This is lecture notes of Marius Dadarlat's talks during the Master class on classification of C^*-algebras at the University of Copenhagen. The material he covered appears to be from the papers *Continuous fields of C*-algebras over finite dimensional spaces* (Advances in Mathematics 222 (2009) 1850-1881) and *Fiberwise KK-equivalence of continuous fields of C*-algebras* (J. K-Theory 3 (2009), 205-219).

1 First Lecture

In general $\mathcal{A}$ will denote a separable C^*-algebra and X will denote a locally compact Hausdorff space.

Definition 1.1 (Kasparov). *$\mathcal{A}$ is a $C_0(X)$ algebra if a $*$-homomorphism from $C_0(X)$ to $Z(\mathcal{M}(\mathcal{A}))$ (the center of the multiplier algebra) is given (this means we can multiply elements from $C_0(X)$ with elements from $\mathcal{A}$) such that*

$$\overline{C_0(X)\mathcal{A}} = \mathcal{A}.$$

Morphisms of $C_0(X)$-algebras $\gamma\colon \mathcal{A} \to \mathcal{B}$ commutes with the multiplication, that is $\gamma(fa) = f\gamma(a)$.

An equivalent definition would be that a surjective $*$-homomorphism going from $C_0(X) \otimes \mathcal{A}$ to $\mathcal{A}$, which is $\mathcal{A}$ linear, is given.

Another equivalent definition is that a continuous map from $\mathrm{Prim}(\mathcal{A})$ to X is given.

Remark 1.2. We can extend the map from $C_0(X)$ to $Z(\mathcal{M}(\mathcal{A}))$ to a map from $C_b(X)$ to $Z(\mathcal{M}(\mathcal{A}))$.

If $U \subseteq X$ is open, then by Cohens lemma $\overline{C_0(U)\mathcal{A}} = C_0(U)\mathcal{A}$. This is an ideal in $\mathcal{A}$ and we denote it by $\mathcal{A}(U)$.

If $Y \subseteq X$ is closed, then we let $\mathcal{A}(Y)$ be the quotient $\mathcal{A}/\mathcal{A}(X \setminus Y)$. If $x \in X$ then the set $\{x\}$ is closed and $\mathcal{A}(x)$ denotes $\mathcal{A}(\{x\})$. This quotient is called the fiber at x of $\mathcal{A}$.

We let π_x denote the quotient map from $\mathcal{A}$ to $\mathcal{A}(x)$. If $a \in \mathcal{A}$ then we write $a(x)$ for $\pi_x(a)$. We have a $*$-homomorphism $\mathcal{A} \to \Pi_{x\in X}\mathcal{A}(x)$ given by $a \mapsto (\pi_x(a))_{x\in X}$.

Lemma 1.3. *For all $a \in \mathcal{A}$ the map $x \mapsto \|\pi_x(a)\| = \|a(x)\|$ is upper semi-continuous.*

Proof. We must show that for all $\alpha > 0$ the set

$$U = \{x \in X \mid \|\pi_x(a)\| < \alpha\}$$

is open. We have

$$\begin{aligned} \|\pi_x(a)\| &= \inf\{\|a + z\| \mid z \in \mathcal{A}(X \setminus \{x\})\} \\ &= \inf\{\|a + fb\| \mid f \in C_0(X \setminus \{x\}), b \in \mathcal{A}\} \\ &= \inf\{\|a + (g - g(x))hb\| \mid g \in C_0(X), b \in \mathcal{A}, h \in C_0(X)\}. \end{aligned}$$

If $x \in U$ then $\|\pi_x(a)\| < \alpha$ so then there must exist $g \in C_0(X), h \in C_0(X), b \in \mathcal{A}$ such that

$$\|a + (g - g(x))hb\| < \alpha.$$

Since that expression is continuous in x, there exists an open set V, $x \in V$ such that for all $y \in V$

$$\|a + (g - g(y))hb\| < \alpha.$$

Hence $x \in V \subseteq U$ and therefore U is open. □

Remark 1.4. For all $x \in X$, $a \in \mathcal{A}$ and $f \in C_0(X)$: $\pi_x(fa) = f(x)\pi(a)$ since $(f - f(x))a \in C_0(X \setminus \{x\})\mathcal{A} = \ker(\pi_x(a))$.

Define for all $a \in \mathcal{A}$ the map $N(a)\colon X \to [0;\infty[$ by $N(a)(x) = \|\pi_x(a)\| = \|a(x)\|$ (N is for norm). By lemma 1.3 this map is lower semi-continuous and by remark 1.4 we have $N(fa)(x) = f(x)N(a)(x)$ for all $x \in X$.

Definition 1.5. *$\mathcal{A}$ is a continuous $C_0(X)$-algebra if $N(a)$ is continuous for all $a \in \mathcal{A}$. In this case $N(a) \in C_0(X)$.*

Such algebras are also called continuous field C^*-algebras.

This definition is equivalent to requiring that the map $\mathrm{Prim}(\mathcal{A}) \to X$ is open.

1.1 Examples

Example 1.6. *$\mathcal{A} = C_0(X, \mathcal{D}) = C_0(X) \otimes \mathcal{D}$. This is called the trivial field. Note that $\mathcal{A}(x) \cong \mathcal{D}$ for all $x \in X$.*

Example 1.7. *Let $\mathcal{D}$ be a C^*-algebra and let $\psi \in End(\mathcal{D})$. Let*

$$\mathcal{A} = \{(\alpha, d) \in C([0,1], D) \oplus \mathcal{D} \mid \alpha(1) = \psi(d)\},$$

$\mathcal{A}$ is $C([0,1])$-algebra with multiplication of an $f \in C([0,1])$ given by

$$f(\alpha, d) = (f\alpha, f(1)d).$$

We will show that $\mathcal{A}(x) \cong \mathcal{D}$ for all $x \in X$. Observe that

$$C_0([0,1] \setminus \{x\})\mathcal{A} = \begin{cases} (\alpha, d) \in \mathcal{A} \text{ with } \alpha(x) = 0, & \text{if } 0 \le x < 1 \\ C_0([0,1), \mathcal{D}) \oplus 0, & \text{if } x = 1 \end{cases}.$$

The extensions

$$0 \to C_0([0,1), \mathcal{D}) \to \mathcal{A} \xrightarrow{(\alpha,d) \mapsto d} \mathcal{D} \to 0$$

and

$$0 \to \{(\alpha, d) \in \mathcal{A} \mid \alpha(x) = 0\} \to \mathcal{A} \xrightarrow{\mathrm{ev}_x} \mathcal{D} \to 0$$

show that indeed all $\mathcal{A}(x)$ are isomorphic to $\mathcal{D}$.

In this example the norm function is

$$N(\alpha, d) = \begin{cases} \|\alpha(x)\|, & \text{if } 0 \le x < 1 \\ \|d\|, & \text{if } x = 1 \end{cases}.$$

N is continuous if and only if $\|d\| = \|\alpha(1)\| = \|\psi(d)\|$ for all $d \in \mathcal{D}$, that is N is continuous if and only if ψ is injective. So we have a continuous field C^*-algebra if and only if ψ is injective.

If ψ is injective, then

$$\mathcal{A} \cong \{\alpha \in C([0,1], \mathcal{D}) \mid \alpha(1) \in \psi(\mathcal{D})\},$$

by an isomorphism that sends (α, d) to α.

We will now try to find out when the field in the second example is trivial, i.e. when $\mathcal{A} \cong C([0,1],\mathcal{D})$.

Lemma 1.8. *Suppose that ψ is injective. Then $\mathcal{A} \cong C([0,1],\mathcal{D})$ if and only if there exists a continuous map $\theta\colon [0,1] \to End(\mathcal{A})$ (where $End(\mathcal{A})$ has the point norm topology) such that $\theta(s) \in Aut(\mathcal{A})$ for all $0 \le s < 1$ and $\theta(1) = \psi$.*

Proof. Suppose θ exists. By identifying $\mathcal{A}$ with $\{\alpha \in C([0,1],\mathcal{D}) \mid \alpha(1) \in \psi(\mathcal{D})\}$ we can define a map $\eta\colon C([0,1],\mathcal{D}) \to \mathcal{A}$ by

$$\eta(\alpha)(s) = \theta(s)(\alpha(s)).$$

This maps into $\mathcal{A}$ since $\eta(\alpha)(1) = \psi(\alpha(1)) \in \psi(\mathcal{D})$. One can check that η is an isomorphism of $C([0,1])$-algebras.

For the other implication, assume that $\eta\colon C([0,1],\mathcal{D}) \to \mathcal{A} \subseteq C([0,1],\mathcal{D})$ is an isomorphism of $C([0,1])$-algebras. This gives us a family of injective homomorphisms $(\eta_s)_{s\in[0,1]}$ from $\mathcal{D}$ to $\mathcal{D}$, such that $s \mapsto \eta_s$ is a continuous map from $[0,1]$ to $\mathrm{End}(\mathcal{D})$, η_s is an automorphism of $\mathcal{D}$ if $0 \le s < 1$ and $\eta_1(\mathcal{D}) = \psi(\mathcal{D})$. By the latter we can define $\gamma \in \mathrm{Aut}(\mathcal{D})$ by $\gamma = \eta_1^{-1}\psi$. We now define $\theta\colon [0,1] \to \mathrm{End}(\mathcal{D})$ by $\theta(s) = \eta_s^{-1}\gamma$. We note that if $0 \le s < 1$ then $\theta(s) \in \mathrm{Aut}(\mathcal{D})$ and that $\theta(1) = \eta_1\eta_1^{-1}\psi = \psi$. □

We can say more if we know more about $\mathcal{D}$.

Corollary 1.9. *Suppose $\mathcal{D}$ is a stable Kirchberg algebra. Then $\mathcal{A} = \{\alpha \in C([0,1],\mathcal{D}) \mid \alpha(1) \in \psi(\mathcal{D})\}$ is trivial if and only if $[\psi] \in KK(\mathcal{D},\mathcal{D})^{-1}$.*

Proof. Suppose $[\psi] \in KK(\mathcal{D},\mathcal{D})^{-1}$. Then by the Kirchberg-Phillips theorem, there exists an automorphism ϕ of $\mathcal{D}$ and a family of unitaries $u_s \in \mathcal{U}(1\mathbb{C} + \mathcal{D}), 0 \le s < 1$ such that $[\psi] = [\phi]$ and

$$\lim_{s\to 1} \|u_s\phi(d)u_s^* - \psi(d)\| = 0,$$

for all $d \in \mathcal{D}$. Now the map $\theta\colon [0,1] \to \mathrm{End}(\mathcal{D})$ given by

$$\theta(s)(d) = \begin{cases} u_s\phi(d)u_s^*, & \text{if } 0 \le s < 1, \\ \psi(d), & \text{if } s = 1 \end{cases} .$$

and the above lemma combines to give the desired conclusion.

The converse is also true, since, by lemma 1.8, we then have that ψ is homotopic to an automorphism. □

Remark 1.10. By the corollary we get: If $\psi_*\colon K_*(\mathcal{D}) \to K_*(\mathcal{D})$ is not bijective then $\mathcal{A}$ is not a trivial field.

As a variation on this example we can fix $x \in (0,1)$ and define

$$\begin{aligned} \mathcal{A} &= \{\alpha \in C([0,1],\mathcal{D}) \mid \alpha(x) \in \psi(\mathcal{D})\} \\ &= \{(\alpha, d) \in C([0,1],\mathcal{D}) \oplus \mathcal{D} \mid \alpha(x) = \psi(\mathcal{D})\}. \end{aligned}$$

The short exact sequence

$$0 \to C_0([0,1] \setminus \{x\},\mathcal{D}) \to \mathcal{A} \underset{\pi_x}{\longrightarrow} \mathcal{D} \to 0$$

where π_x maps (α, d) to d, is split with the split $s\colon \mathcal{D} \to \mathcal{A}$ given by $s(d) \mapsto (\psi(d), d)$ ($\psi(d)$ means a function constantly taking that value). Hence we get a short exact sequence of K_0-groups

$$0 \to K_0(C_0([0,1] \setminus \{x\}), \mathcal{D}) \to K_0(\mathcal{A}) \xrightarrow[(\pi_x)_*]{} K_0(\mathcal{D}) \to 0$$

Since $K_0(C_0([0,1] \setminus \{x\}), \mathcal{D}) = 0$, we get that $(\pi_x)_*$ is an isomorphism. It must have inverse s_*. Consider now some point $y \neq x$. The quotient map $\pi_y\colon \mathcal{A} \to \mathcal{A}(y)$ is given by $\pi_y((\alpha, d)) = \alpha(x)$. Hence we have a map

$$(\pi_y)_*\colon K_0(\mathcal{A}) \to K_0(\mathcal{A}(y)) \cong K_0(\mathcal{D}).$$

We have $(\pi_y)_* s_* \equiv \psi_*\colon \mathcal{K}_0(\mathcal{D}) \to K_0(\mathcal{D})$. Thus ψ_* is not bijective. This implies that $\mathcal{A}$ is not trivial since $K_0(\mathcal{A}) \cong K_0(\mathcal{A}(y))$.

Example 1.11 (Dadarlat & Elliott)**.** *Let $\mathcal{D}$ be a unital Kirchberg algebra such that $K_0(\mathcal{D}) = \mathbb{Z} \oplus \mathbb{Z}$, $[1_{\mathcal{D}}] = (1, 0)$ and $K_1(\mathcal{D}) = 0$. Set*

$$\mathcal{B} = \mathcal{D}^{\otimes\infty} = \varinjlim \left(\mathcal{D} \xrightarrow[d \mapsto d \otimes 1_{\mathcal{D}}]{} \mathcal{D} \otimes \mathcal{D} \to \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D} \to \cdots \right)$$

We will construct a continuous field $\mathcal{A}$ over $[0,1]$ such that $\mathcal{A}(x) \cong \mathcal{B}$ for all $x \in [0,1]$ and such that for all closed intervals $I = [a, b] \subseteq [0,1]$, $a < b$,

$$\mathcal{A}(I) \ncong C(I, \mathcal{B}).$$

Thus $\mathcal{A}$ has all fibers isomorphic but is not locally trivial at any point.

Let ψ be an endomorphism of $\mathcal{D}$ such that $K_0(\psi) = \psi_*\colon K_0(\mathcal{D}) \to K_0(\mathcal{D})$ is given by

$$\psi_* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let (x_n) be a dense sequence in $[0,1]$ with $x_i \neq x_j$ if $i \neq j$. Define

$$\mathcal{D}_n = \{\alpha \in C([0,1], \mathcal{D}) \mid \alpha(x_n) \in \psi(\mathcal{D})\}.$$

Then $\mathcal{D}_n(x) \cong \mathcal{D}$ for all $x \in [0,1]$. Now define $\mathcal{A}$ by

$$\mathcal{A} = \otimes_{n=1}^{\infty} \mathcal{D}_n = \varinjlim(\mathcal{D}_1 \otimes \mathcal{D}_2 \otimes \cdots \otimes \mathcal{D}_n),$$

where all tensor products are taken over $C[0,1]$. That is

$$\mathcal{D}_1 \otimes \cdots \otimes \mathcal{D}_n \cong \{\alpha\colon [0,1] \to \mathcal{D}^{\otimes n} \mid \text{ for } 1 \leq i \leq n \, \alpha(x_i) \in E_i\},$$

where

$$E_i = \mathcal{D} \otimes \mathcal{D} \otimes \cdots \mathcal{D} \otimes \psi(\mathcal{D}) \otimes \mathcal{D} \otimes \cdots \otimes \mathcal{D},$$

with the $\psi(\mathcal{D})$ at the i'th place.

For any $I = [a, b] \subseteq [0,1]$ there exists an $x \notin \{x_1, x_2, \ldots\}$ such that

$$(\pi_x)_*\colon K_0(\mathcal{A}(I)) \to K_0(\mathcal{D}^{\otimes\infty})$$

is not injective. This shows that there can be no I such that $\mathcal{A}(I)$ is trivial, since for such an I all the maps $(\pi_x)_*$ would be isomorphisms.

Theorem 1.12. *Let $\mathcal{D}$ be a stable Kirchberg algebra. Let $\mathcal{A}$ be a stable continuous field of stable Kirchberg algebras over a finite dimensional compact Hausdorff space. Suppose there exists $\sigma \in KK(\mathcal{D}, \mathcal{A})$ such that*

$$[\pi_x]\sigma \in KK(\mathcal{D}, \mathcal{A})^{-1},$$

for all $x \in X$. Then $\mathcal{A} \cong C(X, \mathcal{D})$.

2 Second Lecture

Example 2.1 (Due to Hirshberg, Rørdam & Winter). *Let $f \in \mathcal{M}_2(C(S^2))$ be the Bott projection and let $e = 1_{C(S^2)}$. Denote by p the projection in $\mathcal{M}_3(C(S^2))$ given by*

$$p = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}.$$

For any $x \in S^2$ $p(x)$ is a rank 2 projection. Define

$$\mathcal{A} = \otimes_{n=1}^{\infty} p\mathcal{M}_3(C(S^2))p.$$

This is a continuous field C^*-algebra over $\Pi_{n=1}^{\infty} S^2$ with fibers

$$\otimes_{n=1}^{\infty} \mathcal{M}_2(\mathbb{C}) = UHF(2^{\infty}).$$

So all the fibers have $\mathbb{Z}[\frac{1}{2}]$ as their K_0 group. We will now determine $K_0(\mathcal{A})$. To ease the notation we put $\mathcal{B} = p\mathcal{M}_3(C(S^2))p$. Then $K_0(\mathcal{B}) = K_0(C(S^2))$. Consider the map from $\mathbb{C} \oplus \mathbb{C}$ to $\mathcal{B}$ that sends $(0,1)$ to e and $(0,1)$ to f. It is a unital $*$-homomorphism and it induces a bijection on K_0 and K_1. Hence it is a KK-equivalence. So we get a KK-equivalence

$$\otimes_{n=1}^{\infty} (\mathbb{C} \oplus \mathbb{C}) \to \otimes_{n=1}^{\infty} \mathcal{B} = \mathcal{A},$$

which sends $[1]$ to $[1]$. Letting K denote the set $\Pi_{n=1}^{\infty}\{0,1\}$ (Cantor set) we then get a unital $*$-homomorphism from $C(K)$ to $\mathcal{A}$ that induces a KK-equivalence mapping the class of the unit of $\mathcal{A}$ to the class of the function constantly taking the value 1. Hence

$$K_0(\mathcal{A}) \cong K_0(C(K)) = C(K, \mathbb{Z}).$$

We now consider the C^*-algebra $\mathcal{A} \otimes \mathcal{O}_3$ ($\mathcal{O}_3$ is the Cuntz-algebra with $K_0(\mathcal{O}_3) = \mathbb{Z}/2\mathbb{Z}$ and $K_1(\mathcal{O}_3) = 0$). We have that

$$K_0(\mathcal{A} \otimes \mathcal{O}_3) = C(K, \mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} = C(K, \mathbb{Z}/2\mathbb{Z}).$$

If we let $x \in \Pi_{n=1}^{\infty} S^2$ be given, then we can calculate the fiber at x as

$$(\mathcal{A} \otimes \mathcal{O}_3)(x) \cong \mathcal{A}(x) \otimes \mathcal{O}_3 \cong UHF(2^{\infty}) \otimes \mathcal{O}_3.$$

So all the fibers are Kirchberg algebras, and we can compute their K-theory as

$$K_0(UHF(2^{\infty}) \otimes \mathcal{O}_3) = \mathbb{Z}\left[\frac{1}{2}\right] \otimes \mathbb{Z}/2\mathbb{Z} = 0,$$

and

$$K_1(UHF(2^{\infty}) \otimes \mathcal{O}_3) = 0.$$

Hence all the fibers are $\mathcal{O}_2$. However $\mathcal{A} \otimes \mathcal{O}_3$ is not a trivial continuous field C^*-algebra as it has $K_0(\mathcal{A} \otimes \mathcal{O}_3) \cong C(K, \mathbb{Z}/2\mathbb{Z}) \neq 0$.

The space used in the example to get at non-trivial field with all fibers isomorphic to $\mathcal{O}_2$ were quite large. The following theorems tells us that small spaces can not exhibit that form of behavior.

Theorem 2.2. *Let $\mathcal{A}$ be a separable unital continuous field over a compact Hausdorff space X of finite covering dimension. If $\mathcal{A}(x) \cong \mathcal{O}_2$ for all $x \in X$ then $\mathcal{A} \cong C(X, \mathcal{O}_2)$.*

Theorem 2.3 (Dadarlat-Mayer)**.** *Suppose $\mathcal{A}$ is a separable continuous field of nuclear C^*-algebras over a compact Hausdorff space X. Suppose that for all ideals $\mathcal{J}$ in $\mathcal{A}$ we have $KK(\mathcal{J}, \mathcal{J}) = 0$. Then*

$$\mathcal{A} \sim_{KK_X} C(X, \mathcal{O}_2).$$

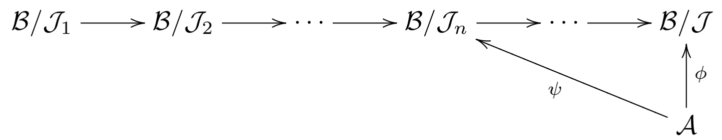
If $\mathcal{A}(x)$ is a Kirchberg algebra for all $x \in X$ then $\mathcal{A} \otimes \mathcal{O}_\infty \otimes \mathcal{K} \cong \mathcal{A} \otimes \mathcal{O}_2 \otimes \mathcal{K}$.

If we have a field of nuclear C^*-algebras then the continuous field C^*-algebra will be nuclear.

The rest of the lecture was devoted to giving a explanation of why the first theorem is true.

The key point is that $\mathcal{O}_2$ is semiprojective, which means that it has good perturbation properties.

Definition 2.4. *A separable C^*-algebra $\mathcal{A}$ is semiprojective, if for any C^*-algebra $\mathcal{B}$ and any increasing chain of ideals $\mathcal{J}_1 \subseteq \mathcal{J}_2 \subseteq \cdots$ in $\mathcal{B}$ and any $*$-homomorphism $\phi\colon \mathcal{A} \to \mathcal{B}/\mathcal{J}$, where $\mathcal{J} = \overline{\cup_n \mathcal{J}_n}$, there exists an $n \in \mathbb{N}$ and a $*$-homomorphism $\psi\colon \mathcal{A} \to \mathcal{B}/\mathcal{J}_n$ such that the following diagram commutes*



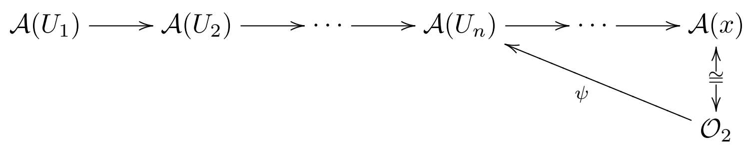
The definition is equivalent to requiring that for all $\mathcal{B}$ and $\mathcal{J}_n$ as above, the canonical map from $\varinjlim \hom(\mathcal{A}, \mathcal{B}/\mathcal{J}_n)$ to $\hom(\mathcal{A}, \mathcal{B}/\mathcal{J})$ is surjective. We say that an algebra is weakly semiprojective if the the map has dense image in the point norm topology.

An algebra is said to be KK-semiprojective if the canonical map from the inductive limit $\varinjlim KK(\mathcal{A}, \mathcal{B}/\mathcal{J}_n)$ to $KK(\mathcal{A}, \mathcal{B}/\mathcal{J})$ is surjective. It turns out that this is equivalent to saying that the map is a bijection.

Example 2.5 (Examples of semiprojective C^*-algebras)**.** *If $\mathcal{A}$ is a Kirchberg algebra satisfying the UCT, then $\mathcal{A}$ is weakly semiprojective if and only if $K_*(\mathcal{A})$ is finitely generated.*

If $K_1(\mathcal{A})$ further is torsion free, then $\mathcal{A}$ is semiprojective. It is an open question whether we need $K_1(\mathcal{A})$ to be torsion free.

From now on we will focus on a separable unital continuous field with fibers $\mathcal{O}_2$ over $[0,1]$. Fix $x \in [0,1]$ and define $U_n = [x - 1/n; x + 1/n] \cap [0,1]$. Then $\varinjlim \mathcal{A}(U_n) = \mathcal{A}(x)$ (non-trivial fact). By the semiprojectivity of $\mathcal{O}_2$ we can get an n and a unital $*$-homomorphism ψ such that



commutes.

Moreover, given any finite set $\mathcal{F} \subseteq \mathcal{A}$ and any $\varepsilon > 0$ we can find a finite set $\mathcal{H} \subseteq \mathcal{O}_2$ such that the isomorphism from $\mathcal{O}_2$ to $\mathcal{A}(x)$ maps $\mathcal{H}$ to $\pi_x(\mathcal{F})$ and such that $\psi(\mathcal{H}) \supseteq_\varepsilon \pi_{U_n}(\mathcal{F})$. We get the latter since $\varinjlim \mathcal{A}(U_n) = \mathcal{A}(x)$.

We can extend ψ to $\tilde{\psi}\colon C(U_n) \otimes \mathcal{O}_2 \to \mathcal{A}(U_n)$ by $C(U_n)$ linearity, and we will have $\pi_{U_n}(\mathcal{F}) \subseteq_\varepsilon \tilde{\psi}(\mathcal{O}_2)$.

Doing this for other x we get closed sets U_k covering all of $[0,1]$ and maps from $C(U_k) \otimes \mathcal{O}_2$ into $\mathcal{A}(U_k)$ as above. The trick is the to paste them together. For that we use elementary fields.

Suppose we have 3 unital C^*-algebras $E_1, \mathcal{D}, E_2$, and $*$-homomorphisms $\gamma_1\colon \mathcal{D} \to E_1$ and $\gamma_2\colon \mathcal{D} \to E_2$. Then the algebra

$$\begin{aligned}\mathcal{A} = \{(\alpha,\beta,\gamma) \quad | \quad & \alpha \in C([0,1],E_1), \beta \in C([1,2],\mathcal{D}), \gamma \in C([2,3],E_2) \text{ such that} \\ & \alpha(1) = \gamma_1(\beta(1)), \gamma_2(\beta(2)) = \gamma(2)\}\end{aligned}$$

is built from elementary fields.

In our case we then have that for all finite sets $\mathcal{F} \subseteq \mathcal{A}$ and all $\varepsilon > 0$ there exists an elementary field $E \subseteq \mathcal{A}$ such that $E(x) \cong \mathcal{O}_2$. The gluing morphisms $\gamma\colon \mathcal{O}_2 \to \mathcal{O}_2$ are KK-equivalent. We have seen that $E \cong C([0,1], \mathcal{O}_2)$. The idea is then to write $\mathcal{A}$ as an inductive limit of elementary fields, and show that things extend nicely.

3 Third Lecture

The main theme of this lecture was the structure of continuous fields, restricted to the case where the fibers are Kirchberg algebras satisfying the UCT.

Definition 3.1. *A sequence of sub-C^*-algebras $(\mathcal{D}_n)$ of a C^*-algebra $\mathcal{D}$ is called exhaustive if for all finite subsets $\mathcal{F} \subseteq \mathcal{D}$ and all $\varepsilon > 0$ there exists n such that $\mathcal{F} \subseteq_\varepsilon D_n$.*

Note that we do not assume $\mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \cdots$. If we did, then $(\mathcal{D}_n)$ would be exhaustive if and only if $\overline{\cup_n \mathcal{D}_n} = \mathcal{D}$.

We will now define n-pullbacks. They are continuous fields obtained by gluing $n+1$ locally trivial fields together.

Definition 3.2. *Suppose we have*

$$X = Y_0 \cup Y_1 \cup \cdots \cup Y_n,$$

where each Y_i is closed. Suppose also that we have locally trivial $C(Y_i)$ algebras E_i and fiberwise injective $C(Y_i \cap Y_j)$ maps $\gamma_{ij} \colon E_i|_{Y_i \cap Y_j} \to E_j|_{Y_i \cap Y_j}$ such that

$$(\gamma_{jk})_x \circ (\gamma_{ij})_x = (\gamma_{ik})_x,$$

for all $x \in Y_i \cap Y_j \cap Y_k$, $i \leq j \leq k$.

Then we define the n-pullback E as

$$E = \{(e_0, \ldots, e_n) \in E_0 \oplus \cdots \oplus E_n \mid e_j(x) = (\gamma_{ij})_x(e_i(x)) \text{ for all } x \in Y_i \cap Y_j\}.$$

Theorem 3.3. *Let $\mathcal{A}$ be a separable nuclear continuous $C(X)$-algebra over a compact metrizable space X of finite covering dimension, $\dim(X) = n$.*

Suppose each fiber $\mathcal{A}(x)$ is a Kirchberg algebra which is KK-equivalent to a commutative C^*-algebra (i.e. satisfies the UCT). Then $\mathcal{A}$ admits an exhaustive sequence $(\mathcal{A}_m)$, where each $\mathcal{A}_m$ is an n-pullback. Moreover, if $K_1(\mathcal{A}(x))$ is torsion free for all x, then one can get $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$. Hence, $\mathcal{A} = \overline{\cup_m \mathcal{A}_m}$.

Outline. Fix a fiber $\mathcal{A}(x)$. Write $\mathcal{A}(x) = \varinjlim \mathcal{D}_k$, where the $\mathcal{D}_k$ are Kirchberg algebras with finitely generated K-theory. By choice of the $\mathcal{D}_k$ they are weakly semiprojective. So for a given k we can find a closed neighborhood V of $\mathcal{A}$ and an approximate lifting $\rho \colon \mathcal{D}_k \to \mathcal{A}(V)$ such that the diagram

$$\begin{array}{ccc} & & \mathcal{A}(V) \\ & \overset{\rho}{\nearrow} & \downarrow \\ \mathcal{D}_k & \longrightarrow & \mathcal{A}(x) \end{array}$$

commutes. Using these liftings in a clever way, we can get n-pullbacks.

If K_1 is torsion free then we can choose the $\mathcal{D}_k$ such that they also have torsion free K_1. Then they will be semiprojective, and the liftings will be exact. □

You do not need Kirchberg algebras. One only needs that every fiber is a limit of direct sums of simple semiprojective algebras, e.g. AF-algebras.

What is KK_X? $\mathcal{A}$ and $\mathcal{B}$ two $C(X)$-algebras, X a compact Hausdorff space, then if ϕ is a $C(X)$-linear $*$-homomorphism it will induce a class $[\phi] \in KK_X(\mathcal{A}, \mathcal{B})$. KK_X is a sort of fiberwise KK-theory. It consists of Fredholm-Kasparov bimodules ${}_{\mathcal{A}}E_{\mathcal{B}}$ subject to the condition $(fa)\xi b = (a)\xi(fb)$ for all $a \in \mathcal{A}, b \in \mathcal{B}, \xi \in E, f \in C(X)$.

Observe that while

$$KK(C_0((0;1]), C_0((0;1])) = 0,$$

we have

$$KK_{[0,1]}(C_0((0;1]), C_0((0;1])) = \mathbb{Z}[id],$$

since one cannot contract fiberwise.

We record the following fact. Suppose $\mathcal{A}$, $(\mathcal{B}_n)_{n=1}^{\infty}$ are nuclear and separable continuous $C(X)$-algebras with injections

$$\mathcal{B}_1 \overset{\gamma_1}{\hookrightarrow} \mathcal{B}_2 \overset{\gamma_2}{\hookrightarrow} \cdots$$

and $\mathcal{B} = \varinjlim \mathcal{B}_n$. Then we have the following short exact sequence

$$0 \to \varprojlim{}^1 KK_X^1(\mathcal{B}_i) \to KK_X(\mathcal{B}, \mathcal{A}) \to \varprojlim KK_X(\mathcal{B}_i, \mathcal{A}) \to 0$$

Recall that if

$$G_1 \overset{\lambda_1}{\leftarrow} G_2 \overset{\lambda_2}{\leftarrow} \cdots \overset{\lambda_i}{\leftarrow} G_{i+1} \overset{\lambda_{i+1}}{\leftarrow} \cdots$$

and we define a map $\mathrm{id} - S \colon \Pi_{i=1}^{\infty} G_i \to \Pi_{i=1}^{\infty} G_i$ by

$$(g_1, g_2, \ldots) \mapsto (g_1 - \lambda_1(g_2), g_2 - \lambda_2(g_3), \ldots),$$

then

$$\ker(\mathrm{id} - S) = \varprojlim (G_i, \lambda_i)$$

and

$$\mathrm{coker}(\mathrm{id} - S) = \varprojlim{}^1 (G_i, \lambda_i).$$

Proposition 3.4. *Let $\mathcal{A}$ be a separable and nuclear continuous field over a compact metriziable space X. Then there exists $\mathcal{A}^{\#}$ a separable nuclear continuous field over X with $\mathcal{A}^{\#}(x)$ Kirchberg for all $x \in X$ and $C(X)$-linear map $\phi \colon \mathcal{A} \hookrightarrow \mathcal{A}^{\#}$ such that*

$$[\phi] \in KK_X(\mathcal{A}, \mathcal{A}^{\#})^{-1}.$$

Theorem 3.5. *Let $\mathcal{A}, \mathcal{B}$ be separable nuclear continuous $C(X)$-algebras over a finite dimensional compact metrizable space X. Let $\sigma \in KK_X(\mathcal{A}, \mathcal{B})$ (e.g. $\sigma = [\phi]$ where ϕ is $C(X)$ linear map from $\mathcal{A}$ to $\mathcal{B}$). Suppose that for all $x \in X$ we have $\sigma_x \in KK(\mathcal{A}(x), \mathcal{B}(x))^{-1}$, then $\sigma \in KK_X(\mathcal{A}, \mathcal{B})$.*

Proof. Consider the mapping cone

$$C_\phi = \{(f, a) \mid f \in C_0((0;1], \mathcal{B}), a \in \mathcal{A}, f(1) = \phi(a)\}.$$

It is a continuous $C(X)$-algebra with fibers $(C_\phi)_x = C_{\phi_x}$. We have a Puppe sequence

$$KK_X(C, C_\phi) \to KK_X(C, \mathcal{A}) \to KK_X(C, \mathcal{B}) \to KK_X^1(C, C_\phi)$$

for all nuclear and separable continuous $C(X)$-algebras C. We have a similar sequence for each ϕ_x:

$$KK(C(x), C_{\phi_x}) \to KK(C(x), \mathcal{A}(x)) \overset{(\phi_x)_*}{\to} KK(C(x), \mathcal{B}(x))$$

By assumption $(\phi_x)_*$ is bijective, so $KK(C(x), C_{\phi_x}) = 0$. Hence $C_{\phi_x} \sim_{KK} \mathcal{O}_2 \otimes \mathcal{K}$. Now $C_\phi \sim_{KK} C_\phi^\#$. The latter will be a field over $\mathcal{O}_2 \otimes \mathcal{K}$. Therefore we have

$$C_{\phi_x} \sim_{KK} (C_\phi^\#)_x \cong \mathcal{O}_2 \otimes \mathcal{K}.$$

By a trivialisation result, we get $C_\phi^\# \cong C(X) \otimes \mathcal{O}_2 \otimes \mathcal{K}$. □

Corollary 3.6. *Let $\mathcal{B}$ be as in the previous theorem. Suppose $\mathcal{D}$ is a separable nuclear C^*-algebra with an element $\sigma \in KK(\mathcal{D}, \mathcal{B})$ such that $\sigma_x \in KK(\mathcal{D}, \mathcal{B}(x))^{-1}$ for all x, then $C(X) \otimes \mathcal{D} \sim_{KK_X} \mathcal{B}$.*

Proof.

$$KK_X(C(X) \otimes \mathcal{D}, \mathcal{B}) \cong KK(\mathcal{D}, \mathcal{B}).$$

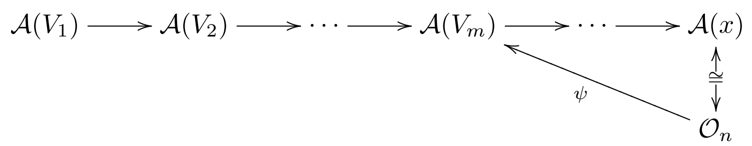
□

Corollary 3.7. *Let $\mathcal{A}$ be a unital separable continuous field over a finite dimensional compact metrizable space X. Suppose $\mathcal{A}(x) \cong \mathcal{O}_n$ for all x (n fixed, $2 \leq n \leq \infty$). Then*

1. *If $n = 2$ or $n = \infty$ then $\mathcal{A} \cong C(X) \otimes \mathcal{O}_n$.*
2. *In all cases $\mathcal{A}$ is locally trivial. Moreover $\mathcal{A} \cong C(X) \otimes \mathcal{O}_n$ if and only if $(n-1)[1_\mathcal{A}] = 0$ in $K_0(\mathcal{A})$.*

"Proof". Locally trivial: Fix $x_0 \in X$. It suffices to find V a closed neighborhood of x_0 such that $C(V) \otimes \mathcal{O}_n \sim^\sigma_{KK_X} \mathcal{A}(V)$ and $\sigma_x[1] = [1]$. For that it suffices to find a closed neighborhood V and a unital $*$-homomorphism $\phi\colon \mathcal{O}_n \to \mathcal{A}(V)$. Indeed if that is the case, then $[\phi] \in KK(\mathcal{O}_n, \mathcal{A}(V))$, and if $x \in X$ then $\phi_x \in KK(\mathcal{O}_n, \mathcal{A}(x))^{-1}$ since the map $K_0(\mathcal{O}_n) \overset{(\phi_x)_*}{\to} \mathcal{K}_0(\mathcal{A}(x)) \cong \mathbb{Z}/(n-1)\mathbb{Z}$ is bijective (it is unital). As there is no K_1 ϕ is a KK-equivalence.

To get such a V, we consider a decreasing set of neighborhoods $V_1 \supseteq V_2 \supseteq \cdots$ such that $\cap_m V_m = \{x\}$. Then, by the semiprojectivity of $\mathcal{O}_n$, we get an n and a unital $*$-homomorphism ψ such that the following diagram commutes



To get global triviality we need to find unital $\phi\colon \mathcal{O}_n \to \mathcal{A}$. For that it is enough to find a map $K_0(\mathcal{O}_n) \to \mathcal{K}_0(\mathcal{A})$ mapping [1] to [1] and then lift it up to the level of algebras. □