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Def^n (Elliott - Kucerovsky) We say that a C^*-algebra A has the Corona Factorization Property (CFP) if any full projection in $M(A \otimes K)$ is properly infinite.

A projection is full if it generates all of $M(A \otimes K)$

" " p is properly infinite if $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \lesssim \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$ in the Murray-von Neumann sense

Theorem (Kucerovsky - Ng) Let A be a separable C^*-algebra. Then A has the CFP iff every norm-full extension is nuclearly absorbing

Theorem (Kucerovsky, Kucerovsky - Ng) Let A be a separable C^*-algebra. TFAE:

(i) A has the CFP

(ii) Every full projection in $M(A \otimes K)/A \otimes K$ is properly infinite

(iii) Every strictly full element in $M(A \otimes K)/A \otimes K$ is properly infinite

Strictly full element: every non-zero element in $C^*(a)$ is full

Properly infinite: $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \lesssim \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ in the Cuntz sense

Examples (i) Simple C^*-algebras with Real Rank 0, stable rank 1 and weakly unperforated K_0 group

(ii) Purely infinite simple C^*-algebras

(iii) X compact Hausdorff metric space with finite dimension $\Rightarrow C(X)$ satisfies the CFP

(iv) $\mathcal{Z}$ - stable algebras

(v) $C(\prod_1^\infty S^2)$ does not have the CFP

(vi) Rørdam's example of a simple C^*-algebra with a finite and an infinite projection does not have the CFP

<u>Stability</u>

Rørdam constructed a C^*-algebra C such that fits into
$$0 \to B \to C \to A \to 0 \quad \text{with } B, A \text{ stable}, \ C \text{ not stable}$$

Rørdam constructed a C^*-algebra A with $M_n(A)$ stable and A not stable.

<u>Theorem</u> (Kucerovsky - Ng)

(i) If B has the CFP, then:

given $0 \to B \to C \to A \to 0$, C is stable when B, A are

(ii) A has the CFP $\iff$ $\left[\begin{array}{l}\forall D \subset A\otimes K \text{ hereditary and full,} \\ \text{then } M_n(D) \text{ is stable for some } n \\ \text{implies } D \text{ is stable}\end{array}\right]$

<u>Outline proof of part (ii)</u>

($\Rightarrow$) Assume A has the CFP. Let $D \subset A\otimes K$ be a ~~hered~~ hereditary ~~subalgebra of~~ and full and such that $M_n(D)$ is stable. We want to show that D is stable

We have
$$D \text{ full} \Rightarrow D\otimes K \cong A\otimes K \Rightarrow D \text{ has the CFP}$$

In $M(D\otimes K)$, take $P = 1\otimes e_{11}$, then $D \cong P(D\otimes K)P$ is full and hereditary in $D\otimes K$.

~~Since~~ $M(D\otimes K)$ is properly infinite

We can embed $O_n \hookrightarrow M(D\otimes K)$ unitally. If $\{s_i\}$ are the isometries coming from O_n, take $Q := \sum_{i=1}^{n} s_i P s_i^* \sim$

$Q(D\otimes K)Q \cong M_n(P(D\otimes K)P) \cong M_n(D)$, full, stable $\boxed{n.P := \begin{pmatrix} P & & 0 \\ 0 & P & \\ & & P\end{pmatrix}}$

This implies by Brown's theorem that $n.P \sim Q \sim 1_{M(D\otimes K)}$

$\Rightarrow P$ is full $\underset{\left(\substack{\text{since } D \text{ has} \\ \text{CFP}}\right)}{\Longrightarrow}$ P properly infinite

~~Ther~~ It follows $1 \sim nP \precsim P \precsim 1 \Rightarrow P \sim 1 \Rightarrow P(D\otimes K)P \cong D$

($\Leftarrow$) Take a full projection P in $M(A \otimes K)$

Then

P full $\Rightarrow$ $n \cdot P \sim 1$ for some n $\Rightarrow M_n(P(A\otimes K)P) \cong A \otimes K$

$\Rightarrow P(A\otimes K)P$ is stable $\Rightarrow M(P(A\otimes K)P)$ is properly infinite with unit P

$\Rightarrow P$ is properly infinite

<u>The CFP in the real rank zero situation</u>

Given a C^*-alg. A.

$V(A) = PM_\infty(A)/\sim$ where $PM_\infty(A)$ = projections in $M_\infty(A)$

$\sim$ = MvN-equivalence of projections

$V(A)$ is an abelian semigroup with zero

$[p]$ equivalence class of p

$[p]+[q] = \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\right] = [p \oplus q]$

$p \precsim q \underset{def}{\Longleftrightarrow} p \sim p' \leq q$ defines an order on $V(A)$ which is algebraic $[p] \leq [q] \Leftrightarrow [p]+[r] = [q]$

<u>Defⁿ</u> An abelian semigroup V is said to be a (Riesz) refinement semigroup if:

$a+b = c+d$ in V then $\exists\ (x_{ij})_{1\leq i,j\leq 2} \in V$

such that $c = x_{11}+x_{21}$ $\quad a = x_{11}+x_{12}$

$d = x_{12}+x_{22}$ $\quad b = x_{21}+x_{22}$

	c	d
a	x_{11}	x_{12}
b	x_{21}	x_{22}

Theorem (Zhang; Ara–Pardo)
~~If~~ A has real rank zero, then $V(A)$ is a refinement semigroup.

- $V = \{0, u, \infty\}$ where $2u = \infty$ is an example of a Riesz semigroup that is not a refinement semigroup

Def^n^ A semigroup V is Riesz if for all $a, c, d \in V$ such that $a \leq c + d$ ~~implies that~~ there are $x_1, x_2 \in V$ such that $a = x_1 + x_2$ and $x_1 \leq c$, $x_2 \leq d$

Weak divisibility

Def^n^ V abelian semigroup (with the order given by the addition operation)
An element $u \in V$, $u \neq 0$ is an order-unit if $\forall x \in V$, then $x \leq nu$ for some n.

Example A C^*-alg., $p \in A$ projection
$[p] \in V(A)$ is an order-unit $\Leftrightarrow$ p is full

Denote by $V^* = \{u \in V \mid u \text{ is an order-unit}\}$

- V^* is a subsemigroup of V

Def^n^ V is simple if $V^* \setminus \{0\}$

Example $V(A)$ is simple if A is a simple C^*-alg.

Def^n^ An element $x \in V$ is weakly divisible if there are $y, z \in V$ such that $x = 2y + 3z$

Say that V has weak divisibility of order-units when every order-unit is weakly divisible

Remark: ~~If~~ V has weak divisibility (w.d.) of order-units, then given $u \in V^*$, $u = 2x + 3y$ and take: $v = x + y$, $w = x + 2y$ then $u = v + w$, $v \leq w$ and $v, w \in V^*$.

- We will show that if A has real rank zero and if 1 is properly infinite, then $V(A)$ has w.d. of order-units
(This applies to $M(A \otimes K)$ when A has real-rank zero because $V(M(A \otimes K))$ has refinement and 1 is properly infinite)

Lemma: V is refinement with w.d. of order-units. Then, given $u, v \in V^*$, $\exists w \in V^*$ such that $w \leq^* u, v$

($u \leq^* v$ if $v = u + t$ and t is an order-unit)

Proof: Split $u = u_1 + u_2$, $v = v_1 + v_2$ with $u_i, v_i \in V^*$

$u_1 \in V^* \Rightarrow v_1 \leq n u_1$ for som n

Thus, $v_1 + t = n u_1$ for some t

$\exists$ refinement such that

	u_1	u_1	$\dots$	u_1
v_1	$v_1' \leq$	$v_2' \leq$	$\dots \leq$	v_n'
t	$t_1 \geq$	$t_2 \geq$	$\dots \geq$	t_n

$u_1 = v_i' + t_i \quad i = 1, \dots, n$
$v_1 = v_1' + \dots + v_n'$
$t = t_1 + \dots + t_n$

Set $v_n' = w$. Then $w \leq v_1, u_1$. w is an order unit because v_1 is

$w + (\underbrace{\quad}_{\in V^*}) = u_1$ $\square$

Def^n • An ideal I in V is a subsemigroup which is order-hereditary ($x \leq y, y \in I \Rightarrow x \in I$).

(Example: I a close ideal of $A \Rightarrow V(I)$ is an ideal of $V(A)$)

• $V/_I = V/_\sim$ where $x \sim y \Leftrightarrow x + z = y + t$ for some $z, t \in I$

$[x]$ the equivalence class of x

(Example: If A has real rank zero: $V(A)/V(I) \cong V(A/I)$)

Def^n An element x is an atom if $x \neq 0$ and $x = a + b \Rightarrow a = 0$ or $b = 0$

— If V has refinement
x is an atom $\Leftrightarrow \langle x \rangle = \{0, x, 2x, 3x, \dots\}$ is an ideal

Thm (Aranda, Goodearl, Perera, Sibs)
If V is refinement, Then u is weakly divisible $\Longleftrightarrow$

$[u]$ in V/I for any I is not an atom

Proposition If V is refinement and contains a properly infinite order unit u
$(2u \le u)$ then V has w.d. of order-units

Proof Take any order-unit v.

u order-unit $\Rightarrow v \le nu \le u$

$u \le mv$ for some m. Then $2(mv) = mv + mv \le u + u \le mv$

$\Rightarrow mv$ is properly infinite

If I is any ideal, look at V/I. $[mv]$ is properly infin

so $[v]$ ~~cannot~~ can not be an atom. Otherwise $\langle [v] \rangle \cong \mathbb{Z}^+$

and contains a properly infinite element. $(\Rightarrow \Leftarrow)$ $\square$

Lecture 2

V is a refinement semigroup $\left(\begin{array}{l} \forall a,b,c,d \in V \text{ s.t. } a+b=c+d \\ \text{Then } \exists (x_{ij})_{1\le i,j\le 2} \text{ s.t.} \\ a = x_{11}+x_{12} \quad c = x_{11}+x_{21} \\ b = x_{21}+x_{22} \quad d = x_{12}+x_{22} \end{array} \right)$

$V^* = \{u \in V \mid u \text{ is an order-unit}\}$

V has weak divisibility of order-units $(\forall u \in V^*,\ u = \cancel{v}+w$ with $v, w \in$ … $w \geq v$

Theorem If V has refinement and weak divisibility of order-units, then V^* has refinement

Outline

Lemma 1 (Goodearl)

If $\exists$ refinement

$$\begin{array}{c|cc} & c_1 \geq & c_2 \\ \hline a & a_1 & a_2 \\ b & b_1 & b_2 \end{array}, \quad b_2 \in V^*$$

then $\exists$ refinement

$$\begin{array}{c|c|c} & c_1 & c_2 \\ a_1 & a_1' \ngeq & a_2' \\ \hline b & b_1' & b_2' \end{array} \quad b_2' \in V^*$$

Lemma 2 If $\exists$ refinement

$$\begin{array}{c|c|c} & c & d \\ \hline a & x_{11} & x_{12} \\ b & x_{21} & x_{22} \end{array} \quad x_{11}, x_{22} \in V^*$$

then $\exists$ refinement

$$\begin{array}{c|c|c} & c & d \\ \hline a & x_{11}' & x_{12}' \\ b & x_{21}' & x_{22}' \end{array} \quad x_{ij}' \in V^* \text{ for } 1 \le i,j \le 2$$

Proof of the theorem

Suppose $a+b = c+d$ with $a,b,c,d \in V^*$

Split $a = x+y,\ x \geq y$, $c = u+v,\ u \geq v$ with $x,y,u,v \in V^*$

by Lemma 1 $\exists$

$$\begin{array}{c|c|c} & v & u+d \\ \hline y & x_{11} & x_{12} \\ \wedge\!\mid & \wedge\!\mid & \\ b+x & x_{21} & x_{22} \end{array} \quad x_{21} \in V^*$$

$v \le u+d$, then by Lemma 1 $\exists$

$$\begin{array}{c|c|c} & v \le & u+d \\ \hline y & x'_{11} \le & x'_{12} \\ b+x & x'_{21} & x'_{22} \end{array}$$

x'_{21}, x

By Lemma 2 $\exists$

$$\begin{array}{c|c|c} & v & u+d \\ \hline y & x_{11} & x_{12} \\ \hline b+x & x_{21} & x_{22} \end{array}$$

with all x_{ij} in V^*

Observe that $b+x = x_{21}+x_{22}$. Find any refinement

$$\begin{array}{c|c|c} & x_{21} & x_{22} \\ \hline x & a_{11} & a_{12} \\ \hline b & a_{21} & a_{22} \end{array}$$

We have

$$\begin{array}{c|c|c} & v & u+d \\ \hline a & x_{11}+a_{11} & x_{12}+a_{12} \\ \hline b & a_{21} & a_{22} \end{array} \quad , \; x_{11}+a_{11} \in V^* \qquad (*)$$

Applying Lemma 1 $\exists$

$$\begin{array}{c|c|c} & v \le & u+d \\ \hline a & t_{11} & t_{12} \\ \hline b & t_{21} \le & t_{22} \end{array} \qquad t_{11}, t_{22} \in V^*$$

After applying Lema 2, may assume that all t_{ij}'s are order-u

We have $u+d = t_{12}+t_{22}$ $\quad u, d, t_{12}, t_{22} \in V^*$

Refine this equality using first part of the proof up to (*)

$$\begin{array}{c|c|c} & u & d \\ \hline t_{12} & w_{11} & w_{12} \\ \hline t_{22} & w_{21} & w_{22} \end{array} \qquad w_{22} \in V^*$$

Now:

$$\begin{array}{c|c|c} & c & d \\ \hline a & t_{11}+w_{11} & w_{12} \\ \hline b & t_{21}+w_{21} & w_{22} \end{array}$$

and Lemma 2 $\Rightarrow$ we can get them all order-units $\square$

$t_{11}+w_{11}, w_{22} \in V^*$

<u>Theorem</u> If V is a refinement, and if $u \in V^*$ such that nu is properly infinite, then $u = s+t$ with $s, t \in V^*$ and ns, nt properly infinite.

Proof (Sketch) (1) Prove it when V is simple ($V = V^+ \cup \{0\}$)
(2) ∞u properly infinite $\Rightarrow V$ has weak divisibility for order units

So by the theorem V^+ is refinement and simple. Now apply step 1.

Corollary If V is refinement and if $u \in V^+$ is such that ∞u is properly infinite. Then $\exists\, t_1, t_2, \ldots \in V^+$ such that ∞t_i is properly infinite and $t_1 + \ldots + t_k \le u \quad \forall k$.

Proof By induction.

The CFP for (algebraically ordered) semigroups

Def^n (i) V has the strong CFP if $\forall x, \{y_n\}, m \ge 1$ and $x \le m y_n \quad \forall n$, then $x \le y_1 + \ldots + y_k$ for some k

(ii) A sequence $x_1 \le x_2 \le \ldots$ in V is called full if $\forall y \in V$, $\exists\, m, n$ s.t. $y \le m x_n$

(the sequence $x, x, \ldots$ is full iff $x \in V^+$)

(iii) V has the CFP if $\forall \{x_n\}$ full, $\forall \{y_n\}$, $\forall m \ge 1$
$x_n \le m y_n \quad \forall n \Rightarrow x_1 \le y_1 + \ldots + y_k$ for some k

Remark Strong CFP $\Rightarrow$ CFP

If V is simple, then they are equivalent

Def^n Given $x, y \in V$, $x <_s y$ if $(k+1)x \le k y$ for $k \ge 1$ ($\Leftrightarrow$ $f(x) <$ ... $\forall f \in S($...)

We say that V has n-comparison if given $x, y_0, \ldots, y_n$ with $x <_s y_j \quad \forall j$, then $x \le y_0 + \ldots + y_n$

0-comparison is the same as almost unperforation

n-comparison $\Rightarrow$ m-comparison $\forall m \geq n$

If V is simple and refinement, then they are equivalent

<u>Example</u> Take $W_n = \langle 0, n+1, n+2 \rangle \leq \mathbb{Z}^+$. Then W_n has n-comparis[on] but not $(n-1)$-comparison.

If V has n-comparison $\Rightarrow$ V has the strong CFP
$\Rightarrow V$ has CFP

<u>Pf.</u> Suppose $X, Y_1, Y_2, \ldots, Y_n, \ldots$ $m \geq 1$ and $X \leq m Y_n$ $\forall n$

Take $Z_j = Y_{j(m+1)+1} + \cdots + Y_{j(m+1)+m+1}$ $\quad j = 0, \ldots, n$

Then $(m+1) X \leq m Z_j$ $\forall j$

$\Rightarrow X \leq Z_1 + \cdots + Z_n$ $\square$

<u>Theorem</u> If A is separable and has RR0, then A has the CFP $\Leftrightarrow$ $V(A)$ has the CFP

<u>Proof</u> (Outline)

($\Rightarrow$) Assume $\underset{V(A\otimes K)}{\overset{V(A)}{\shortparallel}}$ does not have the CFP

Can find $\{X_n\}$, $\{Y_n\}$ $m \geq 1$, $\{X_n\}$ full, $X_n \leq m Y_n$ $\forall n$ but

$X_1 \not\leq Y_1 + \cdots + Y_k$ $\forall k$.

We have

$X_n = [q_n]$, $Y_n = [p_n]$ $\quad q_n, p_n$ can be taken to be pairwise orthogonal. Set

$\sum p_n = P$, $\sum q_n = Q$ in $M(A \otimes K)$

$\{X_n\}$ full $\Rightarrow$ $1_{M(A\otimes K)} \precsim Q$

since Q is full, then Q is properly infinite and $1 \sim Q$

$m P = \sum P_n'$ where $[P_n'] = m[P_n]$

$[q_n] = X_n \leq m Y_n = m[p_n] = [P_n'] \Rightarrow 1 \sim Q \precsim mP \Rightarrow P$ is full $\underset{\text{CFP}}{\Rightarrow}$ P is proper[ly] infini[te]

P full and properly infinite $\Rightarrow Q \precsim P$. But it is not true because we should have: $X_1 = [q_1] \leq [p_1] + \cdots + [p_k] = \sum_{i=1}^{k} Y_i$

($\Leftarrow$) Suppose $V(A)$ has the CFP and P is a full projection in $M(A\otimes K)$. Then $m \cdot p \sim 1$

As before $p = \sum p_n$, and $mp = \sum p_n'$, $[p_n'] = m[p_n]$

Take e_n such that $\sum e_n = 1$. Let $x_n = \sum_{k=1}^{n} [e_k]$. Then $\{x_n\}$ is a full sequence in $V(A)$.

$$1 \sim m \cdot P \Rightarrow x_n \leq [p'_{k_n}] + \dots + [p'_{k_{n+1}-1}]$$
$$= m \underbrace{\left([p_{k_n}] + \dots + [p_{k_{n+1}-1}]\right)}_{y_n}$$

$V(A)$ has CFP $\Rightarrow$ $x_1 \leq y_1 + \dots + y_{l_1}$
$$x_2 \leq y_{l_1+1} + \dots + y_{l_2}$$
$$\vdots$$
$$x_n \leq y_{l_n+1} + \dots + y_{l_{n+1}}$$

This implies $1 \precsim P$.

$\Rightarrow$ P is properly infinite

~~Theorem If A has RR0, and the CFP, and if p is a full projection in A such that mp is properly infinite, then P is properly infinite.~~

~~Proof~~ (~~Sketch~~) A RR0 and CFP $\Rightarrow V(A)$ is refinement and has the ~~property~~ CFP

$u = [p]$ is an order-unit such that mu is properly infinite

$\Rightarrow \exists$ order-units t_i such that mt_i are properly infinite

$t_1 + \dots + t_k \leq u \quad \forall k$

$u \leq mt_i \underset{CFP}{\Rightarrow} \begin{matrix} u \leq t_1 + \dots + t_k \\ u \leq t_{k+1} + \dots + t_l \end{matrix} \Rightarrow 2u \leq t_1 + \dots + t_l \leq u \Rightarrow u$ is properly infinite $\square$

<u>Corollary</u> (Zhang) If A has RR0, the CFP and is simple, then A is either stably finite or purely infinite

<u>Proof</u> If all elements in $V(A)$ are finite

(x finite : $x+y=x \Rightarrow y=0$)

$\Rightarrow A$ is stably finite

If not, $\exists u \in V(A)$, $u \neq 0$ infinite $\Rightarrow u$ is properly infinite.

If $v \in V(A) \setminus \{0\} \Rightarrow u \leq nv \Rightarrow nv$ is properly infinite

$\Rightarrow v$ properly infinite $\Rightarrow A$ purely infinite $\square$

<u>Question</u> If $RR(A)=0 \Rightarrow A$ has the CFP?

<u>Theorem</u> If A has RR0

$V(A)$ has the strong CFP $\Leftrightarrow$ all ideals of A have the CFP

<u>Idea</u> $V(A)$ is refinement, and $I \longmapsto V(I)$ is a lattice isomorphism.

The result follows if:

V refinement : V has the strong CFP $\Leftrightarrow$ all order-ideals have the CFP