MIKAEL RØRDAM'S TALK AT MASTERCLASS ON CLASSIFICATION OF C*-ALGEBRAS

1. Simple C*-algebras

Theorem 1. *For a simple C*-algebra* $A \neq \mathbb{C}$ *TFAE*

(1) $\forall B \subset A$ *heriditary subalgebra,* $B \neq \{0\}$, B *contains an infinite projection.*
(2) $\forall a, b \in A \backslash \{0\} \exists x, y \in A : b = xay$
(3) $RR(A) = 0$ *and all projections in* A *are properly infinite* ($p \oplus p \lesssim p$)
(4) $W(A) = Cu(A) \cong [0, \infty]$

Definition 2. *If one (and hence all) of the statements in the above theorem is true, we say that* A *is a* purely infinite *simple C*-algebra.*

Example 3. *For* $n \in \mathbb{N}$ *set*

$$\mathcal{O}_n = C^*(s_1, s_2, \dots, s_n | \forall j : s_j^* s_j = 1, \sum_{j=1}^{n} s_j s_j^* = 1) \tag{1.1}$$

and set

$$\mathcal{O}_\infty = C^*(s-1, s_2, \dots | \forall j : s_j^* s_j = 1, i \neq j \Rightarrow s_i s_i^* \perp s_j s_j^*). \tag{1.2}$$

Then $\mathcal{O}_n$ *is purely infinite and simple and* $K_1(\mathcal{O}_n) = 0$ *for all* $n \in \mathbb{N} \cup \{\infty\}$ *while* $K_0(\mathcal{O}_n) = \mathbb{Z}_{n-1}$ *for* $n \in \mathbb{N}$ *and* $K_0(\mathcal{O}_\infty) = \mathbb{Z}$.

Theorem 4. *For a simple C*-algebra* A *we have that* A *is purely infinite iff* $T(A) = \emptyset$.

Theorem 5. *It is possible to exhaust* (K_0, K_1) *by purely infinite simple C*-algebras of the following two types:*

- $A = \lim_{\to} \oplus_{j=1}^{n} M_{n_j}(\mathcal{O}_{r_j}) \otimes C(\Pi)$ *where simple implies purely infinite.*
- $A \rtimes \mathbb{Z}$ *for some simple stable* Π*-algebra* A.

Theorem 6. *If* A *is a simple, separable, exact, stable C*-algebra where* $T(A) \neq \emptyset$ *and* $A \otimes \mathcal{Z} \cong A$, *and* $\alpha \in Aut(A)$ *then* $A \rtimes \mathbb{Z}$ *purely infinite iff* A *has no* α*-invariant traces.*

Definition 7. *We say that a C*-algebra* A *is a* Kirchberg algebra *if it is purely infinite, simple, separable and nuclear.*

Definition 8. *We say that a C*-algebra has the* SP *if* $\forall B \subset A$ *hereditary subalgebra,* $B \neq \{0\}$, B *contains a nontrivial projection.*

Theorem 9. *There exists a stabely infinite simple C*-algebra* A *with* $RR(A) \neq 0$ *and not beeing purely infinite having the SP.*

Question 10. *If* A *is a simple C*-algebra, do* $RR(A) = 0$ *and* A *stabely infinite imply* A *purely infinite?*

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Question 11. *If A is simple and all projections in A are infinite, does $P(A) \neq 0$ imply A purely infinite?*

Question 12. *If A is stabely infinite does that imply that A has the SP?*

Theorem 13 (Kirchberg)**.** *If A, B simple not type I C*-algebras then if both are stabely infinite or one is stabely finite and the other stabely infinite $A \otimes_{min} B$ is purely infinite. If both are stabely finite and exact $A \otimes_{min} B$ is stabely finite, if they not both are exact the result is not known.*

Theorem 14 (Kirchberg)**.** *If A is a simple, separable, nuclear C*-algebra A is purely infinite iff $A \cong A \otimes \mathcal{O}_\infty$.*

Remark 15. $K_x(\mathcal{O}_\infty) \cong K_x(\mathbb{C})$

Theorem 16 (Kirchberg)**.** *A C*-algebra A is simple, separable, unitary and nuclear iff $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.*

Remark 17. $K_x(\mathcal{O}_2) = 0$

Theorem 18 (Kirchberg)**.** *A C*-algebra A is separable and exact iff $A \hookrightarrow \mathcal{O}_2$.*

Theorem 19 (Kirchberg, Phillips)**.** *If A, B are Kirchberg algebras:*

- $A \otimes \mathcal{K} \cong B \otimes \mathcal{K} \Leftrightarrow A \sim_{KK} B$
- *If A, B have the UCT then* $A \otimes \mathcal{K} \cong B \otimes \mathcal{K} \Leftrightarrow (K_0(A), K_1(A)) \cong (K_0(B), K_1(B))$.

2. Non-simple C*-algebras

Theorem 20. *If A is a C*-algebra with no nonzero abelian quotient TFAE*

(1) $\forall a, b \in A_+ : a \in \overline{AbA} \Leftrightarrow a \precsim b$
(2) $\forall a \in A_+$ *a is properly infitite* $(a \oplus a \precsim a)$

Definition 21. *If A fullfils one (and hence all) of the statements in the theorem we say that A is* purely infinite.

Definition 22. *If $x \in W(A)$ the we say that x is* properly infinite *if* $2x \leq x$.

Definition 23. *We say that $x \in W(A)$ is* infinite *if* $\exists y \neq 0 \in W(A).x + y \leq x \Rightarrow x + y = x$

Remark 24. *A is purely infinite iff $W(A)$ is properly infinite, where $W(A)$ is properly infinite iff $\forall x \in W(A)$ x is properly infinite.*

Remark 25. *The function $W(A) \to Ideal(A)$ defined by $< a > \to \overline{AaA}$ is welldefined and surjective. And this map is injective (equivalent: this map is an order isomorphism) iff A is purely infinite.*

Example 26. *If $A \in M_n(\{0, 1\})$ then is the Cuntz-Kneyr-algebra*

$$\mathcal{O}_A = C^*(s_1, s_2, \ldots, s_n | \sum_{j=1}^{n} s_j s_j^* = 1, s_j^* s_j = \sum_{i=1}^{n} A_{i,j} s_i s_i^*) \tag{2.1}$$

properly infinite.

Example 27. *If A is a C*-algebra then $A \otimes \mathcal{O}_\infty$ is purely infinite.*

Example 28. $C_0(\mathbb{R}) \otimes \mathcal{O}_\infty$ *is purely infinite.*

Theorem 29. *(1) If we have $0 \to I \to A \to B \to 0$ then I, B are purely infinite iff A is purely infinite.*
(2) If $A = \lim_{\to} A_i$ and all A_i are purely infinite then A is purely infinite.
(3) If A, B are purely inifnite and A is exact then $A \otimes_{min} B$ is purely infinite.

Question 30. *Does A or B purely infinite imply that $A \otimes_{min} B$ is purely infinite?*

Question 31. *Does A purely infinite imply that $A \otimes_{min} C([0,1])$ is purely infinite?*

Definition 32. *We say that A is* strongly purely infinite *if*

$$\forall \begin{pmatrix} a & \lambda^* \\ \lambda & b \end{pmatrix} \in M_2(A)_+ \forall \epsilon > 0 \exists d_1, d_2 \in A :$$

$$\| \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}^* \begin{pmatrix} a & \lambda^* \\ \lambda & b \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^* \| < \epsilon \tag{2.2}$$

Definition 33. *We say that A is weakly purely infinite if $\exists k \in \mathbb{N} \forall x \in W(A) : kx$ properly infinite.*

Remark 34. *$W(A)$ has no dimension function iff $\forall x \in W(A) \exists k \in \mathbb{N} : kx$ is properly infinite.*

Theorem 35. *Let A be a separable, exact C*-algebra and look at these properties:*

(1) $A \cong A \otimes < mathcalO_\infty$
(2) *A strongly purely infinite*
(3) *A purely infinite*
(4) *A weakly purely infinite*
(5) *$l_\infty(A)/C_0(A)$ traceless*
(6) *A traceless*

We have that

- *In general (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Leftrightarrow$ (5) $\Rightarrow$ (6).*
- *If A is seperable and nuclear (2) $\Rightarrow$ (1), but not true in general.*
- *If A is simple, $RR(A) = 0$ or $A \cong A \otimes \mathcal{Z}$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2), unknown if true in general.*
- *If $A \cong A \otimes \mathcal{Z}$ (6) $\Rightarrow$ (5), but not in general.*

Definition 36. *Let A be a C*-algebra the we define $Prim(A) = \{\ker(\pi) | \pi irreducibel\ representation\ of\ A\}$, and enquip it with the Jacobsen topology giving us a T_0-space.*

Example 37. *If X is a locally compact Hausdorff space then $Prim(C(X)) = X$.*

Theorem 38 (Kirchberg)**.** *If A, B are seperable nuclear C*-algebras and $X := Prim(A) = Prim(B)$ we have that $A \otimes \mathcal{O}_\infty \otimes \mathcal{K} \cong B \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ iff $A \sim_{KK} B$. If A, B are strongly purely infinite we can skip $\mathcal{O}_\infty$.*

Corollary 39. *If A, B are separabel nuclear C*-algebras TFAE*

(1) $A \otimes \mathcal{O}_2 \otimes \mathcal{K} \cong B \otimes \mathcal{O}_2 \otimes \mathcal{K}$
(2) $Prim(A) \cong Prim(B)$
(3) $Ideal(A) \cong Ideal(B)$

Question 40. *Which T_0-spaces can arise as $Prim(A)$ for A a separable C*-algebra? Which if A is nuclear?*

Example 41. *Let* $(t_n)_{n\in\mathbb{N}} \subseteq]0,1[$ *and look at the sequence*

$$C_0(]0,1]) \to_{\phi_1} M_2(C_0(]0,1])) \to_{\phi_2} M_4(C_0(]0,1])) \to_{\phi_4} \dots \tag{2.3}$$

where

$$\phi_n(f)(t) = \begin{pmatrix} f(t) & 0 \\ 0 & f(t \wedge t_n) \end{pmatrix}. \tag{2.4}$$

Let A *be the inductive limit of this sequence, then* A *is a* AH_0*-algebra and* $Ideal(A) \approx [0,1]$ *totally ordered.*

We have $A \cong A \otimes M_{2^\infty} \Rightarrow A \cong A \otimes \mathcal{Z}$ *and* $A \cong A \otimes \mathcal{O}_\infty \Leftrightarrow A$ *traceless, which is the case, so we have that* A *is purely infinite.*

$$A \sim_{h,ideal} 0 \Leftrightarrow_{def} \exists \phi_t : A \to A, \phi_t \leq h, t \in [0,1], \phi_1 = id, \phi_0 = 0, \forall J \lhd A, \phi_t(j) \subseteq J \tag{2.5}$$

A *strongly purely infinite, separable, nuclear then* $A \sim_{h,ideal} 0 \Rightarrow A \cong A \otimes \mathcal{O}_2$.

Fact 42. *If* A *has no projections then* $\forall I \lhd A : A/I$ *has no projections.*

Theorem 43. *If* A *is nuclear, separable, stable, strongly purely infinite C*-algebra then* $A \sim_{h,ideal} 0$ *implies that* $A \cong A \otimes \mathcal{O}_2$ *and* A *is an* AH_0*-algebra.*

Fact 44. *If* A *is an* AH_0*-algebra then* A *can be embedded in an* AF*-agebra* D *and* A *is quasidiagonal. If* $A \cong A \otimes \mathcal{O}_2$ *and* B *separabel and exact then* $CB \hookrightarrow C\mathcal{O}_2 \hookrightarrow A \hookrightarrow D$.