The Cuntz semigroup and its relation to classification

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1. Part 1 - Lecture from 16.November 2009

We consider C*-algebras A which are:

- separable
- unital
- nuclear (which is equivalent to being amenable)
- usually simple

A is nuclear if for any other C*-algebra B there is only one way to complete the algebraic tensor product $A \odot B$ to get a C*-algebra.

1.1. **Example** (cross products): Any cross product $A = C(X) \rtimes_\alpha \mathbb{Z}$ is nuclear, where X is a compact Hausdorff space, $\alpha : X \to X$ is a homeomorphism. Recall that $C(X) \rtimes_\alpha \mathbb{Z} = C^*(C(X), u)$ where u is a unitary which implements α, i.e. $ufu^* = f \circ \alpha^{-1}$ for any $f \in C(X) \subset C(X) \rtimes_\alpha \mathbb{Z}$.

1.2. **Example** (recursive subhomogeneous algebras): Any recursive subhomogeneous algebras (RSH-algebra) A is nuclear. Recall that these are defined as iterated pullbacks using the following data:

- compact metric spaces $X_1, \ldots, X_l$
- closed subspaces $X_i^{(0)} \subset X_i$
- numbers $n_1, \ldots, n_l \in \mathbb{N}$
- unital $*$-homomorphisms $\phi_k : A_{k-1} \to M_{n_k}(C(X_k^{(0)}))$ (attaching maps)

such that $A_1 = M_{n_1}(C(X_1))$, and the following is a pullback (for $k = 2, \ldots, l$):

$$\begin{array}{ccc} A_k & \longrightarrow & A_{k-1} \\ \downarrow & & \downarrow \phi_k \\ M_{n_k}(C(X_k)) & \xrightarrow{\partial_k} & M_{n_k}(C(X_k^{(0)})) \end{array}$$

Here ∂_k is induced by the inclusion $X_k^{(0)} \to X_k$. Such a pullback is often written as $A_k = A_{k-1} \oplus_{M_{n_k}(C(X_k^{(0)}))} M_{n_k}(C(X_k))$, and the standard way to define that pullback algebra is as follows:

$$A_k = \{(a,b) \,:\, a \in A_{k-1}, b \in M_{n_k}(C(X_k)), \varphi_k(a) = \partial_k(b) = b_{|X_k^{(0)}}\}$$

These algebras are interesting because one can try to extend results form homogeneous to RSH-algebras. Possibly all stably finite C*-algebras are direct limits of RSH-algebras. Note also that all RSH-algebras are of type I.

What kind of theorem do we want?

1.3. **Theorem:** *Let A, B be simple, unital, separable, nuclear C*-algebras in some class $\mathfrak{C}$. There exists a functor $F : \mathfrak{C} \to \mathfrak{C}'$ such that if $\varphi : F(A) \overset{\cong}{\to} F(B)$ is an isomorphism, then there exists a $*$-isomorphism $\Phi : A \to B$ s.t. $F(\Phi) = \varphi$.*

What is F typically? It is K-theory and traces. (we do not need quasitraces, since we only consider nuclear C*-algebras, where every quasitrace is automatically a trace)

1.4 (K_0-group)**:** For simplicity let us only consider the unital case. For projections $p, q \in A \otimes \mathbb{K}$ say

$$p \sim q :\Leftrightarrow \text{ there exists some } v \in A \otimes \mathbb{K} \text{ s.t. } p = v^*v, vv^* = q$$

Set $V(A) := \{ \text{ the projections in } A \otimes \mathbb{K}\}_{/\sim}$. For a projection $p \in A \otimes \mathbb{K}$ we denote its equivalence class in $V(A)$ by $[p]$. Define an addition on $V(A)$ by $[p] + [q] = \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\right]$. In this way $V(A)$ becomes an abelian semigroup.

Use the Grothendieck completion process Γ to define an abelian group $K_0(A) := \mathrm{Gr}(V(A))$. This comes with a natural map $\Gamma : V(A) \to K_0(A)$ and we denote its image as $K_0(A)^+ := \Gamma(V(A))$. This is also called the positive part (or positive cone) in $K_0(A)$. Then $(K_0(A), K_0(A)^+, [1_A])$ is a pre-ordered, pointed abelian group.

A projection p is called infinite if it is equivalent to a proper subprojection, otherwise it is called finite. We call A stably finite, if all projections in $M_n(A)$ are finite (for all n). In that case K_0 is ordered.

1.5 (K_1-group)**:** Let $\mathcal{U}(A)$ denote the set of unitaries in A, and $\mathcal{U}_0(A) \subset \mathcal{U}(A)$ its connected component containing 1_A. The map $u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1_A \end{pmatrix}$ induces a homomorphism $\varphi_n : \mathcal{U}(M_nA)/\mathcal{U}_0(M_nA) \to \mathcal{U}(M_{n+1}A)/\mathcal{U}_0(M_{n+1}A)$. We set $K_1(A) := \varinjlim_n \mathcal{U}(M_nA)/\mathcal{U}_0(M_nA)$. This is an abelian group with addition defined via $[u] + [v] = [uv]$.

1.6 (Traces): A tracial stat on A is a positive linear functional $t : A \to \mathbb{C}$ such that $\tau(1_A) = 1$, and $\tau(xy) = \tau(yx)$ for all $x, y \in A$. The set $T(A)$ of all traces on A is a metrizable Choquet simplex. A trace defines a state on $K_0(A)$ as follows: first extend τ to a trace $\tau\otimes\mathrm{tr}$ on $M_n(A)$ using the canonical trace $\mathrm{tr} : M_n \to \mathbb{C}$, then for a projection $p \in M_n(A)$ set $\tau([p]) := (\tau \otimes \mathrm{tr})(p)$. We get a map $\rho_A : T(A) \to \mathrm{St}(K_0(A), K_0(A), [1_A])$.

For a unital C*-algebra A the Elliott invariant is:

$$\mathrm{Ell}(A) := (K_0(A), K_0(A), [1_A], K_1(A), T(A), \rho_A)$$

In good cases $(K_0(A), K_0(A), [1_A], T(A), \rho_A)$ is equivalent to the Cuntz semigroup $\mathrm{Cu}(A)$, and then $\mathrm{Ell}(A) \cong (\mathrm{Cu}(A), K_1(A))$, which amounts to a decomposition in a positive and unitary part.

1.7 (The Cuntz semigroup): Let A be unital. For $a, b \in (A \otimes \mathbb{K})_+$ we say a is Cuntz-dominated by b (denoted $a \precsim b$) if there exists a sequence $(r_n) \subset A \otimes \mathbb{K}$ s.t. $r_n b r_n* \to a$ (in norm). Say a is Cuntz-equivalent to b (denoted $a \sim b$) if $a \precsim b$ and $b \precsim a$. On projections this agrees with the earlier defined equivalence for stably finite algebras. Note that for any $\lambda > 0$ and $a \in (A \otimes \mathbb{K})_+$ we have $a \sim \lambda a$.

1.8. **Example:** M_n

Let $A = M_n$. Then $a \precsim b$ iff $\mathrm{rank}(a) \leq \mathrm{rank}(b)$.

1.9. **Example:** $M_n(C[0,1])$

Let $A = M_n(C[0,1])$. Then $a \precsim b$ iff $\mathrm{rank}(a)(t) \leq \mathrm{rank}(b)(t)$ for all $t \in [0,1]$. The reason is that a and b can be approximately unitarily diagonalized.

1.10. **Example:** $M_n(C(X))$

Let $A = M_n(C(X))$ with X a CW-complex of $\dim(X) \geq 3$ and $n \geq 2$. Then there exist $a, b \in M_n(C(X))$ s.t. $\mathrm{rank}(a)(t) = \mathrm{rank}(b)(t)$ for all $t \in [0,1]$, yet $a \nsim b$. The reason is that $\dim(X) \geq 3$ ensures that we can find S^2 in X. We can find projections p, q in $M_2(C(S^2))$ that both have constant rank one, yet $p \nsim q$ (e.g. the trivial line bundle, and the Bott line bundle). Extend this to a small neighborhood of $S^2 \hookrightarrow X$, and then to positive elements $a, b \in M_2(C(X)) \subset M_n(C(X))$.

1.11. **Example:** $C(X)$
Let $A = C(X)$ and $f, g \in A_+$. Then $f \precsim G$ iff $\text{supp}(f) \subset \text{supp}(g)$.

1.12 (The Cuntz semigroup)**:** Define $\text{Cu}(A) := \{$ positive elements in $A \otimes \mathbb{K}\}_{/\sim}$. We denote the equivalence class of $a \in (A \otimes \mathbb{K})_+$ in $\text{Cu}(A)$ by $\langle a \rangle$. As before we define an addition $\langle a \rangle + \langle b \rangle := \left\langle \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix} \right) \right\rangle$. If we define $\langle a \rangle \leq \langle b \rangle$ iff $a \precsim b$, then we get an ordered abelian semigroup.

1.13. **Example:** M_n
Let $A = M_n$. Then $\text{Cu}(A) = \mathbb{N} \cup \{\infty\}$ with $x + \infty = \infty, \infty + \infty = \infty$ and $\langle 1_A \rangle = n \in \mathbb{N}$.

1.14. **Example:** $M_n(C[0,1])$
Let $A = M_n(C[0,1])$. Then $\text{Cu}(A)$ consists of all functions $f : [0,1] \to \mathbb{N} \cup \{\infty\}$ that are the supremum of an increasing sequence of functions $f^{(n)} : [0,1] \to \{0, \dots, n\}\}$.

We denote by $\text{Aff}(T(A))$ the continuous affine $\mathbb{R}$-valued functions on $T(A)$, and by $L(T(A))$ the functions $T(A) \to \mathbb{R} \cup \{\infty\}$ that are the supremum of an increasing sequence of functions $f^{(n)} \in \text{Aff}(T(A))$.

Why are we interested in $\text{Cu}(A)$?

- if $\text{Cu}(A)$ is nice, you can prove classification theorems for such A
- $\text{Cu}(A)$ is more sensitive that K-theory and traces

Assume A is unital, exact and $T(A) \neq \emptyset$. Then every $\tau \in T(A)$ extends to an unbounded trace on $A \otimes \mathbb{K}$ as follows: if $a \in (A \otimes \mathbb{K})_+$, then define $d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n})$.
This is an example of a dimension function on A, i.e. an additive order-preserving map $\varphi : \text{Cu}(A) \to [0, \infty]$ s.t. $\varphi(\langle 1_A \rangle) = 1$. (this gives exactly the lower semicontinuous dimension functions).

1.15. **Example:** For $a \in (M_n)_+$ we get $d_\tau(a) = \text{rank}(a)/n$.

For $\langle a \rangle \in \text{Cu}(A)$ we define $\iota(\langle a \rangle) : T(A) \to [0, \infty]$ by $\iota(\langle a \rangle)(\tau) := d_\tau(a)$. Then:

- $\iota(\langle a \rangle)$ is in $L(T(A))$ since $\tau \mapsto \tau(a^{1/n})$ is continuous and $\tau(a^{1/n}) \leq \tau(a^{1/n+1})$ (if $\|a\| \leq 1$, so rescale a)
- if $a \geq 0$, $f \in C^*(a)$, $f \geq 0$, then $d_\tau(f(a)) = \mu_\tau(\text{supp}(f) \cap \sigma(a))$ where μ_τ is the spectral measure induced by τ
- $a \precsim b$ iff $\forall \varepsilon >\quad \exists \delta > 0$ such that $(a - \varepsilon)_+ \precsim (b - \delta)_+$.

Question: When is $\langle a\rangle = \langle p\rangle$ for some projection p?

1.16. **Lemma:** *If A is unital, simple and $T(A) \neq \emptyset$, then $\langle a\rangle = \langle p\rangle$ for a projection p iff 0 is not a limit point of $\sigma(a)$.*

Proof:

$\Leftarrow$: then $a \sim \chi_X(a)$ where χ_X is the characteristic function on the set $(0,\infty)\cap\sigma(a)$, and $\chi_X(a)$ is a projection

$\Rightarrow$: then $p \sim (p-\varepsilon)_+ \precsim (a-\delta)_+ \precsim a \sim p$, whence $d_\tau((a-\delta)_+) = d_\tau(p)$ for all δ small enough. But $(a-\delta)_+ \leq g(a)+(a-\delta)_+ \leq a$ for some small function g with $\operatorname{supp}(g) \subset [0,\delta]$. Then $d_\tau((a-\delta)_+) = d_\tau(g(a)) + d_\tau((a-\delta)_+)$, and therefore $d_\tau(g(a)) = 0$ for all τ while $g(a) \neq 0$. This is a contradiction. □

Now for A unital, simple with $T(A) \neq \emptyset$ we have

$$\mathrm{Cu}(A) = V(A) \sqcup \mathrm{Cu}(A)_+$$

where $\mathrm{Cu}(A)_+ = \{\langle a\rangle : 0 \text{ is a limit point of } \sigma(a)\}$. $\mathrm{Cu}(A)_+$ is absorbing in the sense that $x + y \in \mathrm{Cu}(A)_+$ whenever $y \in \mathrm{Cu}(A)_+$.

1.17. **Definition:** *Let A be unital. We say A has strict comparison of positive elements (often abbreviated by just saying "strict comparison") if $\precsim b$ whenever $d_\tau(a) < d_\tau(b)$ for all $\tau \in T(A)$ such that $d_\tau(b) < \infty$.*

2. Part 2 - Lecture from 17.November 2009

Let A be simple, unital with $T(A) \neq \emptyset$. Then $\mathrm{Cu}(A) = V(A) \sqcup \mathrm{Cu}(A)_+$. We define a map

$$\varphi : \mathrm{Cu}(A) \to V(A) \sqcup L(TA)$$

as $\varphi(\langle a \rangle) := [p]$ whenever $a \sim p$ for a projection p, and for $\langle a \rangle \in \mathrm{Cu}(A)_+$ we set $\varphi(\langle a \rangle) := \iota(\langle a \rangle)(\tau) := d_\tau(a)$. When is this map injective, when is it surjective?

Suppose A has strict comparison, $\langle a \rangle \in \mathrm{Cu}(A)_+, \langle b \rangle \in \mathrm{Cu}(A)$, and $d_\tau(a) \leq d_\tau(b)$ for all $\tau \in T(A)$ with $d_\tau(b) < \infty$. Since 0 is a limit point of $\sigma(a)$, we have $d_\tau((a-\varepsilon)_+) < d_\tau(b)$ for all $\varepsilon > 0$ small enough. From strict comparison of A we get $(a-\varepsilon)_+ \precsim b$ for all $\varepsilon > 0$ small enough, and therefore also $a \precsim b$.

Thus, if $\langle a \rangle, \langle b \rangle \in \mathrm{Cu}(A)_+$, then $\langle a \rangle = \langle b \rangle$ iff $d_\tau(a) = d_\tau(b)$ for all τ. Now φ is at least injective if A has strict comparison.

When is $\mathrm{im}(\iota) = LT(A)_{>0}$?

2.1. **Proposition:** *Let A be simple, unital with strict comparison and $TA \neq \emptyset$. Suppose that for any $f \in \mathrm{Aff}(TA)$, $\varepsilon > 0$ there exists $a \in (A \otimes \mathbb{K})_+$, s.t. $|d_\tau(a) - f(\tau)| < \varepsilon$ for all $\tau \in TA$. Then for any $g \in LT(A)_{>0}$ there exists $b \in (A \otimes \mathbb{K})_+$ s.t. $d_\tau(b) = g\tau()$.*

Proof:
Let g be given. There exists a sequence $(f_n) \subset \mathrm{Aff}(TA)$ s.t. $f_n > 0$, $f_n < f_{n+1}$ and $\sup_n f_n(\tau) = g(\tau)$, Find a sequence $\varepsilon_n > 0$ s.t. $f_n - \varepsilon_n < f_{n+1} - \varepsilon_{n+1}$. Then find $a_n \in (A \otimes \mathbb{K})_+$ s.t. $|d_\tau(a_n) - f_n(\tau)| < \varepsilon_n$. Then $d_\tau(a_n) < d_\tau(a_{n+1})$ and $\sup_n d_\tau(a_n) = g(\tau)$. By strict comparison $a_n \precsim a_{n+1}$. Suprema of increasing sequences in $\mathrm{Cu}(A)$ exist, and d_τ is sup-preserving. Let $\langle a \rangle = \sup \langle a_n \rangle \in \mathrm{Cu}(A)$. Then $d_\tau(a) = g(\tau)$. □

So when do we have density (in the sense of the proposition)?

2.2. **Definition:** *We say $\mathrm{Cu}(A)$ is **almost divisible** if for any $x \in \mathrm{Cu}(A)$, $n \in \mathbb{N}$ there exists $y \in \mathrm{Cu}(A)$ s.t. $ny \leq x \leq (n+1)y$.*

2.3. **Proposition:** *Let A be simple, unital with $T(A) \neq \emptyset$ and $\mathrm{Cu}(A)$ almost divisible. It follows that for any $f \in \mathrm{Aff}(TA)_{>0}$, $\varepsilon > 0$ there exists $a \in (A \otimes \mathbb{K})_+$, s.t. $|d_\tau(a) - f(\tau)| < \varepsilon$ for all $\tau \in T(A)$.*

Proof:

We can assume $\|f\| \leq 1$. By a theorem of Lin / Cuntz, Pedersen there exists $b \in A_+$ s.t. $\tau(b) = f(\tau)$ and $\|b\| \leq 1 + \varepsilon$. Then:

$$\begin{aligned} f(\tau) &= \tau(b) \\ &\approx \sum_{i=1}^{n} 1/n\tau(\chi_{(i/n,\|b\|]}(b)) \\ &= \sum_{i=1}^{n} 1/nd_\tau(f_i(b)) \qquad \text{for functions } f_i \text{ with } \mathrm{supp}(f_i) = (i/n, \|b\|] \\ &= \sum_{i=1}^{n} d_\tau(c_i) \end{aligned}$$

Set $c = \bigoplus_{i=1}^n c_i$, then $d_\tau(c) \approx f(\tau)$. □

2.4. **Theorem:** *Let A be simple, unital with strict comparison, $T(A) \neq \emptyset$ and* $\mathrm{Cu}(A)$ *almost divisible. Then* $\mathrm{Cu}(A) \cong V(A) \sqcup L(TA)_{>0}$ *is an order-isomorphism. Here addition on the right hand side is as usual in each of $V(A)$ and $L(TA)_{>0}$, and if $x \in V(A)$, $y \in L(TA)_{>0}$ then $x + y = \iota(x) + y$.*

Also, the order on the right hand side is the usual in each of $V(A)$ and $L(TA)_{>0}$, and if $x \in V(A)$, $y \in L(TA)_{>0}$ then $x \leq y$ if $\tau(x) < y$ in $L(TA)_{>0}$, and $y \leq x$ if $y \leq \iota(x)$.

2.5. **Example:** If A is UHF-algebra with $K_0(A) \cong \mathbb{Q}$, then $\mathrm{Cu}(A) \cong \mathbb{Q}^+ \sqcup (\mathbb{R}^+ \setminus \{0\}) \cup \{\infty\}$. Also $\mathrm{Cu}(M_n) = \mathbb{N} \cup \{\infty\}$.

2.6. **Theorem:** *(Winter, Lin-Niu) Let A, B be simple, unital with UCT and locally finite decomposition rank. Also suppose* $\mathrm{Cu}(A) = V(A) \sqcup L(TA)_{>0}$ *(similarly for B) and projections separate traces. If there exists an isomorphism $\varphi : K_*(A) \to K_*(B)$, then there exists a $*$-isomorphism $\Phi : A \to B$ s.t. $K(\Phi) = \varphi$.*

Note that these algebras will have real rank zero after tensoring with an UHF algebra.

2.7. **Example:** Let A be simple, unital, exact, finite, $\mathcal{Z}$-stable. Then A has strict comparison (the proof uses that strict comparison is equivalent to almost unperforation of $\mathrm{Cu}(A)$, i.e. if $x, y \in \mathrm{Cu}(A)$ with $(n+1)x \leq yn$ for some n, then $x \leq y$).

Also $\mathrm{Cu}(A)$ is almost divisible. The proof uses:

(1) Under the isomorphism $A \otimes \mathcal{Z} \cong A$ we have $\langle a \otimes 1_{\mathcal{Z}} \rangle = \langle a \rangle$
(2) There exists an embedding $\gamma : C[0,1] \hookrightarrow \mathcal{Z}$ s.t. the image of $\tau \in T(\mathcal{Z}) = \{\tau\}$ is the Lebesgue measure on $[0,1]$. Thus, for any $0 < \lambda < 1$ there exists $a_\lambda \in C[0,1]$ s.t. $d_\tau(a_\lambda) = \lambda$ for all $\tau \in T(A)$

(3) Compute $d_\tau(a \otimes a_\lambda) = \lambda d_\tau(a)$ (so $\mathrm{Cu}(A)$ is a cone)

2.8. **Theorem:** *If A is a simple, unital ASH-algebra with slow dimension growth, then* $\mathrm{Cu}(A) \cong C(A) \sqcup L(TA)_{>0}$

2.9. **Definition:** *A has slow dimension growth (s.d.g.) if there exist RSH-algebras A_k and connecting maps $\varphi_k : A_k \to A_{k+1}$ s.t. $A \cong \varinjlim_k A_k$, and for the underlying spaces $X_{k1}, X_{k2}, \ldots$ and matrix sizes $n_{k1}, n_{k2}, \ldots$ of the RSH-algebras A_k we have:*

$$\limsup_k (\max_i \dim X_{ki}/n_{ki}) = 0$$

How to prove strict comparison? Does s.d.g. imply $\mathcal{Z}$-stability for ASH-algebras?

For projections $p, q \in M_n(C(X))$ with $\mathrm{rank}(p) + (\dim(X) - 1)/2 < \mathrm{rank}(q)$, we have $p \precsim q$. We want to show that a similar result holds for positive elements.

Assume $A = \varinjlim A_k$, $A_k = M_{n_k}(C(X_k))$. Then s.d.g. means $\dim(X_k)/n_k \to 0$. Assume $(n+1)\langle a \rangle \leq n\langle b \rangle$ for $a, b \in A_k$. Does it follow that $\mathrm{rank}(a(x)) \leq \mathrm{rank}(b(x))$ for all $x \in X_k$?

2.10. **Theorem:** *If* $\mathrm{rank}(a(x)) + \dim(X)/2 < \mathrm{rank}(b(x))$ *for all $x \in X_k$, then $a \precsim b$.*

The proceeding is a sketch why strict comparison holds for simple, unital ASH-algebras with s.d.g.

Why is $\iota(\mathrm{Cu}(A)_+)$ "dense" in $\mathrm{Aff}(TA)_{>0}$ (in the above sense)? Consider $M_n(C(X))$, and $f \in \mathrm{Aff}(T(M_n(C(X))))_{>0} \cong C_{\mathbb{R}}(X)$ (since $T(...)$ is a Bauer simplex, with compact boundary X). We want $a \in M_n(C(X))_+$ s.t. $|d_\tau(a) - f(\tau)| < 1/n$. Can assume $\tau = \delta_x$ for some $x \in X$, so $d_\tau(a) = \mathrm{rank}(a(x))/n$. Thus want $|\,\mathrm{rank}(a(x))/n - f(x)| < 1/n$. Take $p = e_{11} \otimes \mathrm{id}_x$, and fix $f \in C(X)$ s.d. $\mathrm{supp}(f_i) = U_i$. Set $a_i = f_i(p)$. Then $a = a_1 \oplus \ldots \oplus a_n$ does the trick.

3. Part 3 - Lecture from 18.November 2009

Are there simple, unital, separable, nuclear C*-algebras with the same K-theory and traces, but which are not isomorphic?

Yes, first examples have been given by Rørdam, and there are even examples in the stably finite case.

Strategy: Construct A as inductive limit $A = \varinjlim M_{n_k}(C(X_k))$ with each X_k contractible. Then $K_0(A_k) = \mathbb{Z}$ and $K_1(A_k) = 0$, so also $K_1(A) = 0$. Assume we can achieve that the elements of $K_0(A_k)$ get divisible in the limit, i.e. for each n and k there is some $N > k$ such that $1 \in K_0(A_k)$ is divisible by n in A_N. Then $K_0(A) = \mathbb{Q}$, and hence $\mathrm{St}(K_0(A)) = \{\tau\}$, so the pairing between traces and K_0 is uninteresting.

Let Q be the universal UHF-algebra (i.e. $K_0(Q) = \mathbb{Q}$), then

$$(K_0(A \otimes Q), K_1(A \otimes Q), T(A \otimes Q), \rho_{A\otimes Q}) \cong (K_0(A), K_1(A), T(A), \rho_A)$$

For a counterexample we just need $A \ncong A \otimes Q$. We will show that AUP (almost unperforation property) fails in $\mathrm{Cu}(A)$, but $\mathrm{Cu}(A \otimes Q)$ has AUP.

Let us first see how AUP can fail in $M_n(C(X))$ using the fact that AUP is equivalent to:

$$(n+1)x \leq ny \quad \Rightarrow x \leq y$$

How do we show that $p \not\precsim q$ for projections $p, q \in M_n(C(X))$? View p, q as VB (vector bundles) over X: the fibre of p at $x \in X$ is $p(x)\mathbb{C}^n$. Villadsen used Chern classes to get comparability obstructions.

3.1 (Chern classes)**:** The (full) Chern class is a map $c(\cdot) : \mathrm{Vect}(X) \to H^{\mathrm{ev}}(X : \mathbb{Z})$ with the following properties:

(i) $c(\xi \oplus \xi') = c(\xi) \cup c(\xi')$
(ii) $c(e_r) = 1 \in H^0(X)$ where $e_r = X \times \mathbb{C}^r$ is the trivial VB
(iii) if $f : X \to Y$ is continuous, then $c(f^*(\xi)) = f^*(c(\xi))$
(iv) $c(\xi) = 1 + c_1(\xi) + \ldots + c_{\dim \xi}(\xi)$ with $c_i(\xi) \in H^{2i}(X)$

3.2. **Lemma:** *(Villadsen) Let γ, e_r be VB over X. Assume $c_j(\gamma) \neq 0$ for some $k > \dim(\gamma) - r$. Then $e_r \not\precsim \gamma$.*

Proof:

If $e_r \precsim \gamma$, then there exists ω s.t. $e_r \oplus \omega \cong \gamma$. Then $c(e_r \oplus \omega) = c(e_r) \cup c(\omega) = c(\omega) = c(\gamma)$, but $\dim(\omega) < \dim(\gamma) - i$. □

On the other hand, if $\mathrm{rank}(\omega) + (\dim(X) - 1)/2 < \mathrm{rank}(\gamma)$, then $\omega \precsim \gamma$. Thus, if $\mathrm{rank}(\omega) < \mathrm{rank}(\gamma)$, then $(n+1)\langle\omega\rangle \leq n\langle\gamma\rangle$ for large enough n.

3.3. **Example:** Let ρ be the Bott bundle over S^2. Then $c(\rho) = 1 + 1 \in H^0(S^2) \oplus H^2(S^2)$. $\rho \times \rho$ is a bundle over $S^2 \times S^2$ defined by $\pi_1^*(\rho) \oplus \pi_2^*(\rho)$ where $\pi_i : S^2 \times S^2 \to S^2$ are the coordinate projections. Then

$$\begin{aligned} c(\pi_1^*(\rho) \oplus \pi_2^*(\rho)) &= \pi_1^*(c(\rho))\pi_2^*(c(\rho)) \\ '' &='' (1+1)(1+1) \end{aligned}$$

in particular $c_2(\rho \times \rho) \neq 0$: Thus $e_1 \not\precsim \rho \times \rho$.

Consider $S^2 \times S^2 \subset [0,1]^3 \times [0,1]^3 = X_1$. Extend $\rho \times \rho$ to an open neighborhood U of $S^2 \times S^2$, choose $f : X_1 \to [0,1]$ with $f = 1$ on $S^2 \times S^2$ and $f = 0$ on U^c (the complement of U). Set $a = f \cdot e_1$, $b = f \cdot \rho \times \rho$. Then $a, b \in M_n(C(X_1))_+$ and $(n+1)\langle a\rangle \leq n\langle b\rangle$ for large n, but $\langle a\rangle \not\leq \langle b\rangle$ since otherwise $\langle a_{|S^2\times S^2}\rangle = \langle e_1\rangle \leq \langle\rho\times\rho\rangle = \langle b_{|S^2\times S^2}\rangle$.

Set $X_2 := X_1^{\times m_2}$. Define $\varphi_1 : M_{n_1}(C(X_1)) \to M_{n_2}(C(X_2))$ as:

$$\varphi_1(f) = \begin{pmatrix} f\circ\pi_1 & & & & \\ & \cdots & & & \\ & & f\circ\pi_{m_1} & & \\ & & & f(x_i) & \\ & & & & \cdots \end{pmatrix}$$

Note that we add the evaluations at points x_i to ensure simplicity of the limit. (so want these points to be eventually dense). Then:

$$\varphi_1(b)_{|(S^2\times S^2)^{\times m_1}} = (\rho\times\rho)^{\times m_1}$$

and $c_{2m_1}((\rho\times\rho)^{\times m_1}) \neq 0$. Thus $\langle\varphi_1(a)\rangle \not\leq \langle\varphi_1(b)\rangle$. If we proceed this way, a similar result will hold for all forward images. In fact there exists $\delta > 0$ such that for all i and $x \in A_i$: $\|x\varphi_{1,i}(b)x^* - \varphi_{1,i}(a)\| \geq \delta$, so $\langle\varphi_{1,\infty}(a)\rangle \not\leq \langle\varphi_{1,\infty}(b)\rangle$. Thus AUP fails in A.

3.4. **Definition:** *Let A be unital, exact. Define the radius of comparison for A to be:*

$$\mathrm{rc}(A) := \inf\{r > 0 \,:\, a \precsim b \text{ whenever } d_\tau(a) + r < d_\tau(b) \forall\tau\}$$

(where τ runs over all normalized traces, and $a, b \in (A\otimes\mathbb{K})_+$).

One can show that

$$\mathrm{rc}(A) = \inf\{m/n \,:\, a \precsim b \text{ whenever } na + m\langle 1_A\rangle \leq ny\}$$

3.5. **Proposition:** *If X is a CW-complex with $\dim(X) = d < \infty$, then:*

$$(d-2)/2 \leq \mathrm{rc}(C(X)) \leq (d-1)/2$$

Proof:
The upper bound was already discussed (and it works for all X, not just CW-complexes). To get the lower bound note that one can immerse $S^{2d'}$ into X (for some large d'). □

If A is simple, then $\mathrm{rc}(A) = 0$ if and only if $\mathrm{Cu}(A)$ is almost unperforated. We also have the following properties:

(i) $\mathrm{rc}(\varinjlim_k A_k) \leq \liminf_k \mathrm{rc}(A_k)$
(ii) $\mathrm{rc}(A/I) \leq \mathrm{rc}(A)$
(iii) $\mathrm{rc}(M_n(A)) = 1/n \, \mathrm{rc}(A)$

3.6. **Theorem:** *There exists a family A_r of simple AH-algebras indexed over $r \in [0, \infty]$ s.t.:*

(1) The Elliott invariant of A_r (K-theory and traces) is the same for all r
(2) $\mathrm{rc}(A_r) = r$, so the algebras are pairwise not isomorphic

The algebras A_r of the theorem are all shape equivalent, since they are constructed as AH-algebras over contractible spaces, so all homotopy invariant continuous functors agree on the A_r. Further $K_0(A_r) = \mathbb{Q}$ and $\mathrm{sr}(A_r) = 1$. This means we have uncountably many different Morita equivalence classes among the A_r.

3.7 (Mean dimension)**:** Let X be compact, metric, $\alpha : X \to X$ a homeomorphism, and $\mathcal{U}$ an open cover of X. Define

$$\mathrm{ord}(\mathcal{U}) := \sup\{(\sum_{U \in \mathcal{U}} \chi_U(x)) - 1 \ : \ x \in X\}$$

and write $\mathcal{V} > \mathcal{U}$ if $\mathcal{V}$ refines $\mathcal{U}$. Set:

$$D(\mathcal{U}) := \min\{\mathrm{ord}(\mathcal{V}) \ : \ \mathcal{V} > \mathcal{U}\}$$

We have $D(\mathcal{U} \cup \mathcal{V}) \leq D(\mathcal{U}) + D(\mathcal{V})$, since one can show that $D(\mathcal{U}) \leq d$ if and only if there exists a continuous map $f : X \to K$ with $\dim(K) \leq d$ such that f is compatible with $\mathcal{U}$.

Set $\mathcal{U}^n := \mathcal{U} \vee \alpha^{-1}(\mathcal{U}) \vee \ldots \vee \alpha^{-(n-1)}(\mathcal{U})$ where $\mathcal{V} \vee \mathcal{W}$ means the union and also all intersections of set in $\mathcal{V}, \mathcal{W}$. Set

$$\mathrm{mdim}(X, \alpha) := \sup_{\mathcal{U}} \lim_{n \to \infty} D(\mathcal{U}^n)/n$$

3.8. **Example:** Let Y be a CW-complex, $X = Y^{\mathbb{Z}}$, and $\alpha : X \to X$ the bilateral shift. Then $\mathrm{mdim}(X, \alpha) = \dim(Y)$.

Problem: If $\dim(X) < \infty$, then $\mathrm{mdim}(X, \alpha) = 0$ for all α.

3.9. **Theorem:** *(Kerr, Giol) For any $k > 0$ there exists a minimal system (X_k, α_k) s.t. $k \leq \mathrm{rc}(C(X_k) \rtimes_{\alpha_k} \mathbb{Z})$. Also* $\mathrm{mdim}(X, \alpha)/2 \approx k$.

If $\alpha : Y^\infty \to Y^\infty$ is the bilateral shift, then let Y_{2^n} be the 2^n-periodic points. Then:

$$\begin{array}{ccccccc} & & & C(Y_{2^n}) \rtimes_\alpha \mathbb{Z} & & & \\ & & \swarrow & & \searrow & & \\ \cdots \longrightarrow & C(Y_{2^n}) \rtimes_\alpha \mathbb{Z} & & \longrightarrow & & C(Y_{2^{n+1}}) \rtimes_\alpha \mathbb{Z} & \longrightarrow \cdots \end{array}$$

Proposal: Define a dynamical dimension $\mathrm{ddim}(X, G)$ for a countable, discrete group G acting on X via:

$$\mathrm{ddim}(X, \alpha) := \mathrm{rc}(C(X) \rtimes_\alpha G)$$

The reasons are:

(1) It looks like one could recover mdim for the bilateral shift
(2) If $G = \{1\}$, then $\mathrm{ddim}(X, G) \approx \dim(X)/2$
(3) If $G = \mathbb{Z}$ acting trivially, then $\mathrm{ddim}(X, G) = (\dim(X) + 1)/2$
(4) If $X = Y^m$ with α the cyclic shift, then $\mathrm{ddim}(X, \alpha) \approx \dim(Y)/2$

Outlook: Hopefully for minimal systems (X, α) we have $\mathrm{ddim}(X, \alpha) \leq \mathrm{mdim}(X, \alpha)/2$ and that this is sharp (see results of Kerr and Giol). Why are we hopeful?

We have that $C^*(C(X), uC(X \setminus \{y\})) = A_{\{y\}}$ is ASH, but the RSH-subalgebras have infinite dimension. Idea: fix $a, b \in A_{\{y\}+}$, $a = \sum_{i=1}^N f_i u$. Take $\mathcal{U}$ a finite open cover, iterate under α^{-1}, get covers $\mathcal{V}_n$ s.t. $\mathrm{ord}(\mathcal{V}_n) = n \cdot \mathrm{ddim}$, thus u corresponds to the size of the matrices.