Towards the classification of outer actions of finite groups on Kirchberg algebras

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Herzliche Glückwünsche zum Geburtstag!

Comments for the version for the conference website:

- This version does not include the pictures. (The files are large.)
- One misprint has been corrected. Please let me know if you find any others!
- The talk actually only went through page 33.

Introduction

Possible subtitle: Initiating the Elliott classification program for group actions.

This is work in progress. The intended main theorem has not yet been proved.

Caution: Some of the results stated have not been carefully checked. Don't quote them yet!

Rough outline

- Goal, results, and background.
  - The hoped for main theorem.
  - Some ingredients of the statement: Equivariant K-theory and E-theory.
  - Previous results.
  - Examples.
- Outline of the intended proof.
  - Step 1: Equivariant Kirchberg absorption.
  - Step 2: Homotopy implies asymptotic unitary equivalence.
  - Step 3: Equivariant asymptotic morphisms are equivariantly asymptotically unitarily equivalent to an equivariant homomorphisms.
  - Step 4: Equivariant unsuspended E-theory.
  - Step 5: KK^G-equivalence implies conjugacy.
  - Step 6: Isomorphism of the invariants implies conjugacy.
- Equivariant semiprojectivity.
  - Definition of equivariant semiprojectivity.
  - Equivariant semiprojectivity: Results and spplications.
  - Equivariant semiprojectivity for finite dimensional C*-algebras.

The goal

The intended main theorem is as follows. (Some items are described afterwards.)

Conjecture

Let G be a cyclic group of prime order. Let A and B be Kirchberg algebras (purely infinite simple separable nuclear C*-algebras) which are unital and satisfy the Universal Coefficient Theorem. Let $\alpha \colon G \to \mathrm{Aut}(A)$ and $\beta \colon G \to \mathrm{Aut}(B)$ be pointwise outer actions of G which belong to a suitable bootstrap class (defined by Manuel Köhler). Suppose the extended K-theory of α (as defined by Köhler) is isomorphic to that of β. Then α and β are conjugate.

Conjugacy is isomorphism of dynamical systems: there exists an isomorphism $\varphi \colon A \to B$ such that $\beta_g = \varphi \circ \alpha_g \circ \varphi^{-1}$ for all $g \in G$.

The action $\alpha \colon G \to \mathrm{Aut}(A)$ is called pointwise outer if for every $g \in G \setminus \{1\}$, the automorphism α_g is not inner. (Without this, the theorem is not true.)

Extended K-theory

Conjecture

Let G be a cyclic group of prime order. Let $\alpha \colon G \to \mathrm{Aut}(A)$ and $\beta \colon G \to \mathrm{Aut}(B)$ be pointwise outer actions of G which belong to a suitable bootstrap class. Suppose the extended K-theory of α is isomorphic to that of β. Then α and β are conjugate.

The extended K-theory $EK_*^G(A)$ consists of three groups:

- $K_*(A)$.
- $K_*^G(A)$.
- With M being the mapping cone of the unital embedding of $\mathbb{C}$ in $C(G)$, the group $KK_G^*(M, A)$.

(See the next slides for more on equivariant K-theory and KK-theory.) $EK_*^G(A)$ has additional structure, given by various operations, which must be preserved by isomorphisms.

If the actions are in Köhler's bootstrap class, then the algebras automatically satisfy the UCT.

A one page summary of equivariant K-theory

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a compact group G on a unital C*-algebra A. The ordinary K_0-group of A is made from finitely generated projective modules over A. (If we use right modules, the projection $p \in M_n(A)$ corresponds to the module pA^n.)

In a similar way, the equivariant K_0-group of A, written $K_0^G(A)$, is made from finitely generated projective modules over A which carry a compatible action of G. (It is a bit more complicated than just G-invariant projections in $M_\infty(A)$.)

One generalizes to nonunital algebras and to $K_1^G(A)$ in the usual way, by unitizing and suspending. (The action of G in the suspension direction is trivial.)

The Green-Julg Theorem tells us that $K_*^G(A) \cong K_*(C^*(G, A, \alpha))$.

$K_*^G(A)$ is a module over the representation ring $R(G) = K_0^G(\mathbb{C})$, the Grothendieck group made from finite dimensional representations of G. (Tensor the A-module with the representation.)

Equivariant KK-theory and E-theory

There are also equivariant versions of KK-theory and E-theory, denoted $KK_G^*(A, B)$ and $E_*^G(A, B)$. We will use E-theory. The convenient definition is in terms of asymptotic morphisms.

Definition

Let A and B be separable C*-algebras. An *asymptotic morphism* from A to B is a family $\varphi = (\varphi_t)_{t \in [0,\infty)}$ of functions $\varphi_t \colon A \to B$ such that:

- $t \mapsto \varphi_t(a)$ is continuous for all $a \in A$.
- For all $a, b \in A$ and $\lambda \in \mathbb{C}$, as $t \to \infty$ the quantities
$$\varphi_t(a+b) - \varphi_t(a) - \varphi_t(b), \qquad \varphi_t(\lambda a) - \lambda\varphi_t(a),$$
$$\varphi_t(ab) - \varphi_t(a)\varphi_t(b), \qquad \text{and} \qquad \varphi_t(a^*) - \varphi_t(a)^*$$
all converge to zero.

Equivariant KK-theory and E-theory (continued)

An asymptotic morphism from A to B is a family of functions $\varphi_t \colon A \to B$ such that:

- $t \mapsto \varphi_t(a)$ is continuous for all $a \in A$.
- For all $a, b \in A$ and $\lambda \in \mathbb{C}$, as $t \to \infty$ the quantities

$$\varphi_t(a+b) - \varphi_t(a) - \varphi_t(b), \qquad \varphi_t(\lambda a) - \lambda \varphi_t(a),$$
$$\varphi_t(ab) - \varphi_t(a)\varphi_t(b), \qquad \text{and} \qquad \varphi_t(a^*) - \varphi_t(a)^*$$

all converge to zero.

In other words, $(\varphi_t)_{t \in [0,\infty)}$ is asymptotically a homomorphism.

If G is compact second countable and $\alpha \colon G \to \mathrm{Aut}(A)$ and $\beta \colon G \to \mathrm{Aut}(B)$ are actions of G on A and B, for an equivariant asymptotic morphism we ask in addition that

$$\beta_g \circ \varphi_t(a) - \varphi_t \circ \alpha_g(a) \to 0$$

for all $a \in A$ and $g \in G$.

The set of homotopy classes of equivariant asymptotic morphism from A to B is written $[[A, B]]_G$.

Equivariant KK-theory and E-theory (continued)

For an equivariant asymptotic morphism we ask in addition that

$$\beta_g \circ \varphi_t(a) - \varphi_t \circ \alpha_g(a) \to 0$$

for all $a \in A$ and $g \in G$.

$[[A, B]]_G$ is the set of homotopy classes of equivariant asymptotic morphisms from A to B. Then define

$$E^G(A, B) = [[SA,\ K \otimes SB]]_G$$

with suitable actions of G on the suspension part and on K: trivial in the suspension direction, on K, conjugation by the direct sum of infinitely many copies of the regular representation of G.

We take $E_0^G(A, B) = E^G(A, B)$ and $E_1^G(A, B) = E^G(SA, B)$. One has Bott periodicity, so take $E_*^G(A, B) = E_0^G(A, B) \oplus E_1^G(A, B)$.

Equivariant KK-theory and E-theory (continued)

$[[A, B]]_G$ is the set of homotopy classes of equivariant asymptotic morphisms from A to B.

$E^G(A, B) = [[SA,\ K \otimes SB]]_G$.

With some care, one can compose homotopy classes of equivariant asymptotic morphisms from A to B and from B to C. This gives a product $E^G(A, B) \times E^G(B, C) \to E^G(A, C)$. In particular, $E_0^G(A, A)$ is a ring, and one can make $E_*^G(A, A)$ into a ring.

$E_0^G(\mathbb{C}, \mathbb{C}) = R(G)$, the representation ring mentioned above.

For G cyclic of prime order, one can view $EK_*^G(A)$ as $E_*^G(\mathbb{C} \oplus C(G) \oplus M,\ A)$, which is a module over

$$R_G = E_*^G\big(\mathbb{C} \oplus C(G) \oplus M,\ \mathbb{C} \oplus C(G) \oplus M\big).$$

Note that $R(G)$ is contained in R_G, but R_G is much bigger.

The Universal Coefficient Theorem

$EK_*^G(A) = E^G(\mathbb{C} \oplus C(G) \oplus M,\ A)$, which is a module over $R_G = E_*^G(\mathbb{C} \oplus C(G) \oplus M,\ \mathbb{C} \oplus C(G) \oplus M)$.

Without the group: for A in a suitable class (the bootstrap class), there is a natural short exact sequence

$$0 \longrightarrow \mathrm{Ext}_1^{\mathbb{Z}}(K_*(A), K_*(B)) \longrightarrow E_*(A, B) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K_*(A), K_*(B)) \longrightarrow 0.$$

(Rosenberg-Schochet.) The first map has degree 1.

Köhler proved a similar result for C*-algebras with an action of a cyclic group G of prime order, computing $E_*^G(A, B)$, using $EK_*^G(-)$ in place of $K_*(-)$, and using extensions over R_G instead of over $\mathbb{Z}$. For actions of G on A in a suitable bootstrap class, and arbitrary actions on B:

$$\begin{aligned} 0 \longrightarrow \mathrm{Ext}_1^{R_G}\big(EK_*^G(A),\ EK_*^G(B)\big) &\longrightarrow E_*^G(A, B) \\ &\longrightarrow \mathrm{Hom}_{R_G}\big(EK_*^G(A),\ EK_*^G(B)\big) \longrightarrow 0. \end{aligned}$$

Previous result: Classification of Rokhlin actions

The following is Theorem 4.2 of M. Izumi, *Finite group actions on C*-algebras with the Rohlin property. II*, Adv. Math. **184**(2004), 119–160.

Theorem

Let A be a unital UCT Kirchberg algebra, and let G be a finite group. Let $\alpha, \beta \colon G \to \mathrm{Aut}(A)$ be actions with the Rokhlin property. Then α is conjugate to β if and only if the actions of G they induce on $K_*(A)$ are equal.

We omit the definition of the Rokhlin property.

Interpreted as a theorem about conjugacy of dynamical systems, the invariant involved includes A, equivalently, it includes $K_*(A)$ and $[1_A] \in K_0(A)$.

There are severe restrictions on the possible actions of G on $K_*(A)$.

Previous result: Classification of actions of $\mathbb{Z}_2$ on $\mathcal{O}_2$

The following is essentially a restatement of part of Theorem 4.8 of M. Izumi, *Finite group actions on C*-algebras with the Rohlin property. I*, Duke Math. J. **122**(2004), 233–280.

$\mathcal{O}_2$ is the Cuntz algebra.

Theorem

Let $\alpha, \beta \colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathcal{O}_2)$ be actions which are pointwise outer but strongly approximately inner. Then α is conjugate to β if and only if $K_*^{G,\alpha}(\mathcal{O}_2) \cong K_*^{G,\beta}(\mathcal{O}_2)$ via an isomorphism which sends [1] to [1].

$K_*^{G,\alpha}(A)$ is the equivariant K-theory of A with respect to the group action α.

$K_*(\mathcal{O}_2)$ isn't needed in the invariant, since it is zero.

We omit the definition of strong approximate innerness.

Previous result: Quasifree actions on $\mathcal{O}_\infty$

Quasifree actions are described below.

Theorem

Let G be a finite group. Then any two quasifree actions of G on $\mathcal{O}_\infty$ coming from injective representations of G are conjugate.

This is in a recent preprint of Goldstein and Izumi, written after I started this project but based on work done earlier.

We take the generators of the Cuntz algebra $\mathcal{O}_d$ to be $s_1, s_2, \ldots, s_d$, satisfying

$$s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1 \quad \text{and} \quad s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1.$$

(In the second condition, if $d = \infty$, just the $s_j s_j^*$ are orthogonal.)

Examples: Quasifree actions on Cuntz algebras

Relations: $s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1$ and $s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1$.

Let $\rho \colon G \to L(\mathbb{C}^d)$ be a unitary representation of G. Write

$$\rho(g) = \begin{pmatrix} \rho_{1,1}(g) & \cdots & \rho_{1,d}(g) \\ \vdots & \ddots & \vdots \\ \rho_{d,1}(g) & \cdots & \rho_{d,d}(g) \end{pmatrix}$$

for $g \in G$. Then there exists a unique action $\beta^\rho \colon G \to \mathrm{Aut}(\mathcal{O}_d)$ such that

$$\beta_g^\rho(s_k) = \sum_{j=1}^{d} \rho_{j,k}(g) s_j$$

for $j = 1, 2, \ldots, d$. (This can be checked by a computation.)

Examples:

- For $G = \mathbb{Z}_n$, choose n-th roots of unity $\zeta_1, \zeta_2, \ldots, \zeta_d$ and let a generator of the group multiply s_j by ζ_j.
- Take $d = \mathrm{card}(G)$, and label the generators s_g for $g \in G$. Then define $\beta^G \colon G \to \mathrm{Aut}(\mathcal{O}_d)$ by $\beta_g^G(s_h) = s_{gh}$ for $g, h \in G$.

Examples: Quasifree actions (continued)

Relations: $s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1$ and $s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1$.

$\rho\colon G \to L(\mathbb{C}^d)$ is a unitary representation.

$$\rho(g) = \begin{pmatrix} \rho_{1,1}(g) & \cdots & \rho_{1,d}(g) \\ \vdots & \ddots & \vdots \\ \rho_{d,1}(g) & \cdots & \rho_{d,d}(g) \end{pmatrix} \quad \text{and} \quad \beta_g^{\rho}(s_k) = \sum_{j=1}^{d} \rho_{j,k}(g) s_j.$$

An analogous construction gives actions on $\mathcal{O}_\infty$.

Example: Label the generators of $\mathcal{O}_\infty$ as $s_{g,j}$ for $g \in G$ and $j \in \mathbb{Z}_{>0}$. Define $\iota\colon G \to \mathrm{Aut}(\mathcal{O}_\infty)$ by $\iota_g(s_{h,j}) = s_{gh,j}$ for $g \in G$ and $j \in \mathbb{Z}_{>0}$.

This is the quasifree action coming from the direct sum of infinitely many copies of the regular representation. One can compute its equivariant K-theory, getting $K_0^G(\mathcal{O}_\infty) \cong R(G)$ (recall that this is the representation ring of G), with $[1] \mapsto 1$, and $K_1^G(\mathcal{O}_\infty) = 0$.

Example: The tensor flip

Define $\varphi\colon \mathcal{O}_\infty \otimes \mathcal{O}_\infty \to \mathcal{O}_\infty \otimes \mathcal{O}_\infty$ by $\varphi(a \otimes b) = b \otimes a$ for $a, b \in \mathcal{O}_\infty$. Using $\mathcal{O}_\infty \otimes \mathcal{O}_\infty \cong \mathcal{O}_\infty$, this defines an action of $\mathbb{Z}_2$ on $\mathcal{O}_\infty$, the tensor flip.

Is this action conjugate to the action ι above? (It is equivariantly strongly selfabsorbing. I don't yet know the equivariant K-theory, but I suspect it is $R(G)$.)

More generally, subgroups of the symmetric group S_n act on $(\mathcal{O}_\infty)^{\otimes n}$.

Methods

Recall the classification conjecture:

Conjecture

Let G be a cyclic group of prime order. Let A and B be unital UCT Kirchberg algebras. Let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be pointwise outer actions of G which are in Köhler's bootstrap class. Suppose $EK_*^G(A) \cong EK_*(B)$. Then α and β are conjugate.

Three basic methods go into the work:

- Reduction to known results in the case in which there is no group.
- Imitating known arguments from the case in which there is no group.
- New arguments.

We outline the intended method (which follows my proof of the nonequivariant case), and say what has been done. Unless otherwise specified, G is an arbitrary finite group.

Step 1: Equivariant Kirchberg absorption

The nonequivariant versions:

Theorem

Let A be a simple separable unital nuclear C*-algebra. Then $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$.

Theorem

Let A be a Kirchberg algebra. Then $\mathcal{O}_\infty \otimes A \cong A$.

(In fact, there is an isomorphism from A to $\mathcal{O}_\infty \otimes A$ which is asymptotically unitarily equivalent to the map $a \mapsto 1 \otimes a$.)

Also, trivially:

Theorem

Let A be a purely infinite simple C*-algebra, and let $p \in A$ be a nonzero projection such that $[p] = 0$ in $K_0(A)$. Then there exists a unital homomorphism $\mathcal{O}_2 \to pAp$.

Step 1 (continued): Equivariant Kirchberg absorption

1. $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$ for A simple separable unital nuclear.
2. $\mathcal{O}_\infty \otimes A \cong A$ for a Kirchberg algebra A.
3. $\mathcal{O}_2 \to pAp$ unitally when $[p] = 0$ and A is purely infinite simple.

These are true equivariantly with the following choices:

1. In (1), there is a unique (up to conjugacy) action of G on $\mathcal{O}_2$ with the Rokhlin property. (Definition omitted.) Use this action. (Result can then be reduced to the nonequivariant version.)
2. In (2), take the action on A to be pointwise outer and the action on $\mathcal{O}_\infty$ to be the quasifree action from the direct sum of infinitely many copies of the regular representation of G. (Done; also by Goldstein-Izumi.)
3. Take the action on A to be pointwise outer and $[p] = 0$ in $K_0^G(A)$. (Done.)

(2) works for the trivial action on $\mathcal{O}_\infty$ and any action on A. But, with our choice of action on $\mathcal{O}_2$, using the trivial action on $\mathcal{O}_\infty$ spoils (3).

Step 2: Homotopy implies asymptotic unitary equivalence

A is a unital Kirchberg algebra with a pointwise outer action of G. The algebra $\mathcal{O}_\infty$ has the same quasifree action as in Step 1. Asymptotic morphisms are equivariant and go from A to a C*-algebra of the form $K \otimes \mathcal{O}_\infty \otimes D$, with D unital, with any action on D, and they are "full" (not asymptotically in any proper ideal).

Two equivariant asymptotic morphisms $t \mapsto \varphi_t$ and $t \mapsto \psi_t$ are *equivariantly asymptotically unitarily equivalent* if there exists a continuous path $t \to u_t$ of invariant unitaries in (the unitization of) the codomain such that $\lim_{t\to\infty} \| u_t \varphi_t(a) u_t^* - \psi_t(a) \| = 0$ for all a.

The following result has not quite been proved:

Conjecture

Let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be a pointwise outer action on a unital Kirchberg algebra. Let $t \mapsto \varphi_t$ and $t \mapsto \psi_t$ be equivariantly homotopic equivariant asymptotic morphisms as above. Then they are equivariantly asymptotically unitarily equivalent.

Step 2 (continued)

Conjecture

Let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be a pointwise outer action on a unital Kirchberg algebra. Let $t \mapsto \varphi_t$ and $t \mapsto \psi_t$ be equivariantly homotopic equivariant asymptotic morphisms as above. Then they are equivariantly asymptotically unitarily equivalent.

Equivariant Kirchberg absorption and a messy technical argument (not yet completely checked) reduce to the cases $A = \mathcal{O}_2$ and $A = \mathcal{O}_\infty$.

The case $A = \mathcal{O}_2$ follows from the nonequivariant case: Suppose $t \mapsto \varphi_t$ and $t \mapsto \psi_t$ are equivariant and nonequivariantly asymptotically unitarily equivalent, and that the action on the domain has the Rokhlin property (true here). Then $t \mapsto \varphi_t$ and $t \mapsto \psi_t$ are equivariantly asymptotically unitarily equivalent. (This has been done.)

Step 2 (continued)

We still want equivariant asymptotic unitary equivalence for two homotopic equivariant asymptotic morphisms $t \mapsto \varphi_t$ and $t \mapsto \psi_t$ from $\mathcal{O}_\infty$. (In fact, we want this under a weaker K-theoretic condition.)

For $\mathcal{O}_2$, we could reduce to the known (nonequivariant) case because the action on $\mathcal{O}_2$ has the Rokhlin property. But the action on $\mathcal{O}_\infty$ doesn't have the Rokhlin property. (There is no action on $\mathcal{O}_\infty$ with the Rokhlin property.)

Despite this, we will use the Rokhlin property!

We reduce to the case in which the image of [1] is zero in equivariant K-theory, which allows approximation by homomorphisms from finite Cuntz algebras with suitable actions. Details omitted (a few are still to be checked), but one needs to glue together different paths of equivariant homomorphisms constructed over different intervals, which don't quite agree at the endpoints of the intervals.

This requires equivariant semiprojectivity (described later).

Step 3: Equivariant asymptotic morphisms are equivariantly asymptotically unitarily equivalent to an equivariant homomorphisms

As in Step 2: A is a unital Kirchberg algebra with a pointwise outer action of G. The algebra $\mathcal{O}_\infty$ has the same quasifree action as in Step 1. Asymptotic morphisms are equivariant and go from A to a C*-algebra of the form $K \otimes \mathcal{O}_\infty \otimes D$, with D unital, with any action on D, and they are "full" (not asymptotically in any proper ideal).

Using the fact that an equivariant asymptotic morphism $t \mapsto \varphi_t$ is equivariantly homotopic to its reparametrizations, and that its reparametrizations are therefore equivariantly asymptotically unitarily equivalent to $t \mapsto \varphi_t$, one can show that $t \mapsto \varphi_t$ equivariantly asymptotically unitarily equivalent to an equivariant homomorphism.

Step 4: Equivariant unsuspended E-theory

Fix a unital Kirchberg algebra A with a pointwise outer action of G. For a unital C*-algebra D with an action of G, define $\widetilde{E}_A^G(D)$ to be the set of homotopy classes of full equivariant asymptotic morphisms from A to $K \otimes \mathcal{O}_\infty \otimes D$. Make it an abelian semigroup (in fact, group) using a direct sum operation. Extend to nonunital D the "same" way one does for K-theory.

Now prove that this is part of a generalized equivariant cohomology theory. (Details not all checked.)

This uses Steps 2 and 3, since for homotopy invariance we need to use homotopy, but for middle exactness we need to take the equivalence relation to be equivariant asymptotic unitary equivalence.

Step 4 (continued): Equivariant unsuspended E-theory

For a unital Kirchberg algebra A with a pointwise outer action of G and a unital C*-algebra D with an action of G, $\widetilde{E}_A^G(D)$ is the group of homotopy classes of full equivariant asymptotic morphisms from A to $K \otimes \mathcal{O}_\infty \otimes D$. It is part of a generalized equivariant cohomology theory.

Use the equivariant version of Higson's uniqueness theorem to show that $\widetilde{E}_A^G(D) \cong E^G(A, D)$. The equivariant Higson uniqueness theorem has already been proved, by Klaus Thomsen.

Conclusion: $E^G(A, D)$ is the set of asymptotic unitary equivalence classes of full homomorphisms from A to $K \otimes \mathcal{O}_\infty \otimes D$.

Step 5: KK^G-equivalence implies conjugacy

For a unital Kirchberg algebra A with a pointwise outer action of G and a unital C*-algebra D with an action of G, $E^G(A, D)$ is the set of asymptotic unitary equivalence classes of full homomorphisms from A to $K \otimes \mathcal{O}_\infty \otimes D$.

If A and D are both Kirchberg algebras with pointwise outer actions of G, and if $KK_G^0(A, D)$ contains an invertible element, then we get equivariant homomorphisms

$$\varphi\colon K \otimes A \to K \otimes D \quad \text{and} \quad \psi\colon K \otimes D \to K \otimes A$$

such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are both equivariantly asymptotically unitarily equivalent to the appropriate identity maps. Then the Elliott approximate intertwining argument shows that $K \otimes A$ is equivariantly isomorphic to $K \otimes D$.

If the class of the identity is preserved, cutting down shows that A is equivariantly isomorphic to D.

Step 6: $EK_*^G(A) \cong EK_*^G(D)$ implies conjugacy

For the first time, we must restrict G.

Let A and D be unital Kirchberg algebras with pointwise outer actions of G. Suppose that $EK_*^G(A) \cong EK_*^G(D)$. If G is cyclic of prime order, and both actions are in the bootstrap class, then Köhler's Universal Coefficient Theorem implies that there is an element in $KK_G^0(A, D)$ which induces this isomorphism.

It will be invertible, so it follows that A is equivariantly isomorphic to D.

Equivariant semiprojectivity

I will talk about equivariant semiprojectivity because (after Köhler's Universal Coefficient Theorem) it seems to be the place where the methods for the equivariant case differ the most from the ordinary case.

For short, a *G-algebra* (G, A, α) is a C*-algebra A together with a continuous action $\alpha\colon G \to \mathrm{Aut}(A)$.

The following definition (next slide) is just semiprojectivity, but with a group action put in everywhere.

Probably equivariant semiprojectivity is only interesting when G is compact. (Blackadar has examples.)

Equivariant semiprojectivity (continued)

Definition

A G-algebra (G, A, α) is *equivariantly semiprojective* if whenever (G, C, γ) is a G-algebra, $J_0 \subset J_1 \subset \cdots$ are G-invariant ideals in C, $J = \overline{\bigcup_{n=0}^{\infty} J_n}$, and $\varphi\colon A \to C/J$ is unital equivariant, then there exists n and a unital equivariant $\psi\colon A \to C/J_n$ such that the following diagram commutes:

$$\begin{tikzcd} & C \arrow[d, dotted] \arrow[dd, bend left] \\ & C/J_n \arrow[d, dotted] \\ A \arrow[r, "\varphi"'] \arrow[ur, dotted, "\psi"] & C/J \end{tikzcd}$$

At least when G is finite, one can prove that one has the equivariant analog of the usual relation between semiprojectivity and stable relations.

Equivariant semiprojectivity: Results

These results have been proved.

Theorem

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a compact group G on a finite dimensional C*-algebra A. Then (G, A, α) is equivariantly semiprojective.

Theorem

Let $d \in \{2, 3, \ldots\}$. Let $\alpha\colon G \to \mathrm{Aut}(\mathcal{O}_d)$ be a quasifree action of a compact group G. Then $(G, \mathcal{O}_d, \alpha)$ is equivariantly semiprojective.

Theorem

Let $d \in \{2, 3, \ldots\}$. Let $\alpha\colon G \to \mathrm{Aut}(E_d)$ be a quasifree action of a compact group G. Then (G, E_d, α) is equivariantly semiprojective.

Equivariant semiprojectivity: Results (continued)

Theorem

Let $\alpha\colon G \to \mathrm{Aut}(\mathcal{O}_\infty)$ be a quasifree action of a finite group G. Then $(G, \mathcal{O}_\infty, \alpha)$ is equivariantly semiprojective.

(In the last one, I don't know what happens if G is compact but not finite.)

Some applications:

- If G is finite, an equivariant asymptotic morphism from $\mathcal{O}_\infty$ (with the action used above) is asymptotically equal to a continuous path of homomorphisms.
- In the Rokhlin and tracial Rokhlin properties for finite groups, one can take the projections to be exactly orthogonal and exactly permuted by the group.

The previous page was the last one shown at the talk. (I had expected to get further, but not all the way to the end.)

Equivariant semiprojectivity of finite dimensional C*-algebras

The hardest part is:

Theorem

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a compact group G on a finite dimensional C*-algebra A. Then (G, A, α) is equivariantly semiprojective.

This is needed for the other parts.

We describe some ideas of the proof.

Rather than describing the lifting problem, we describe how to show that an approximately equivariant unital approximate homomorphism from A to some G-algebra B is close to an exactly equivariant unital true homomorphism.

Finite dimensional C*-algebras (continued)

Let (G, A, α) be a finite dimensional G-algebra, let (G, B, β) be a unital G-algebra, and let $\varphi\colon A \to B$ be unital, approximately equivariant, and an approximate homomorphism. We want to find a nearby equivariant unital homomorphism $\psi\colon A \to B$.

The usual method is to "straighten out" φ step by step, using functional calculus. It seems not to be possible to get equivariance this way.

Step 1: Restrict φ to the unitary group $U(A)$. Its values are then nearly unitary, and hence at least invertible.

Step 2: Average over G. Let μ be normalized Haar measure on G, and for $u \in U(A)$ set

$$\sigma(u) = \int_G \left(\beta_g \circ \varphi \circ \alpha_g^{-1}\right)(u)\, d\mu(g).$$

Then σ is exactly equivariant but is only approximately unitary and only approximately a group homomorphism. It is close to $\varphi|_{U(A)}$.

Finite dimensional C*-algebras (continued)

So far, we have an exactly equivariant approximately unitary approximate group homomorphism $\sigma \colon U(A) \to B$ which is close to $\varphi|_{U(A)}$.

Step 3: Set $\rho_0(u) = \sigma(u)\big[\sigma(u)^*\sigma(u)\big]^{-1/2}$. Then $\rho_0 \colon U(A) \to U(B)$ (its values are exactly unitary), and ρ is exactly equivariant and approximately a group homomorphism.

Steps 4 and 5 below (without the equivariance) have been independently discovered by Grove, Karcher, and Ruh (1972), and by Kazhdan (1982).

Step 4(1): Let ν be normalized Haar measure on $U(A)$. For $u \in U(A)$ set

$$\rho_1(u) = \rho_0(u)\exp\left(\int_{U(A)} \log\left(\rho_0(u)^*\rho_0(uw)\rho_0(w)^*\right)d\nu(w)\right).$$

Then ρ_1 is still exactly unitary and exactly equivariant. It is still only approximately a group homomorphism, but (see below) the error is less than before. It is also close to ρ_0.

Finite dimensional C*-algebras (continued)

Step 4(1): Let ν be normalized Haar measure on $U(A)$. For $u \in U(A)$ set

$$\rho_1(u) = \rho_0(u)\exp\left(\int_{U(A)} \log\left(\rho_0(u)^*\rho_0(uw)\rho_0(w)^*\right)d\nu(w)\right).$$

Then ρ_1 is still exactly unitary and exactly equivariant. It is still only approximately a group homomorphism, but (see below) the error is less than before. It is also close to ρ_0.

Step 4(n): Given ρ_{n-1}, set

$$\rho_n(u) = \rho_{n-1}(u)\exp\left(\int_{U(A)} \log\left(\rho_{n-1}(u)^*\rho_{n-1}(uw)\rho_{n-1}(w)^*\right)d\nu(w)\right).$$

This is close to ρ_{n-1}. It is still exactly unitary and exactly equivariant, and is yet closer to being a true homomorphism.

Finite dimensional C*-algebras (continued)

Step 4(n): Given ρ_{n-1}, set

$$\rho_n(u) = \rho_{n-1}(u)\exp\left(\int_{U(A)} \log\left(\rho_{n-1}(u)^*\rho_{n-1}(uw)\rho_{n-1}(w)^*\right)d\nu(w)\right).$$

Step 5: The maps ρ_n are exactly unitary and exactly equivariant. They form a Cauchy sequence, uniformly in $u \in U(A)$, and as $n \to \infty$, the errors $\|\rho_n(uv) - \rho_n(u)\rho_n(v)\|$ converge uniformly to zero. (See the next slide.) Therefore $\rho(u) = \lim_{n\to\infty}\rho_n(u)$ is an exactly equivariant homomorphism from $U(A)$ to $U(B)$. Moreover, ρ is uniformly close to $\varphi|_{U(A)}$.

Step 6: Since ρ and $\varphi|_{U(A)}$ are uniformly close, they are unitarily equivalent. It follows that ρ extends to a unital homomorphism from A to B. This is the equivariant homomorphism which is close to φ.

Finite dimensional C*-algebras (continued)

Recall:

$$\rho_n(u) = \rho_{n-1}(u)\exp\left(\int_{U(A)} \log\left(\rho_{n-1}(u)^*\rho_{n-1}(uw)\rho_{n-1}(w)^*\right)d\nu(w)\right).$$

Why is ρ_n better than ρ_{n-1}?

- One can check that if everything commutes, then ρ_n will in fact be exactly a homomorphism. (This comes from group cohomology.)
- There are constants C_1 and C_2 such that, if $\|a\|, \|b\| \leq r$ with r small enough, then
  $$\big\|\exp(a+b) - \exp(a)\exp(b)\big\| \leq C_1 r^2$$
  and
  $$\big\|\log\big((1+a)(1+b)\big) - \big(\log(1+a) + \log(1+b)\big)\big\| \leq C_2 r^2.$$
  (The linear terms in the power series cancel out.)
- Even with these estimates, things must work out just right.

Equivariant semiprojectivity of quasifree actions

Theorem

Let $d \in \{2, 3, \ldots\}$. Let $\alpha\colon G \to \mathrm{Aut}(\mathcal{O}_d)$ be a quasifree action of a compact group G. Then $(G, \mathcal{O}_d, \alpha)$ is equivariantly semiprojective.

Theorem

Let $d \in \{2, 3, \ldots\}$. Let $\alpha\colon G \to \mathrm{Aut}(E_d)$ be a quasifree action of a compact group G. Then (G, E_d, α) is equivariantly semiprojective.

Theorem

Let $\alpha\colon G \to \mathrm{Aut}(\mathcal{O}_\infty)$ be a quasifree action of a finite group G. Then $(G, \mathcal{O}_\infty, \alpha)$ is equivariantly semiprojective.

On the next two slides, there is a sketch of the proof for the special case coming from the regular representation. The general case requires a version of the argument above, but for cocycles.

Theorem

Let G be a finite group. Set $d = \mathrm{card}(G)$, and label the generators of $\mathcal{O}_d$ as s_g for $g \in G$. Then the quasifree action $\beta_g(s_h) = s_{gh}$ is equivariantly semiprojective.

Sketch of proof.

Let C, J_n, and J be as before (so G acts on everything and $J = \overline{\bigcup_{n=0}^{\infty} J_n}$), and let $\varphi\colon \mathcal{O}_d \to C/J$ be unital and equivariant.

The elements $s_g s_g^*$ generate a unital copy of $C(G)$ in $\mathcal{O}_d$, on which G acts by translation. Choose n such that one can lift $\varphi|_{C(G)}$ equivariantly to $\psi_0\colon C(G) \to C/J_n$. Increasing n, we may assume that $\psi_0(s_1 s_1^*)$ is Murray-von Neumann equivalent to 1. That is, there exists $t \in C/J_n$ such that $t^* t = 1$ and $t t^* = \psi_0(s_1 s_1^*)$. Increasing n further and modifying t, we may assume its image in C/J is $\varphi(s_1)$. Set $t_g = (\gamma_n)_g(t)$ for $g \in G$.

Equivariance of ψ_0 implies that $t_g t_g^* = \psi_0(s_g s_g^*)$ for all $g \in G$. Thus $\sum_{g \in G} t_g t_g^* = 1$. We can define an equivariant unital homomorphism $\psi\colon \mathcal{O}_d \to C/J_n$ by $\psi(s_g) = t_g$ for $g \in G$, and ψ lifts φ. □

Equivariant semiprojectivity of further quasifree actions

Theorem

Let G be a finite group. Set $d = \mathrm{card}(G)$, and label the generators of $\mathcal{O}_d$ as s_g for $g \in G$. Then the quasifree action $\beta_g(s_h) = s_{gh}$ is equivariantly semiprojective.

One can use similar methods to get equivariant semiprojectivity for the quasifree action coming from the direct sum of finitely many copies of the regular representation of G, for the corresponding quasifree actions on the Cuntz-Toeplitz algebras, and, following Blackadar, for the quasifree action on $\mathcal{O}_\infty$ coming from the direct sum of infinitely many copies of the regular representation of G.