Spectra of C^*-algebras, classification.

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Conventions and notations:

- the C^*-algebras are separable (except multipliers and corona algebras).

- To spaces are 2^{nd} countable.
- $O(X)$ is the lattice of open sets of X
  $F(X)$ " " " " closed " " X.

- Let A be a C^*-algebra, $X = \mathrm{Prim}(A)$.
Recall that $\mathrm{Prim}(A \otimes B) = \mathrm{Prim}(A)$ if B is simple and exact, A is separable. We may also require that A be exact and B simple. One of them must be exact.

- If A is p.i. then $W(A) \cong \mathrm{Lat}(A) = O(X)$. (the lattice of open sets).
- $W(A)$ does not detect if a p.i A tensorially absorbs O_∞ or not (among exact C^*-algebras)
- No p.i amenable has been found of A s.t $W(A)$ does not detect $\otimes O_\infty$ (Is p.i = s.p.i. ?) for nuclear algebras.

- X is T_0, sober (i.e. point-complete), locally quasicompact, and 2nd countable (by separability).
Point-complete: each prime closed set is the closure of a point.
Sobriety comes from the fact that X is an open and continuous image of a Polish space (the pure states).

- Dini function.
  $f : X \to [0, \infty)$ is Dini if for any upward directed net of l.s.c. function g_i such that $\vee g_i = f$, it also converge uniformly (on compact sets) to f.

- The supports of Dini functions on $X = \mathrm{Prim}(A)$, form a base for the topology.

- the generalized Gelfand transform
$$a \longmapsto \hat{a}, \text{ where}$$
$$\hat{a}(J) := \| a + J \|, \quad J \in \mathrm{Prim}(A)$$
is a surjection onto the Dini functions on X (A is separable).

three basic questions:

1) Is every 2nd countable, locally quasi-compact, sober T_0 space X homeomorphic to $Prim(A)$ for some A (A separable).

2) Is there a topological characterization of $Prim(A)$ for A nuclear?

3) Is there a uniqueness for the corresponding algebra A s.t. $Prim(A) = X$ assuming that $A \otimes O_2 \cong A$ (or some other such property).

Strategies and partial results.

Lemma 1 For a relation that is a ~~partial order~~ on a locally compact Polish space such that the projection onto ~~X are open~~ the first variable is open, and onto the 2nd is closed

$\Rightarrow \exists\ H_0 : C_0(P, K(H)) \to M(C_0(P, K(H)))$

$*$-monomorphism, that characterizes the relation as follows:

$(x, y) \in R \iff V_y \otimes id$ is weakly contained in $M(V_x \otimes id) \circ H_0$.

V_x, V_y irreducible representations.

Proof.

ν_x and ν_y are point-evaluation maps. (?)
By Michael's selection, ∃ sufficiently many c.p. maps

$$V : C_0(P) \longrightarrow C_0(P)$$

such that the states $\nu_x \circ V$ have support in
$\bar{x}^R := \{ y \in P \mid x \le y \}$. . . . .

- If an order relation is given on P, one can define the Scott topology on P.

U is open $\iff$ U is ~~open~~ upward hereditary and open.

- Define $x \sim y$ as the symmetrization of the partial order. then

$$x \sim y \iff \forall\, U \text{ open in the Scott topology } \; x \in U \iff y \in U$$

It follows that

$$X := P/\sim$$

is a T_0-space and ~~..~~ satisfies that

Scott top = ~~preimage~~ preimages of open sets of $P/\sim$.

Let $\mathcal{H} = C_0(P, K(H))$. Let $T_{\mathcal{H}}$ be the Cuntz-Pimsner algebra associated to $C_0(P, K(H))$ (viewed as a $C_0(P, K(H))$ bimodule, the left action given by H_0 of Lemma).

Proposition (Harnisch, Kirchberg).

$T_{\mathcal{H}}$ is separable, nuclear, s.p.i.

$\mathrm{Lat}(T_{\mathcal{H}}) \cong \mathcal{O}_R(P) \cong \mathcal{O}(P/\sim)$.

This proposition leads to the problem of finding, for given A, a loc. comp. Polish space P and ~~...Prim(A)~~ $\lambda : P \to \mathrm{Prim}(A) := X$, such that the relation

$(x, y) \in R \Leftrightarrow y \in \overline{\{x\}}$ satisfies the conditions of Lemma 1, and that $\lambda(P)$ is sufficiently dense in X, in the sense that

$$\lambda^{-1} : \mathcal{O}(X) \to \mathcal{O}(P) \text{ is injective.}$$

Passage to sober T_0 spaces.

~~Def~~ X ~~be~~ T_0, then $F(X)$ (the lattice of closed subsets) is anti-isomorphic to $O(X)$. $F(X)$ is T_0 with respect to the topology generated by the complements of the intervals $[\phi, F]$, F closed.

The map $\eta : X \to F(X)$, $\eta(x) = \overline{\{x\}}$ is a homeomorphism from X onto $\eta(X)$.

$\eta(X) \subseteq X^c$, where X^c is the $\vee$-prime elements of $F(X)$.

⋮

Regular subalgebras

$C \subseteq A$ is a regular C^*-subalgebra if

- $C \cap (J_1 + J_2) = C \cap J_1 + C \cap J_2$
- C separates the ideals of A.

$C \cap J_1 = C \cap J_2 \Rightarrow J_1 = J_2$.

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Assume that C is abelian.

The map $J \to C \cap J$, gives rise to a map

$$\pi : X(C) \to \mathrm{Prim}(A)$$

that is pseudo-open, pseudo-epi.

- For AF algebras (and AH algebras) one finds regular C^*-subalgebras (AF if A is AF).

- Regular commutative subalgebras $\mathcal{U}$ is in general not maximal. $C \cap J$ may not contain an a.u. of J.

- For every w.p.i algebra B, and $E \subseteq Q(\mathbb{R}_+, B)$, E separable, there is a separable subalgebra A, $E \subseteq A \subseteq Q(\mathbb{R}_+, B)$, with $EAE = A$, such that A contains a regular abelian subalgebra.

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Theorem (Harnisch, Kirchberg, Rørdam).

X is a point-complete, T_0-space. TFAE

(i) $X \cong \operatorname{Prim}(E)$ for some E exact C^*-algebra

(ii) $\mathcal{O}(X)$ is isomorphic to a sup inf invariant sub-lattice of a loc. comp. Polish space Y.

(iii) there is Y, Polish, and pseudo-open and pseudo-epi map $\pi : Y \to X$.

Under the above conditions there is A, stable, nuclear C^* algebra A s.t. $\operatorname{Prim}(A) \cong X$, $A \otimes \mathcal{O}_2 \cong A$. Moreover, there is

$$\psi : X \to \operatorname{Prim} A, \quad \text{homeo.}$$

with the universal property that:

If $B \otimes \mathcal{O}_2 \cong B$, $\phi : X \to \operatorname{Prim} B$ is homeo, then $\exists\, \alpha : A \to B$, isomorphism, such that $\hat{\alpha} \circ \psi = \phi$.

α is unique up to unitary homotopy.

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Pseudo open: π is pseudo open if

$$\bigcup_\lambda \pi^{-1}(F_\lambda) = \pi^{-1}\left(\overline{\bigcup_\lambda F_\lambda}\right)$$

for a family (F_λ) of closed subsets.

Pseudo epi:

For $G \subseteq F$ closed, $G \neq F$, $\Rightarrow$

$$\pi(Y) \cap (F \setminus G) \neq \emptyset.$$

$[0,1]_{lsc}$ is $[0,1]$ with the Scott topology of $x R y$ if $y \leq x$.

The open sets are $\emptyset$, $[0,1]$ and $(\alpha, 1]$, $\alpha > 0$.

The map $[0,1]_{\text{Hausdorff}} \longrightarrow [0,1]_{lsc}$

is pseudo-open but not open.

A similar example may be given of a pseudo-epi map that is not epi.

Def. X subset T_0 is called **coherent** if the intersection $C_1 \cap C_2$ of saturated quasi-compact subsets is again quasi compact.

A subset $C \subseteq X$ is saturated if ....

Spectra of C^*-algebras, classification

Lecture 2.

Recall from Lect. 1: (4)

• Q denotes the Hilbert cube (with the coordinate-wise order)

the basic result ~~from~~ Lect 1. was:

X is homeomorphic to $\mathrm{Prim}(A)$ for A nuclear iff.

. $\exists P$ polish l.c. space and $\pi: P \to X$ continuous s.t. $\pi^{-1}: \mathcal{O}(X) \to \mathcal{O}(P)$ is

injective, pseudo-epi, and pseudo open.
the algebra $A \otimes O_2$ is .....

Remark:
A continuous epi $\pi : P \to X$ is not pseudo-open. There is no epi from the Cantor set to $[0,1]$. (5)

We call a map $\psi: \mathcal{O}(X) \to \mathcal{O}(Y)$ lower semicontinuous if

$$\left(\bigcap_n \psi(U_n)\right)^\circ = \psi\left(\left(\bigcap_n U_n\right)^\circ\right)$$ for

each sequence $(U_n) \in \mathcal{O}(X)$. (that is, ψ preserves countable infs).
(then π pseudo-open iff π^{-1} is l.s.c.)

Recall: $C \subseteq X$ is saturated if (6)

$C = \mathrm{Sat}(C)$, where $\mathrm{Sat}(C) = \bigcap \{U \in \mathcal{O}(X) \mid C \subseteq U\}$.

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<u>Definition</u>: A sober T_0 space is <u>coherent</u> if $C_1 \cap C_2$ is quasi-compact for C_1, C_2 saturated and quasi-compact.

<u>Question</u> $\exists$ a every (2nd countable) . . .

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Let X be loc. quasi-compact, T_0.

$F(X)$ is the lattice of closed sets.

<u>Def</u>. $F(X)_{sc}$ is $F(X)$ with the Scott topology. ~~It is~~ generated by

$$F(X) \setminus [\phi, F] = \{G \in F(X) \mid G \cap U \neq \phi\} = M_U.$$

The Fell-Vietoris top is generated by M_U and the sets

$$M_C = \{G \in F(X) \mid G \cap C = \phi\}$$

for C quasi-compact.

The space $F(X)_{lsc}$ is a coherent 8

2nd countable, loc. quasi-compact, sober

T_0-space.

The space $F(X)_H$ is a compact Polish

space (It seems that the Fell-Vietoris

topology is the Lawson topology).

<u>Def</u>. A map $f : X \longrightarrow [0, \infty)$ is a

Dini function if it is l.s.c. and

$$\sup f|_F = \inf \{ \sup f|_{F_n} \}, \ \bigcap F_n = F.$$

~~There are several other~~

For X sober . . . .

For Q the Hilbert cube

$F(Y)$ is nothing;

On the blackboard:

$\mathcal{O}([0,1]_{lsc}) = \{\emptyset, [0,1], (t,1] ; t \in [0,1)\}$.

$[0,t] = \overline{\{t\}}$.

Sat $\{t\} = [t,1]$

$[0,1)_{lsc}$ open in $[0,1]_{lsc}$.

The Fell-Vietoris top of Q (the Hilbert Cube) is the ordinary Hausdorff topology.

On the other hand, Q with Scott topology is Prim(A) for some nuclear A, since $Q_{lsc} = \prod [0,1]_{lsc}$ ($[0,1]_{lsc}$ ~~appear~~ arise in Bordoni's example.

In a T_0 space it may not be that quasi-G_δ is the same as G_δ. 10

But for $\pi : P \to X$ continuous, $\pi^{-1}(Z)$ is G_δ for Z quasi-G_δ, hence $\pi^{-1}(Z)$ is Polish.

Quasi G_δ : $Z = \bigcap Z_i$, $Z_n = U_n \cup F_n$, U_n open, F_n closed.

The Scott ~~topology~~ on Q induces the Scott topology on $F(X)$, in which X becomes a quasi G_δ of $F(X)$ and Q. Since $Q = \mathrm{Prim}(A)$ for some A, there is a Polish P and $\pi : P \to X$ s.t. $\pi^{-1}(x)$ is a disjoint union of ∞-dimensional projective spaces.

On the blackboard:

$\iota(\eta(X))$ is quasi-G_δ in Q_{sc}.

In this way $X \subseteq \overline{X}^H \setminus \{0\} \subseteq F(X) \subseteq Q$ 17

as Polish spaces.

Below, we denote by $Y = \overline{X}^H \setminus \{\emptyset\} \subset F(X) \setminus \{0\}$

the closure of X in $Q \setminus \{0\}$.

Proposition.

The image $\eta(X) \cong X$ in $F(X) \setminus \{\emptyset\}$ of a l.q.c., 2nd countable, sober T_0 ~~space~~ X is closed in $F(X) \setminus \{0\}$ with respect to the Fell-Vietoris topology on $F(X)$ iff X is coherent iff the set of Dini functions is convex iff $D(X)$ is min-closed iff $D(X)$ is multiplicatively closed.

(12)

<u>Lemma</u>. Each closed $F \subseteq Q_H$ is a coherent sober subspace F_{lsc} of Q_{lsc}.

If $F = \bigcap F_n$, for $F_1 \supseteq F_2 \supseteq \cdots$ in $\mathcal{F}(Q_H)$ and if $F_{n\,lsc} = \mathrm{Prim}(A_n)$ for some nuclear A_n, then the same holds for F.

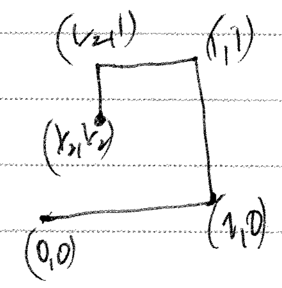
<u>Corollary</u>. If $\exists$ X coherent, l.c., sober, s.t. $X \neq \mathrm{Prim}(A)$ for A nuclear, then there is $n \in \mathbb{N}$ and Y, a finite union of cubes in $[0,1]^n$, s.t. Y (with the order ~~topology~~) is not $\mathrm{Prim}(A)$ for nuclear A.

I do not know if the following subset of $[0,1]^2$ (with the subset topology induced by $[0,1]^2_{lsc}$) is the Prim(A) for A nuclear.

$$F = \bigcup \overline{(0,0),(1,0)},\ \overline{(1,0)(1,1)},\ \overline{(1/2,1)(1,1)},\ \overline{(1/2,1/2)(1/2,1)}.$$

the nice subspaces of $[0,1]_{lsc}$ are all of the form Prim(A) with A nuclear.

F =

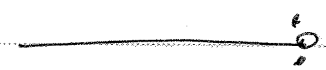


Examples of non-coherent and coherent Prim(A).

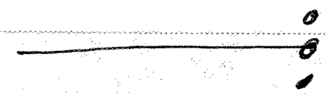
Let $X = \mathrm{Prim}(A)$ for

$$A = \{ f : [0,1] \to M_2 \mid f(1) \text{ diagonal} \}.$$

then

X 

then the closure Y of X in F_H then

Y 

the Dini functions on X are given by the nonnegative continuous functions $g \in C(Y)$ with $g(1) = \max(g(2), g(3))$.

the closed subset F_1 of X that corresponds to 1 in $\{2,3\}$.

The topology Y_{sc} generated by the supports of Dini functions is given by the lattice of open subsets V of Y s.t. $1 \in V$ if $V \not\subseteq [0,1)$.

With this topology,

$$Y \cong \mathrm{Prim}(B)$$

with B as follows:

$$D = K + \mathbb{C}\cdot 1 \oplus \mathbb{C}\cdot 1 \subseteq B(\ell_2 \oplus \ell_2)$$

. . . .

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the Dini function on Y_{sc} are given by $g \in C(Y)_+$ with $g(1) \geq \max(g(2), g(3))$. It follows that $D(Y_{sc})$ is invariant under min (i.e., Y is coherent). Thus

$$C(Y) \subseteq C^*(D(Y_{sc})) = C^*(D(X)) \subseteq l_\infty(X).$$

- - - - - -
- - - -

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the map

$$\psi : [0,1] \cup [4,5] \longrightarrow X$$
$$\psi(t) = \psi(4+t) = t \quad \text{for } t \in [0,1)$$
$$\psi(1) = 2, \; \psi(5) = 3$$

is continuous and pseudo-open.

Since $Y_{sc} = \cdot$ - - - -

The spaces X and Y_{sc} are subspaces (18)

of $[0,1]^3_{lsc}$.

. . . . .

Let $\mathcal{O}(X)$ be the lattice of open (19)

subsets of X.

Action on T_0 spaces.

Definition: $\Psi : \mathcal{O}(X) \longrightarrow \mathrm{Ideal}(A)$ is called

an action of X on A if it is increasing.

Notation: $A(U) := \Psi(U)$, $A|_F = A/\Psi(X \setminus F)$,

$$a|_F := a + \Psi(X \setminus F) \in A|_F .$$

the action is l.s.c. if

$$x \longmapsto \| a|_{\overline{\{x\}}} \|$$

is l.s.c for all a.

Ψ is non-degenerate if $\Psi(\phi) = \{0\}$ and $\Psi^{-1}(A) = \{X\}$.

The action is upper semicontinuous if

$$\psi(\vee U_i) = \vee \psi(U_i).$$

(Lower semicontinuous is

$$\psi(\wedge U_i) = \wedge \psi(U_i).)$$

Example If A is a $C_0(X)$-algebra then

$$\psi(U) = C_0(U) A$$

is upper semicontinuous. If it is l.s.c we have a continuous field.

Example $X = \mathrm{Prim}(B)$. Then ψ_B is the natural action of $\mathrm{Prim}(B)$ on B, with

$$\psi_B(U) = \bigcap_{J \notin U} J.$$

Example 3. If $S \subseteq CP(A, B)$, $X = \mathrm{Prim}(B)$ then . . . . .

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Definition. We call $S \subseteq CP(A,B)$ non-degenerate if $\{T(a) \mid a \in A, T \in S\}$ is a dense ideal of B

...

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M.O.C. cones

Definition. A subset $C \subseteq CP(A,B)$ is a matricially operator convex cone if

(i) C is closed, convex $\subseteq CP(A,B)$

(ii) for $V \in C$, $(a_i)_{i=1}^n \in A$, $(b_i)_{i=1}^n \in B$,

$$W : a \longmapsto \sum_{j,k} b_j^* V(a_j^* a a_k) b_k$$

is in C.

...

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~~Rec~~

A m.o.c cone defines a l.o.c. action $\psi : \mathcal{O}(\mathrm{Prim}(B)) \to \mathcal{I}(A)$.

$\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{\mathcal{I}(B)}$.

Let F_∞ denote the free group, $E = C^*(F_\infty)$

<u>Theorem</u>. If $C \subset CP(A, B)$ is given, and ψ' is the action defined above, then

<u>Corollary</u>. If B is nuclear or A is exact and $C \subseteq CP_{nuc}(A, B)$ then for $\forall \in CP$

- - - - .

Hilbert A-B-modules vs MOC cones.

We say that a Hilbert A-B-module H_B is C-compatible if the maps

$$a \longmapsto \langle \lambda(a) y, y \rangle$$

are in C for all y.

C-compatible modules