive elements (strict comparison) if $a \lesssim b$, $a, b \in (A \otimes \mathcal{K})_+$ whenever

$$d_\tau(a) < d_\tau(b) \quad \forall \tau \ \text{ s.t. } d_\tau(b) < \infty$$

Andrew Toms II

A simple unital, $T(A) \neq \phi$

~~Then~~ $\cancel{Cu(A) = V(A) \sqcup W(A)}$

$$Cu(A) = V(A) \sqcup W(A)$$

$$\iota(\langle a \rangle)(\tau) = d_\tau(a) = \lim_{n \to \infty} \tau(a^{1/n})$$

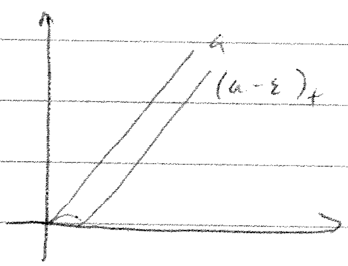
So we get a map $\varphi : Cu(A) \to V(A) \sqcup \mathrm{Im}(\iota)$
$\varphi(\langle a \rangle) = [p]$ if $a \sim p$ a proj.
$\varphi(\langle a \rangle) = \iota(\langle a \rangle)$ otherwise

$\mathrm{Im}(\iota) \subseteq L(T(A))$

Definition. A has strict comparison if $a, b \in (A \otimes K)_+$ satisfy $a \precsim b$ whenever $d_\tau(a) < d_\tau(b)$ $\forall \tau$ s.t. $d_\tau(b) < \infty$

Suppose $\langle a \rangle \in Cu(A)$, $\langle b \rangle \in Cu(A)$ and $d_\tau(a) \leq d_\tau(b)$ $\forall \tau$ such that $d_\tau(b) < \infty$

Since 0 is a limit point of $\sigma(a)$, $d_\tau((a-\varepsilon)_+) < d_\tau(a)$



If A has strict comparison
$\Rightarrow (a-\varepsilon)_+ \precsim b \quad \forall \varepsilon > 0 \Rightarrow a \precsim b$

Thus if $\langle a \rangle, \langle b \rangle \in Cu(A)_+$ then $\langle a \rangle = \langle b \rangle$
$\iff d_\tau(a) = d_\tau(b)$

Now φ is at least injective (assuming strict comparison)

When is $\mathrm{Im}(\iota) = L(T(A))_{>0}$?

<u>Proposition</u>: Let A be unital, simple, $T(A) \neq \emptyset$, strict comparison. Suppose that for any $f \in \mathrm{Aff}(T(A))$, $f > 0$ and any $\varepsilon > 0$, $\exists a \in (A \otimes K)_+$ s.t. $|d_\tau(a) - f(\tau)| < \varepsilon$ $\forall \tau \in T(A)$. Then for any $g \in L(T(A))_{>0}$, $\exists b \in (A \otimes K)_+$ st $d_\tau(b) = g(\tau)$.

<u>Proof</u>: Let g be given $\exists (f_n) \in \mathrm{Aff}(T(A))$ such that $f_n > 0$, $f_n < f_{n+1}$, $\sup_n f_n(\tau) = g(\tau)$.

Find sequence $\varepsilon_n \searrow 0$ such that $f_n + \varepsilon_n < f_{n+1} - \varepsilon_{n+1}$
Then find $a_n \in (A \otimes K)_+$ such that

$$|d_\tau(a_n) - f_n(\tau)| < \varepsilon_n$$

Then $d_\tau(a_n) < d_\tau(a_{n+1})$ and $\sup_n d_\tau(a_n) = g(\tau)$.
By strict comparison $a_n \precsim a_{n+1}$

<u>Fact</u> (Coward, Elliott, Ivanescu): Suprema of increasing sequences exist in $Cu(A)$ and $d_\tau(\cdot)$ is sup-preserving.

Let $\langle a \rangle = \sup_n \langle a_n \rangle$, then $d_\tau(\langle a \rangle) = \sup_n d_\tau(\langle a_n \rangle)$

$$= g(\tau). \quad \blacksquare$$

So when do we have density in the sense of the Proposition?

<u>Definition</u>: $Cu(A)$ is <u>almost divisible</u> if for any $x \in Cu(A)$, any $n \in \mathbb{N}$, $\exists y \in Cu(A)$ such that $ny \leq x \leq (n+1)y$.

Proposition: Let A be unital, simple, $T(A) \neq \emptyset$. Assume $Cu(A)$ is almost divisible. It follows that for any $f \in Aff(T(A))$, $f > 0$ and any $\varepsilon > 0$, $\exists a \in (A \otimes \mathcal{K})_+$ such that $|d_\tau(a) - f(\tau)| < \varepsilon$ $\forall \tau$.

ssume $\|f\| \leq 1$

Proof: By theorem of Lin / Cuntz - Pedersen, $\exists b \in A_+$ such that $\tau(b) = f(\tau)$ and $\|b\| \leq 1 + \varepsilon$

$$f(\tau) = \tau(b) = \sum_{i=1}^{N} \frac{1}{N} \tau\left(\chi_{(\frac{i}{n}, \|b\|]}(b)\right)$$

(Remark: not in the C^*-alg in general)

$$\approx \sum_{i=1}^{N} \frac{1}{N} d_\tau(b_i) \quad \text{where } b_i = f_i(b),\ supp(f_i) = (\tfrac{i}{N}, \|b\|],\ f_i > 0$$

$$\approx \sum_{i=1}^{N} d_\tau(b_i') \quad \text{(almost divisibility)}$$

Set $a = \bigoplus_{i=1}^{N} b_i'$, $d_\tau(a) = \sum_{i=1}^{N} d_\tau(b_i') \approx_\varepsilon f(\tau)$ ∎

Theorem: Let A be unital, simple, $T(A) \neq \emptyset$, strict comparison, $Cu(A)$ almost divisible.
It follows that

$$Cu(A) = V(A) \sqcup L(T(A))_{>0}$$

- where addition in RHS is as usual in each of $V(A)$, $L(T(A))_{>0}$ and if $x \in V(A)$, $y \in L(T(A))_{>0}$, then $x + y = \iota(x) + y$
- where order in $V(A)$, $L(T(A))_{>0}$ is as usual and if $x \in V(A)$ $y \in L(T(A))_{>0}$ then $x \leq y$ if $\iota(x) < y$ in $L(T(A))_{>0}$ and $y \leq x$ if $y \leq \iota(x)$ in $L(T(A))_{>0}$.

Ex: A UHF, $K_0(A) = \mathbb{Q}$, $Cu(A) = \mathbb{Q}^+ \sqcup (\mathbb{R}^+ \setminus \{0\} \cup \{\infty\})$
but $A = \lim_{n \to \infty} M_n(\mathbb{C})$, $Cu(M_n(\mathbb{C})) = \mathbb{N} \cup \{\infty\}$

Theorem (Winter, Lin-Niu) Let A, B unital UCT simple, separable with locally finite decomposition rank. Also suppose $Cu(A) = V(A) \sqcup L(T(A))_{>0}$, similarly for B, and that projections separate traces. If $\exists$ isomorphism

$$\varphi : K_*(A) \longrightarrow K_*(B)$$

then $\exists$ *-isomorphism $\Phi : A \to B$ such that $K_*(\Phi) = \varphi$.

Ex: ~~At~~ A simple unital exact finite, $A \otimes Z = A$ (Z is the Jiang-Su algebra). Then A has strict comparison (Rordam) (Proof uses that strict comparison is equivalent to almost unperforation of the Cuntz semigrp ie if $x, y \in Cu(A)$, $(n+1)x \leq ny$ $n \in \mathbb{N}$ then $x \leq y$)

$Cu(A \otimes Z)$ is almost divisible:
1. $A \otimes Z \cong A$ and (it turns out) $\langle a \otimes 1_Z \rangle = \langle a \rangle$
2. $\exists$ embedding $\gamma : C[0,1] \hookrightarrow Z$ such that the image of $\tau \in T(Z) = \{pt\}$ under $\gamma^\#$ is Lebesgue. Thus, for any $0 < \lambda \leq 1$, $\exists$ ~~a~~ $a_\lambda \in C[0,1]$ such that $d_\tau(\gamma(a_\lambda)) = \lambda$ $\forall \tau \in T(A)$
3. One computes $d_\tau(a \otimes \gamma(a_\lambda)) = \lambda d_\tau(a)$

Theorem. $Cu(A) = V(A) \sqcup L(T(A))_{>0}$ if A is a unital simple ASH algebra with slow dimension growth.

Ex) $C(M) \rtimes_\alpha \mathbb{Z}$, M compact manifold α minimal diffeomorphism (Q. Lin, Phillips)

Definition: A has slow dimension growth if $\exists$ Recursive subhomogeneous algebras A_i, unital $\varphi_i : A_i \to A_{i+1}$ such that $A = \varinjlim (A_i, \varphi_i)$ and

$$\left(A_i \overset{data}{\rightsquigarrow} X_{i,1}, \dots, X_{i,l_i}, \; n_{i,1}, \dots, n_{i,l_i} \right)$$

$$\limsup_i \max_{1 \le j \le l_i} \left(\frac{\dim X_{i,j}}{n_{i,j}} \right) = 0$$

~~How approach~~

How to prove strict comparison?

Fact: If p, q projections in $M_n(C(X))$ and ~~rank~~ $\operatorname{rank}(p) + \frac{\dim(X) - 1}{2} < \operatorname{rank} q$, then $p \precsim q$.

Assume $A = \varinjlim (A_i, \varphi_i)$, $A_i = M_{n_i}(C(X_i))$, $\frac{\dim X_i}{n_i} \to 0$

Assume $(n+1)\langle a \rangle \le n \langle b \rangle$, $a, b \in A_i$

$\Rightarrow \operatorname{rank}(a(x)) + 1 \le \operatorname{rank}(b(x)) \quad \forall x \in X$

Theorem: If $\operatorname{rank} a(x) + \frac{\dim(X) - 1}{2} < \operatorname{rank}(b(x)) \; \forall x \in X$
$\Rightarrow a \precsim b$

$$\varphi_{ij} : A_i \to A_j, \quad \varphi_{ij}(a)(\underset{\underset{X_j}{\uparrow}}{y}) = \bigoplus_{k=1}^{n_j/n_i} a(x_k) \qquad x_k \in X_k$$

$$\Rightarrow \operatorname{rank} \varphi_{ij}(a)(y) + {}^{n_j}/_{n_i} \le \operatorname{rank}(\varphi_{ij}(b)(y)) \quad \forall y \in X_j$$

$$\langle \varphi_{i,\infty}(a) \rangle \le \langle \varphi_{i,\infty}(b) \rangle$$

The preceding is a sketch of why strict comparison holds for A unital, simple, ASH algebra with ~~strict~~ slow dimension growth.

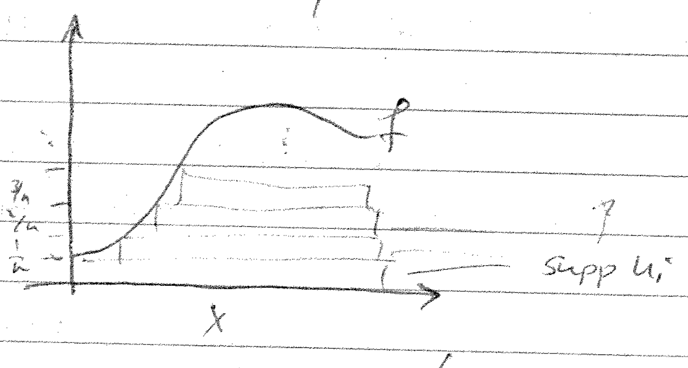
Why is $\iota(Cu(A)_+)$ "dense" in $Aff(T(A))_{>0}$?

Consider $M_n(C(X))$, $f \in Aff(T(M_n(C(X))))_{>0}$

$$\xrightarrow[\text{to extreme boundary}]{\text{restrict}} \cong C_{\mathbb{R}}(X)_{>0}$$

Want $a \in M_n(C(X))_+$ such that $|d_\tau(a) - f(\tau)| \leq \frac{1}{n}$.
Can assume $\tau = \delta_x$, $x \in X$ so that $d_\tau(a) = \frac{rank(a(x))}{n}$

Thus want $\left| \frac{rank(a(x))}{n} - f(x) \right| \leq \frac{1}{n}$



Take $e_{11} \otimes id_X =: p$ (rank one projection) and fix $f_i \in C(X)$ such that $supp(f_i) = U_i$. Set $a_i = f_i(p)$
Then $a := \bigoplus_{i=1}^{N} a_i$ does the trick.

Andrew Toms III 18/11/2009

Q: Are there separable simple unital nuclear C^*-algebras with the same K-theory and traces but not isomorphic?

A: Yes (Rørdam) and even in the stably finite case.

Strategy: Construct $A = \varinjlim M_{n_i}(C(X_i)) = \varinjlim A_i$

- X_i contractible $\Rightarrow K_1(A) = \{e\}$, $K_0(A_i) = \mathbb{Z}$
- for any $k \in \mathbb{N}$, $k | n_i$ for all i large $\Rightarrow K_0(A) = \mathbb{Q}$

$\mathcal{Q}$ = universal UHF algebra ($K_0(\mathcal{Q}) = \mathbb{Q}$)

$$(K_0(A\otimes\mathcal{Q}), K_1(A\otimes\mathcal{Q}), T(A\otimes\mathcal{Q}), \rho_{A\otimes\mathcal{Q}})$$
$$= (K_0(A), K_1(A), T(A), \rho_A)$$

$\rightarrow$ just need $A \not\cong A\otimes\mathcal{Q}$

We will show that almost unperforation fails in $Cu(A)$ but $Cu(A\otimes\mathcal{Q})$ has almost unperforation property.

~~We need~~

First see how A.U.P can fail in $M_n(C(X))$ using projections

($AUP \Leftrightarrow (n+1)x \le ny \Rightarrow x \le y$)

How to show $p \not\lesssim q$ for projections $p, q \in M_n(C(X))$

View p, q as vector bundles over X: the fibre of p at x is $p(x)\mathbb{C}^n$. Villadsen used Chern class to get comparability obstructions

Chern class $\quad c(\ ): Vect(X) \longrightarrow H^{2*}(X;\mathbb{Z})$

$\quad\quad\quad\quad\quad\quad$ complex topological vector bundles over X

(i) $c(\gamma \oplus \omega) = c(\gamma)c(\omega)$
(ii) $c(\theta_r) = 1 \in H^0(X)$, θ_r trivial of rank r, ie $\theta_r \cong X \times \mathbb{C}^r$
(iii) $f: Y \to X$ continuous then $c(f^*(\gamma)) = f^*(c(\gamma))$
(iv) $c(\gamma) = 1 + c_1(\gamma) + c_2(\gamma) + \cdots + c_{dim(\gamma)}(\gamma) \quad c_i(\gamma) \in H^{2i}(X)$.

Lemma (Villadsen): Let γ, θ_i be bundles over X. Assume $c_j(\gamma) \neq 0$ for some $j > dim(\gamma) + i$. Then $\theta_i \not\leq \gamma$.

Proof: If $\theta_i \leq \gamma$ then $\exists$ bundle ω such that $\theta_i \oplus \omega = \gamma \Rightarrow c(\theta_i \oplus \omega) = c(\theta_i)c(\omega) = c(\omega) = c(\gamma)$ ↯ ∎
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ 1

On the other hand, if $\mathrm{rank}(\omega) + \frac{dim(X)}{2} - 1 < \mathrm{rank}(\gamma)$, then $\omega \leq \gamma$. Thus if $\mathrm{rank}(\omega) < \mathrm{rank}(\gamma)$, then $(n+1)\langle\omega\rangle \leq n\langle\gamma\rangle$ for large enough n

Ex. ρ Bott bundle over S^2 then $c(\rho) = \underset{H^0(X)}{1} + \underset{H^2(X)}{1}$

$\rho \times \rho$ is a bundle over $S^2 \times S^2$ isomorphic to $\pi_1^*(\rho) \oplus \pi_2^*(\rho)$ $\quad \pi_j: S^2 \times S^2 \longrightarrow S^2$ j-order projections

$c(\pi_1^*(\rho) + \pi_2^*(\rho)) = \pi_1^*(c(\rho))\pi_2^*(c(\rho)) = (1+1)(1+1)$
$\quad\quad\quad\quad \rightsquigarrow c_2(\ ) \neq 0$

Thus $\theta_1 \not\leq \rho \times \rho$.

Consider $S_2 \times S_2 \subseteq [0,1]^3 \times [0,1]^3 = X_1$

Extend $\rho \times \rho$ to an open neighbourhood $\mathcal{U}$ of $S_2 \times S_2$ $\quad \theta_1$

Choose $f: X_1 \to [0,1]$ $\quad f|_{S^2 \times S^2} \equiv 1 \quad f|_{\mathcal{U}^c} \equiv 0$

Set $a = f\theta$, $b = f(\rho \times \rho)$ $\quad a, b \in M_n(C(X))_+$ $\quad\longrightarrow$

and $(n+1)\langle a\rangle \leq n\langle b\rangle$ $\forall n$ Large

But $\langle a\rangle \not\leq \langle b\rangle$ since $\langle a\rangle \leq \langle b\rangle \Rightarrow \langle a|_{S^2\times S^2}\rangle \leq \langle b|_{S^2\times S^2}\rangle$ ↯

$X_2 = X_1^{\times m_1}$ What should $\varphi: M_{n_1}(C(X_1)) \to M_{n_2}(C(X_2))$ be?

$$\varphi_1(f) = \begin{bmatrix} f\circ\pi_1 & & & & \\ & f\circ\pi_2 & & & \\ & & \ddots & & \\ & & & f\circ\pi_{m_1} & \\ & & & & f(x_1) \end{bmatrix}$$

$$\varphi_1(b)\Big|_{(S^2\times S^2)^{m_1}} = (\rho\times\rho)^{m_1}$$

$C_{2m_1}((\rho\times\rho)^{m_1}) \neq 0$ (same argument)

Thus $\langle \varphi_1(a)\rangle \not\leq \langle \varphi_1(b)\rangle$ and similarly for all forward images. In fact $\exists\, \delta > 0$ such that $\forall i$, $\forall x \in A_i$, $\|x\varphi_{1,i}(b)x^* - \varphi_{1,i}(a)\| \geq \delta$

So $\langle \varphi_{1,\infty}(a)\rangle \not\leq \langle \varphi_{1,\infty}(b)\rangle$ and AUP fails on A. ∎

<u>Definition</u>: A unital, exact. Define the radius of comparison for A to be

$$rc(A) = \inf\left\{ r > 0 \;\middle|\; \begin{array}{l} a \leq b \ (a, b \in A\otimes K) \text{ whenever} \\ d_\tau(a) + r < d_\tau(b) \ \forall \tau \text{ q.t.} \end{array} \right\}$$

$$rc(A) = \inf\{ m/n \mid nx + m\cdot\langle 1\rangle \leq ny \Rightarrow x \leq y \quad x, y \in Cu(A) \}$$

If A is simple,

$$rc(A) = \inf\{ m/n \mid (n+1)x + m\cdot\langle 1\rangle \leq ny \Rightarrow x \leq y \quad x, y \in Cu(A) \}$$

Proposition If X (a CW-complex) has dimension $d < \infty$ then
$$\frac{d-2}{2} \leq rc(C(X)) \leq \frac{d-1}{2}$$

Sketch of Proof: Upper bound $\Rightarrow$ already discussed
Lower bound $\rightarrow$ can immerse S^{d-2} $\rightarrow$ made even
Build positive elements from n-dimensional Bott bundle ξ_n and θ_1. These are not comparable but differ in rank by $\frac{d-2}{2}$

Properties
(i) $rc(\varinjlim(A_i, \varphi_i)) \leq \liminf rc(A_i)$
(ii) $rc(A/I) \leq rc(A)$
(iii) $rc(M_n(A)) = \frac{1}{n} rc(A)$

Theorem: $\exists$ a family A_r, $r \in [0, \infty]$ of simple AH algebras such that
(i) K-theory and traces same $\forall r$
(ii) $rc(A_r) = r$

$K_0(A_r) = \mathbb{Q}$, $sr(A) = 1$
$\rightsquigarrow$ uncountably many Morita equivalence classes among $(A_r)_{r \in [0,\infty]}$

Mean dimension

(X, α) X compact metric, α homeomorphism
$\mathcal{U}$ an open cover of X. Define $\operatorname{ord}(\mathcal{U}) = \sup_{x \in X} \sum_{U \in \mathcal{U}} \chi_U(x) - 1$

Write $\mathcal{V} \succ \mathcal{U}$ if $\mathcal{V}$ is an open cover which refines $\mathcal{U}$.
$D(\mathcal{U}) = \min_{\mathcal{V} \succ \mathcal{U}} \operatorname{ord}(\mathcal{V})$

Fact (Lindenstrauss): $D(\mathcal{U} \vee \mathcal{V}) \leq D(\mathcal{U}) + D(\mathcal{V})$.
since one can show that $D(\mathcal{U}) \leq d$ iff $\exists$ cts map $f: X \to K$, K is d-dimensional, f compatible with $\mathcal{U}$.

Set ~~the~~ $\mathcal{U}^n = \mathcal{U} \vee \alpha^{-1}(\mathcal{U}) \vee \cdots \vee \alpha^{-(n-1)}(\mathcal{U})$
($\mathcal{U}$ finite)

$$\mathrm{mdim}(X, \alpha) = \sup_{\text{such } \mathcal{U}} \lim_{n \to \infty} \frac{D(\mathcal{U}^n)}{n}$$

Ex Y a CW-complex, $X = Y^{\times\infty}$, α the bilateral shift on X. Then $\mathrm{mdim}(X, \alpha) = \dim Y$

Problem: If $\dim(X) < \infty$ then the mean dimension $\mathrm{mdim}(X, \alpha) = 0$ $\forall \alpha$.

Theorem (Giol-Kerr): For any $k > 0$, $\exists$ a minimal system (X_k, α_k) such that
$\frac{\mathrm{mdim}(X_k, \alpha)}{2} \approx k \leq rc(C(X_k) \rtimes_{\alpha_k} \mathbb{Z})$

If $\alpha: Y^{\times\infty} \to Y^{\times\infty}$ is the bilateral shift, then
$C(Y^{\times\infty}) \rtimes_\alpha \mathbb{Z}$
Y_{2^n} - 2^n periodic points

~~$C(Y_{2^n}) \rtimes_\alpha \mathbb{Z}$~~

$$\begin{array}{c} C(Y^{\times\infty}) \rtimes_\alpha \mathbb{Z} \\ \swarrow \quad \downarrow \quad \searrow \\ C(Y_{2^n}) \rtimes_\alpha \mathbb{Z} \to C(Y_{2^{n+1}}) \rtimes_\alpha \mathbb{Z} \to \end{array}$$

$$rc(C(Y_{2^n}) \rtimes_\alpha \mathbb{Z}) = \frac{\dim Y}{2} = \frac{\mathrm{mdim}(Y^\infty, \alpha)}{2}$$

Proposal: Define a dynamic ~~domain~~ dimension
$ddim(X, G)$ (G countable, discrete) by
$$ddim(X, G) = rc(C(X) \rtimes G)$$

Why? ① Looks ~~tat~~ like we could recover
mdim for unilateral shift
② If $G = \{e\}$ then $dim(X, G) = \frac{\dim X}{2}$
③ If $G = \mathbb{Z}$ ~~acting~~ acting triv. then
$ddim(X, G) = \frac{\dim X + 1}{2}$.

④ $X = Y^{\times n}$, α a cyclic shift on co-ords, then
$ddim(X, \alpha) \approx \frac{\dim Y}{2}$

Outlook: Hopeful for minimal systems (X, α)
that $ddim(X, \alpha) \leq \frac{mdim(X, \alpha)}{2}$ and that this
is sharp (Giol–Kerr)

Why hope?
$C^*(C(X), uC_0(X \setminus \{y\})) = A_{\{y\}}$ is ASH, but RSH
subalgebras have infinite dimensional spectrum.
Idea: for $a, b \in A_{\{y\}+}$ $\quad a = \sum_{i \leq N} f_i u^i$

Take $\mathcal{U}$ finite open cover, iterate under α^{-1}
$\rightsquigarrow$ cover $\mathcal{V}$, ord $= n(m\dim)$