Characterization of locally reflexive algebras

by an “inner” version of exactness

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Contents

Some Notation

Let A and B C*-algebras.

$\| \cdot \| = \| \cdot \|_{\min}$ means the spatial tensor product norm on the *algebraic* tensor product $A^{**} \odot B^{**}$,

We denote $A^{**} \otimes B^{**}$ the the C*-algebra completion of $A^{**} \odot B^{**}$ with $\| \cdot \|$.

Let $X \subset \mathcal{L}(H_1)$ and $Y \subset \mathcal{L}(H_2)$ closed subspaces (operator spaces). $X \otimes Y$ is the operator space that is the closure of vector space tensor product $X \odot Y$ in $\mathcal{L}(H_1 \otimes H_2)$.

Locally reflexive C*-algebras

Let $X \subset \mathcal{L}(H)$ a unital linear subspace of finite dimension, and $V\colon X \to A^{**}$ a unital completely contractive map.

In general, it is not possible to find a family $\{V_\gamma\}$ of completely contractive maps $V_\gamma\colon X \to A$, such that V is the point-$\sigma(A^{**}, A^*)$ limit of the V_γ.

Definition (1)

A is **locally reflexive** (in a matricial sense) if, for every subspaces $X \subset A^{**}$ and $F \subset A^*$ of finite dimension and for every $\varepsilon > 0$ there exists a completely contractive linear map $T\colon X \to A$ with

$$|x(f) - f(T(x))| \leq \varepsilon \|x\| \cdot \|f\| . \quad \forall x \in X, f \in F .$$

$\mathcal{L}(\ell_2)$ and $C^*(SL(\mathbb{Z}))$ are not locally reflexive.

The C-norm $\|\cdot\|_C$ on $A^{**} \odot B^{**}$

The algebraic tensor product $A^{**} \odot B^{**}$ is a *-subalgebra of $(A \otimes B)^{**}$ in a natural way. The induced C*-norms on $A^{**} \odot B^{**}$, $A \odot B^{**}$ or $A^{**} \otimes B$ will be denoted by $\|\cdot\|_C$.

Lemma (2)

For $a_1, \ldots, a_n \in A^{**}$, $b_1, \ldots, b_n \in B^{**}$, and $w = \sum_k a_k \otimes b_k$ holds

$$\|w^* w\|_C = (\|w\|_C)^2 = \sup_\lambda \sum_{j,k} \lambda((a_j^* a_k) \otimes (b_j^* b_k)) = \sup_\lambda \lambda(w^* w)$$

where λ runs over all unital positive linear functionals λ on $A^{**} \odot B^{**}$ that are partially normal on A^{**} and on B^{**}, and are continuous on $A \odot B$ with respect to $\|\cdot\|_{\min}$ on $A \odot B$.

It means, that $\lambda(a \otimes b) = \langle d_1(a)d_2(b)x, x\rangle$ where $d_1 : A^{**} \to \mathcal{L}(H)$ and $d_2 : B^{**} \to \mathcal{L}(H)$ are commuting non-degenerate ($=$ unital) normal *-representations of the von Neumann algebras A^{**} and B^{**}, and that the *-representation

$$\sum_k a_k \otimes b_k \mapsto \sum_k d_1(a_k)d_2(b_k)$$

is continuous on $A \odot B$ with respect to the minimal C*-norm on $A \odot B$, and the vector $x \in H$ has norm $\|x\| = 1$.

Lemma (3) (Reduction to separable case)

Suppose that $X \subset A$, $Y \subset A^*$ and $Z \subset A^{**}$ are (norm-)separable subspaces. Then there exist a separable C-subalgebra B of A and a normal completely positive map*

$$V\colon A^{**} \to B^{**} \cong \overline{B}^{\mathrm{strong}}$$

and a projection $P \in V(A^{**})' \cap B^{**}$ such that

- $X \subset B$ *and* $V(x) = x$ *for all* $x \in C^*(X)$,
- $\rho(V(a)) = \rho(a)$ *for all* $a \in A^{**}$ *and* $\rho \in Y$, *i.e.*, V_* *fixes* Y,
- $a \mapsto V(a)P$ *is multiplicative on* $C^*(X \cup Z)$, *and* P *is countably decomposable (in* B^{**}*),*
- $Py = y = yP$ *for all* $y \in Y$.

In particular, the separable C^*-subalgebra $B \subset A$ and the C^*-morphism $\phi\colon C^*(Z \cup X) \to B^{**} \subset A^{**}$ given by $\phi(a) := V(a)P$ satisfy $X \subset B$, $\rho(\phi(z)) = \rho(z)$ for $\rho \in Y$, $z \in Z$, and $\phi(x) = x$ for $x \in X$.

It yields (the non-trivial part of the proof of):

Lemma (4)

$\| \cdot \|_C = \| \cdot \|_{\min}$ on $A^{**} \odot B$, if and only if,
$\| \cdot \|_C = \| \cdot \|_{\min}$ on $D^{**} \odot E$ for all separable C^*-subalgebras $D \subset A$ and $E \subset B$,
if and only if,
$\| \cdot \|_C = \| \cdot \|_{\min}$ on $(\mathbb{K} \otimes A)^{**} \odot B$, if and only if,
$\| \cdot \|_C = \| \cdot \|_{\min}$ on $(A + \mathbb{C} \cdot 1)^{**} \odot B$.

Notice that $\| \cdot \|_C = \| \cdot \|$ on $A \odot B$.

Let $L \subset A$ a closed left-ideal and $p_L \in A^{**}$ its open support projection, and define linear $\pi : A \odot B \to (A/L) \odot B \subset (A/L) \otimes_{min} B$ by

$$\pi(a \otimes b) := (a + L) \otimes b \, .$$

Lemma (5)

If $w \in A \odot B$ *then* $w((1 - p_L) \otimes 1) \in A^{**} \odot B$,

$$\operatorname{dist}(w, L \otimes_{\min} B) = \operatorname{dist}(w, L \odot B) = \| w((1 - p_L) \otimes 1) \|_C$$

and

$$\| \pi(w) \| = \| w((1 - p_L) \otimes 1) \|_{\min} \, .$$

Properties (C), (C $'$) and (C $''$)

Definition (6)

The algebra A has *property*

(C) if $\|\,.\,\|_C = \|\,.\,\|_{min}$ on $A^{**} \odot B^{**}$ for every C^*-algebra B,

(C $'$) if $\|\,.\,\|_C = \|\,.\,\|_{min}$ on $A \odot B^{**}$ for every C^*-algebra B,

(C$''$) if $\|\,.\,\|_C = \|\,.\,\|_{min}$ on $A^{**} \odot B$ for every C^*-algebra B.

Effros and Haagerup: All this properties pass to subalgebras, $E \subset A$, and properties (C) and (C $''$) pass to quotients A/J.
A has property (C $''$) if and only if A is locally reflexive.

E.K.(in Crelle J.): (C $'$) $\Rightarrow$ exactness $\Rightarrow$ (C). In particular, (C)=(C $'$).

It is known:

- Each exact C*-algebra is locally reflexive.
- A C*-algebra is locally reflexive, if and only if, all its separable C*-subalgebras are locally reflexive.
- Locally reflexive C*-algebras with WEP (of Lance) are nuclear.
- Locally reflexive C*-algebras with a matricial variant of the Grothendieck approximation property are exact.
- Extensions of locally reflexive C*-algebras are locally reflexive, if and only if, the Busby invariant is locally liftable.
- Locally reflexive algebra A is exact, if and only if, A^{**} is a weakly exact W*-algebra.

Open problems concerning local reflexivity:

Let A, $A_1 \subset A_2 \subset \cdots$ locally reflexive (=: l.r.) C*-algebras.

(a) Is A exact? In particular: Let G a Gromov example ($=$ discrete finitely presented group that is not uniformly embeddable into a Hilbert space with respect to its word length metric).
Is $C_r^*(G)$ not l.r.? (It is not exact by a result of Ozawa.)

(b) Is $M_{2^\infty} \otimes A$ l.r.? (Equivalent to: Is $B \otimes A$ l.r. if B is exact?)

(c) Are inductive limits of l.r. algebras A_n again l.r.?

(d) Is the crossed product $A \rtimes_\alpha \mathbb{Z}$ of A by $\alpha \in \mathrm{Aut}(A)$ again l.r.?

(f) Are reduced free products of l.r. algebras again l.r.?

(g) Suppose A does not have the WEP. Are there states λ on A and μ on C[0, 1] such that $A * 1 \subset A *_{\rho,\mu} \mathrm{C}[0,1]$ is *not* relatively weakly injective in the reduced free product $A *_{\rho,\mu} \mathrm{C}[0,1]$?

(h) Is A embeddable into simple l.r. C*-algebra B?

Possible positive answers to questions (a – h) have the following implications:

(c) $\Rightarrow$ (b),

(d) $\Rightarrow$ (b, f),

(a) $\Rightarrow$ (b – f, h),

(d) $\Rightarrow$ (f),

(c, f, g) $\Rightarrow$ (a),

(f) $\Rightarrow$ (h).

Definition (7)

We call a C^*-algebra A **inner exact for** B, if the sequence

$$0 \to L \otimes B \to (\mathcal{K} \otimes A) \otimes B \to ((\mathcal{K} \otimes A)/L) \otimes B \to 0$$

is exact (in the metric sense) for every closed left-ideal $L \subset \mathcal{K} \otimes A$.
The algebra A is **inner exact** if A is inner exact for *every* C*-algebra B.

By Lemma 5, $\| \cdot \|_C = \| \cdot \|_{\text{min}}$ on $A^{**} \otimes B$ implies that A is *inner exact for* B.

A is inner exact $\quad \Leftrightarrow \quad \forall \ X \subset (\mathbb{K} \otimes A)/L$ with $\mathrm{Dim}(X) < \infty$ and $\varepsilon > 0$,
$\exists \ T \colon X \to \mathbb{K} \otimes A$ with $\pi_L \circ T = \mathrm{id}_X$ and $\|T\|_{cb} \leq 1 + \varepsilon$.

Reduction to the separable and unital case

Lemma (8)

TFAE:

- *A is inner exact for B.*
- *Every separable C*-subalgebra of A is inner exact for B.*
- $\mathbb{K} \otimes A$ *is inner exact for B.*
- *The unitization* $\widetilde{A}$ *is inner exact for B.*

Since the same happens for the property that $\|\cdot\|_C = \|\cdot\|_{\min}$ on $A^{**} \otimes B$, it suffices to prove the following Proposition 9 only for separable unital A.

Proposition (9)

$\| \cdot \|_C = \| \cdot \|_{\min}$ *on* $A^{**} \otimes B$*, if and only if, A is inner exact for B.*

If we combine this with the work of Effros and Haagerup, we get:

Theorem (10) (characterization of locally reflexive agebras)

A C-algebra A is locally reflexive, if and only if, the sequence of operator spaces*

$$0 \to L \otimes B \to (\mathcal{K} \otimes A) \otimes B \to ((\mathcal{K} \otimes A)/L) \otimes B \to 0$$

is exact – in the complete metric sense – for every closed left-ideal L of $\mathcal{K} \otimes A$ *and every C*-algebra B.*

It is an open question if every locally reflexive algebra is exact. If this question would have a positive answer, then we could get from Theorem 10, or even better from a proof of the following Conjecture 11, an "algebraic" inner characterization of exactness.

Conjecture (11)

If, for every C-subalgebra $E \subset A$ and every closed ideal J of E the Busby invariant of the extension $0 \to J \to E \to E/J \to 0$ is locally liftable, then A is locally reflexive.*

On the Proof of Proposition 9:

Lemmata 4 and 8 show that it suffices to consider the case of **separable and unital** A.

Proof of "$\Rightarrow$" follows from Lemmata 4 and 5.

The proof of "$\Leftarrow$" needs some ideas related to the nc Lusin Theorem and to desired non-commutative versions of the Egorov theorem.

Lemma 2, repeated use of the nc Lusin Theorem (M.Tomita 1959, see book of G.K. Pedersen, Thm. 2.7.3), and $N(x^*x) = N(xx^*)$ for each C^*-norm N on $A^{**} \odot B$ together prove the following lemma.

Lemma (12)

If $\|v^*(p \otimes 1)v\|_C \leq \|v^*(p \otimes 1)v\|_{\min}$ for all $v \in A \odot B$ and all countably decomposable projections $p \in A^{**}$, then $\| \cdot \|_C = \| \cdot \|_{\min}$ on $A^{**} \odot B$

Lemmata 12 and Lemma 2 show that $\| \cdot \|_C = \| \cdot \|_{\min}$ on $A^{**} \odot B$, if for each (fixed) $v \in A \odot B$ and each (fixed) positive partially normal state λ on $A^{**} \odot B^{**}$, that is continuous on $A \odot B$ with respect to $\| \cdot \|_{\min}$, holds

$$\lambda(v^*(p \otimes 1)v) \leq \|v^*(p \otimes 1)v\|_{\min}$$

for *all* countably decomposable projection $p \in A^{**}$.

Fix $v \in A \odot B$, λ as above. Find countably decomposable projection c in the center of A^{**} with $pc = p$, $\lambda(c \otimes 1) = 1$, and $\|ac\| = \|a\|$ for all $a \in A$. Then $A^{**}c$ has a faithful normal unital representation on a separable Hilbert space. Thus, the Up-Down Theorem of G.K. Pedersen applies. Since $A \cong Ac \subset A^{**}c$ is *unital*, we get that each element of A^{**}_{+} with norm ≤ 1 (in particular our projection p) is in

$$\left((A^1_+)_{\sigma-\text{down}}\right)^{\sigma-\text{up}}.$$

We define the set $S = S(A, \lambda, v)$ of $a \in A^{**}$ with $0 \leq a \leq 1$ and the property and $\lambda(v^*(a \otimes 1)v) \leq \|v^*(ac \otimes 1)v\|_{\text{min}}$.

Since $a \in A^{**}_{+} \mapsto \|v^*(a \otimes 1)v\|$ is order preserving, and since $a \in A^{**} \to \lambda(v^*(a \otimes 1)v)$ is a normal positive functional, we get that $a \in S$ if $a = \sup_n a_n$ for $a_1 \leq a_2 \leq \cdots \in S$.

Intermediate result:

If $\lambda(v^*(a \otimes 1)v) \leq \|v^*((ca) \otimes 1)v\|_{\min}$ with c as above selected (depending on λ and v) for all $a \in ({A_+}^1)_{\sigma-\text{down}}$, each λ and $v \in A \odot B$, then $\| \cdot \|_C = \| \cdot \|_{\min}$ on $A^{**} \odot B$.

The proof of Proposition 9 becomes complete, if we can find a *closed projection* $q \in (\mathbb{K} \otimes A)^{**}$ such that
$e_{11} \otimes (v^*(a \otimes 1)v) = (e_{11} \otimes v)(q \otimes 1)(e_{11} \otimes v)$.

Let A a unital C*-algebra and $0 \leq a_1 \leq a_2 \leq \cdots$ an increasing sequence of contractions in A_+, and $b := \sup_n a_n \in A^{**}$ its $\sigma(A^{**}, A^*)$-limit (i.e., weak limit). Denote by e_{ij} the matrix units in $\mathcal{K}$.

Lemma (13) (Dilation of increasing sequences)

*There is a untal *-morphism*

$$h\colon \; \widetilde{\mathcal{K}} := \mathcal{K} + \mathbb{C} \cdot 1 \to \mathcal{M}(\mathcal{K} \otimes A)$$

such that

$$e_{11} \otimes a_n = (e_{11} \otimes 1)\, h(p_n)\, (e_{11} \otimes 1)\,,$$

with $\quad p_n := e_{11} + e_{22} + \cdots + e_{nn}$ *for all* $\; n \in \mathbb{N}\,.$

Idea of proof: *Modify the Stinespring dilation of the unital c.p. map $V\colon \widetilde{\mathcal{K}} \to A$ with $V(e_{ij}) := (a_j - a_{j-1})\delta_{ij}$* . Here $a_{-1} := 0$.

Remark (14)

Consider the hereditary C*-subalgebra

$$D := \overline{\bigcup_n h(p_n)(\mathcal{K} \otimes A)h(p_n)} \subset \mathcal{K} \otimes A .$$

The open projection $p_D \in (\mathcal{K} \otimes A)^{**} \cong \mathcal{L}(\ell_2)\overline{\otimes}A^{**}$ corresponding to D satisfies $(e_{11} \otimes 1)p_D(e_{11} \otimes 1) = e_{11} \otimes b$ and, for $w \in A \odot B$:

$$e_{11} \otimes (w^*((1-b) \otimes 1)w) = (e_{11} \otimes w)^*((1-p_D) \otimes 1)(e_{11} \otimes w) .$$

The last equation **completes the proof** of Proposition 9.

A noncommutative Egorov problem

Let A a unital or stable separable C^*-algebra, and let $\mu \in A^*$ a positive linear functional on A. The **central support** $c \in \mathcal{Z}(A^{**})$ of μ is defined as the smallest projection c in the center $\mathcal{Z}(A^{**})$ of A^{**} with $\mu(c) = \|\mu\|$. The usual support projection $p_\mu \in A^{**}$ of μ is not necessarily in the center of A^{**}.

Question (15) (nc Egorov)

Let $p \in A^{**}$ a projection, $\varepsilon > 0$.
Does there exists a *closed* projection $q \in A^{**}$ such that

$$qc \leq p \qquad \text{and} \qquad \mu(q) + \varepsilon > \mu(p)\,.$$

Recall here that a projection $q \in A^{**}$ is *closed* if $1 - q$ is the *open* support projection p_D of a closed hereditary C*-subalgebra $D \subset A$.

If $A = C(\Omega)$ is commutative and unital, then the answer is positive and is equivalent to a theorem of Egorov in Measure theory.

There exists partial results that are related to a possible positive answer of question 15. But they are only generalization of a theorem of Lusin.

The above Lemma 13 (together with Remark 14) is a step towards a partial result, but only after stabilizing A with the compact operators.

We obtain in a similar way a “stable” version of a non-commutative Egorov theorem (where we identify $e_{11} \otimes a$ with $a \in A^{**}$):

Theorem (16) (nc Egorov)

Let A a separable unital C-algebra and $\mu \in A^*$ a positive linear functional with central support (-projection) $c \in Z(A^{**})$, $T \in A^{**}_+$ with $\|T\| \leq 1$.*
Then, for every $\varepsilon > 0$, there exists a ***closed*** *projection $q \in (\mathcal{K} \otimes A)^{**}$ (i.e., $1 - q$ is the open support projection of an hereditary C*-subalgebra $D \subset \mathcal{K} \otimes A$) such that*

$$(e_{11} \otimes c)q(e_{11} \otimes c) \leq T \qquad \text{and} \qquad \mu((e_{11} \otimes 1)q(e_{11} \otimes 1)) + \varepsilon > \mu(T)\,.$$

What about **a non-stable version of the characterization of local reflexivity**, or only of $\| \cdot \|_C = \| \cdot \|$ on $A^{**} \otimes B$? (We may assume again that A is unital and separable.)

The basic assumption is (equivalent to the assumption), that there is a universal constant $\rho < \infty$ with the property $\|(q \otimes 1)w\|_C \leq \rho \|(q \otimes 1)w\|$ for all closed projections $p \in A^{**}$ and all $w \in A \odot B$.

Then we need as a property of A that, for each countably decomposable projection $z \in \mathcal{Z}(A^{**})$ and each $b \in (A^1_+)_{\sigma-\text{down}}$, normal state $\mu \in A^* = (A^{**})_*$ on A^{**} and every $\varepsilon > 0$, there is a closed projection $q \in A^{**}$ and an element $a \in A$ (both depending on ε) such that $za^*qa \leq \varepsilon 1 + (1+\varepsilon) \sup_n b^{1/n}$ and $\mu(b) \leq 2\varepsilon + \mu(a^*qa)$.

The proof of the conclusion $\| \cdot \|_C = \| \cdot \|_{\min}$ on $A^{**} \odot B$ under this assumptions is similar to the above given proof of Proposition 9.

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Proof of "⇒" for Proposition 9:

If $\| \cdot \|_C = \| \cdot \|_{\min}$ on $A^{**} \otimes B$, then $\| \cdot \|_C = \| \cdot \|_{\min}$ on $(\mathcal{K} \otimes A)^{**} \otimes B$, by Lemma 4.

For closed left-ideals $L \subset \mathcal{K} \otimes A$ with open support projection $p_L \in (\mathcal{K} \otimes A)^{**}$ and $w \in (\mathcal{K} \odot A)) \odot B$ holds
$\mathrm{dist}(w, L \odot B) = \|w((1 - p_L) \otimes 1)\|_C$ and
$\|w((1 - p_L) \otimes 1)\|_{\min} = \|(\pi_L \otimes \mathrm{id}_B)(x)\|$, by Lemma 5. □

Proof of Lemma 12:

We make repeated use of the non-commutative Lusin theorem (from M.Tomita in 1959, see book of G.K. Pedersen, Thm. 2.7.3): Given $x \in H$, $a_1, \ldots, a_n \in A^{**}$, $d_1 : A^{**} \to \mathcal{L}(H)$ normal *-representation $\varepsilon > 0$, then exist countably decomposable $p \in A^{**}$, $a'_k \in A$ with $a'_k p = a_k p$ and $\|x - px\| < \varepsilon$. Then $w = \sum_k a_k \otimes b_k \in A^{**} \odot B$ and $v := \sum_k a'_k \otimes b_k \in A \odot B$ satisfy $w(p \otimes 1) = v(p \otimes 1)$.

The distance $|\rho(w^*w) - \rho((p \otimes 1)w^*w(p \otimes 1))|$ between $\|(d_1 \cdot d_2)(w)x\|^2$ and $\|(d_1 \cdot d_2)(w(p \otimes 1))x\|^2$ is $\leq \varphi(\varepsilon)$ for some increasing continuous function φ with $\varphi(0) = 0$ (if given x and w are fixed).

$\lambda((p \otimes 1)w^*w(p \otimes 1))^{1/2} \leq \|w(p \otimes 1)\|_C = \|v(p \otimes 1)\|_C = \|v(p \otimes 1)\|$.

Since $\|v(p \otimes 1)\| = \|w(p \otimes 1)\| \leq \|w\|$ it implies

$$\lambda(w^*w) \leq f(\varepsilon) + \|w^*w\|_{\min},$$

for each $\varepsilon > 0$, hence $\lambda(w^*w) \leq \|w^*w\|_{\min}$.
Now Lemma 2 says $\|w\|_C = \|w\|_{\min}$. □

More about the proof of Lemma 13: *Consider the unital c.p. map* $V\colon \mathcal{K} \to A$ *with* $V(e_{ij}) := (a_j - a_{j-1})\delta_{ij}$. Here $a_{-1} := 0$.

The Kasparov-Stinespring dilation defines a countably generated Hilbert A-module H and a unital *-representation $k_1 \colon \widetilde{\mathcal{K}} \to \mathcal{L}(H)$ and a vector $x \in \mathcal{H}$ such that $\langle x, k(c)x\rangle = V(c)$ for all $c \in \widetilde{\mathcal{K}}$.

Then Kasparov triviality theorem gives an Hilbert A-modul isomorphism γ from $H \oplus H_A$ onto the Kasparov standard module H_A. Consider H_A as the set $(\mathcal{K} \otimes A)(e_{11} \otimes 1)$ of first columns $\sum e_{n1} \otimes y_n$ in $\mathcal{K} \otimes A$. Then $\mathcal{L}(H_A)$ becomes naturally isomorphic with the multiplier algebra $\mathcal{M}(\mathcal{K} \otimes A)$ of $\mathcal{K} \otimes A$.

Define $k_2 \colon \widetilde{\mathcal{K}} \to \mathcal{L}(H_A)$ by

$$k_2(c) := \gamma(k_1(c) \oplus \chi(c)1)\gamma^{-1},$$

where $\chi \colon \widetilde{\mathcal{K}} \to \mathbb{C}$ is the unique non-zero character.

The element $w := \gamma(x)$ is a partial isometry in $\mathcal{K} \otimes A$ with $w^*w = e_{11} \otimes 1$. Since A is unital, its stabilization $\mathcal{K} \otimes A$ has stable rank one, and – therefore – we find a unitary

$$U \in (\mathcal{K} \otimes A) + \mathbb{C} \cdot 1 \subset \mathcal{M}(\mathcal{K} \otimes A)$$

with $\gamma(x) = U(e_{11} \otimes 1)$.

The desired unital *-morphism $h \colon \widetilde{\mathcal{K}} \to \mathcal{M}(\mathcal{K} \otimes A)$ is given by $h(c) := U^* k_2(c) U$. □