Algebraic actions, entropy, and operator algebras

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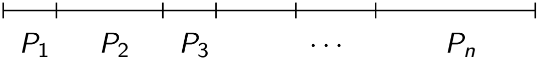
Shannon entropy

The *Shannon entropy* of a partition $\mathcal{P}$



of a probability space (X, μ) is defined as

$$H(\mathcal{P}) = -\sum_{i=1}^{n} \mu(P_i) \log \mu(P_i),$$

which can be viewed as the integral of the information function

$$I(x) = -\log \mu(P_i)$$

where i is such that $x \in P_i$.

Kolmogorov-Sinai entropy

For a single measure-preserving transformation $T : X \to X$ we set

$$h_\mu(T, \mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} H(\mathcal{P} \vee T^{-1}\mathcal{P} \vee \cdots \vee T^{-n+1}\mathcal{P})$$
$$h_\mu(T) = \sup_{\mathcal{P}} h_\mu(T, \mathcal{P}).$$

Kolmogorov-Sinai theorem

If $\mathcal{P}$ is a finite generating partition then $h_\mu(T) = h_\mu(T, \mathcal{P})$.

As a consequence, the entropy of the Bernoulli shift on $(X_0, \mu_0)^{\mathbb{Z}}$ is equal to the Shannon entropy of the base.

Theorem (Ornstein)

Bernoulli shifts are classified by their entropy.

This theory applies most generally to **amenable** acting groups.

Bowen's measure entropy

Basic idea

Replace internal averaging (information theory) by the counting of discrete models (statistical mechanics).

Let $\mathcal{P}$ be a partition of X whose atoms have measures $c_1, \dots, c_n$. In how many ways can we approximately model this ordered distribution of measures by a partition of $\{1, \dots, m\}$ for a given $m \in \mathbb{N}$? By Stirling's formula, the number of models is roughly

$$c_1^{-c_1 m} \cdots c_n^{-c_n m}$$

for large m, so that

$$\frac{1}{m} \log(\#\text{models}) \approx -\sum_{i=1}^{n} c_i \log c_i = H(\mathcal{P}).$$

Bowen's measure entropy

Let $G \curvearrowright (X, \mu)$ be a measure-preserving action, and let Σ be a sequence of maps $\sigma_i : G \to \mathrm{Sym}(m_i)$ into finite permutation groups which are asymptotically multiplicative and free (the existence of such a sequence defines a **sofic** group).

Let $\mathcal{P}$ be a finite ordered partition of X. For a finite set $F \subseteq G$ and $\varepsilon > 0$ we write $\mathrm{AP}(\mathcal{P}, F, \varepsilon, \sigma_i)$ for the number of ordered partitions $\mathcal{Q}$ of $\{1, \ldots, m_i\}$ such that the measures of the atoms of $\bigvee_{s \in F} s^{-1}\mathcal{P}$ and $\bigvee_{s \in F} s^{-1}\mathcal{Q}$ which correspond to each other under the dynamics are summably ε-close. Set

$$h_{\Sigma,\mu}(\mathcal{P}) = \inf_F \inf_{\varepsilon > 0} \limsup_{i \to \infty} \frac{1}{m_i} \log \#\mathrm{AP}(\mathcal{P}, F, \varepsilon, \sigma_i)$$

Theorem (Bowen)

$h_{\Sigma,\mu}(\mathcal{P})$ has a common value for generating partitions $\mathcal{P}$.

A generator-free definition of sofic entropy

We seek a general generator-free definition of sofic entropy in the spirit of what Sinai furnished for single transformations in response to Kolmogorov's generator-based definition.

Basic idea

Measure the exponential growth of the number of sofic models as before but for **each** *partition at some* **fixed observational scale**, *and then take a supremum of these growth rates as the scale becomes finer and finer.*

The observational scale is determined by a second partition, and so the variables in the definition now include two partitions playing different roles.

A generator-free definition of sofic entropy

Define $\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma_i)$ to be the set of all homomorphisms from the algebra generated by $\mathcal{P}$ to the algebra of subsets of $\{1, \ldots, m_i\}$ which, to within δ,

- are approximately F-equivariant, and
- approximately pull back the uniform probability measure on $\{1, \ldots, m_i\}$ to μ.

For a partition $\mathcal{Q} \leq \mathcal{P}$, write $|\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma_i)|_{\mathcal{Q}}$ for the cardinality of the set of restrictions of elements of $\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma_i)$ to $\mathcal{Q}$.

Definition

$$h_{\Sigma,\mu}(X, G) = \sup_{\mathcal{Q}} \inf_{\mathcal{P} \geq \mathcal{Q}} \inf_{F,\delta} \limsup_{i \to \infty} \frac{1}{m_i} \log |\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma_i)|_{\mathcal{Q}}$$

A generator-free definition of sofic entropy

We then have the following Kolmogorov-Sinai theorem, which enables us to compute the entropy as we have defined it.

Theorem

In the definition of $h_{\Sigma,\mu}(X, G)$, one can equivalently restrict the partitions $\mathcal{P}$ and $\mathcal{Q}$ to range within a given generating σ-algebra. In particular, if there is a finite generating partition then $\mathcal{P}$ and $\mathcal{Q}$ need not range over any partitions except this one.

As a consequence, the entropy of a Bernoulli action $G \curvearrowright (X_0, \mu_0)^G$ is equal to the Shannon entropy of the base.

The above theorem also permits us to show that our definition is equivalent to Bowen's in the presence of a generating partition.

Application to Bernoulli actions

In the case that G is amenable the following theorem is a well-known consequence of classical entropy theory. In the case that G contains the free group F_2 it was proved by Lewis Bowen. Note that there exist countable sofic groups that lie outside of these two classes. Indeed Ershov showed the existence of a countable nonamenable residually finite torsion group.

Theorem (K.-Li)

Let G be a countable sofic group. Let (X, μ) be a standard probability space with $H(\mu) = +\infty$. Then the Bernoulli action $G \curvearrowright (X^G, \mu^G)$ has no generating partition with finite Shannon entropy.

Embedded sofic approximation viewpoint

In the case that X is a compact metrizable space and G acts by measure-preserving homeomorphisms, we can formulate an equivalent definition of sofic measure entropy by measuring the exponential growth, relative to the sofic approximation sequence $\Sigma = \{\sigma_i : G \to \mathrm{Sym}(m_i)\}$, of the number of maps

$$\{1, \ldots, m_i\} \to X$$

which

- are approximately equivariant, and
- approximately push forward the uniform probability measure to μ.

The images of such a map can be viewed as a system of interlocking partial orbits.

Topological entropy

For a finite set $F \subseteq G$ and a $\delta > 0$ define $\mathrm{Map}(d, F, \delta, \sigma_i)$ to be the set of all maps $\{1, \dots, m_i\} \to X$ which are approximately F-equivariant to within δ. Set

$$h_\Sigma(d) = \sup_{\varepsilon > 0} \inf_F \inf_{\delta > 0} \limsup_{i \to \infty} \frac{1}{m_i} \log N_\varepsilon(\mathrm{Map}(d, F, \delta, \sigma_i))$$

where $N_\varepsilon(\cdot)$ denotes the maximal cardinality of an ε-separated set w.r.t. the pseudometric $\rho(\varphi, \psi) = \max_{a=1,\dots,m_i} d(\varphi(a), \psi(a))$.

Proposition

$h_\Sigma(d)$ has a common value over all compatible metrics d on X.

Definition

Define $h_\Sigma(X, G)$ to be this common value.

Application to surjunctivity

Gottschalk's surjunctivity problem asks which countable groups G are *surjunctive*, which means that, for each $k \in \mathbb{N}$, every injective G-equivariant map $\{1, \dots, k\}^G \to \{1, \dots, k\}^G$ is surjective.

Theorem (Gromov)

Every countable sofic group is surjunctive.

One can give an entropy proof of Gromov's theorem by showing the following.

Theorem (K.-Li)

Let G be a countable sofic group and $k \in \mathbb{N}$. Then with respect to every sofic approximation sequence the shift $G \curvearrowright \{1, \dots, k\}^G$ has entropy $\log k$ and all proper subshifts have entropy less than $\log k$.

Amenable case

The entropy of a homeomorphism $T : X \to X$ is given by

$$h_{\mathrm{top}}(T) = \sup_{\varepsilon>0} \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{sep}(n,\varepsilon)$$

where $\mathrm{sep}(n,\varepsilon)$ is the maximal cardinality of an ε-separated set of partial orbits from 0 to $n-1$. By averaging over Følner sets one can define this more generally for amenable acting groups.

When G is amenable, every sofic approximation approximately decomposes into Følner sets. Thus an approximately equivariant map $\{1,\dots,m_i\} \to X$ approximately decomposes into partial orbits over Følner sets. Consequently:

Theorem (K.-Li)

Suppose that G is amenable. Then $h_{\Sigma,\mu}(X,G) = h_\mu(X,G)$ in the measure case and $h_\Sigma(X,G) = h(X,G)$ in the topological case.

Variational principle

The classical variational principle asserts that, for a homeomorphism $T : X \to X$ of a compact Hausdorff space,

$$h_{\mathrm{top}}(T) = \sup_{\mu} h_{\mu}(T)$$

where μ ranges over all invariant Borel probability measures on X. This is moreover true for actions of any countable amenable group.

Theorem (K.-Li)

Let $G \curvearrowright X$ be a continuous action of a countable sofic group on a compact metrizable space. Then

$$h_{\Sigma}(X, G) = \sup_{\mu} h_{\Sigma,\mu}(X, G)$$

where μ ranges over all invariant Borel probability measures on X.

Algebraic actions

By an *algebraic action* we mean an action of a group G on a compact Abelian group by automorphisms.

Example: toral automorphisms

A matrix $A \in \mathrm{GL}(n, \mathbb{Z})$ defines an automorphism T_A of $\mathbb{T}^n$, and

$$h_{\mathrm{top}}(T_A) = h_\mu(T_A) = \sum_{|\lambda_i|>1} \log|\lambda_i|$$

where μ is the Haar measure and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A with multiplicity.

However, for $d > 1$ a $\mathbb{Z}^d$-action generated by d commuting matrices in $\mathrm{GL}(n, \mathbb{Z})$ always has **zero entropy**.

Algebraic actions

Is entropy relevant for algebraic actions of groups other than $\mathbb{Z}$?

By Jensen's formula, the entropy of a toral automorphism T_A can be expressed as the logarithm of the Mahler measure

$$M(f) := \exp \int_0^1 \log |f(e^{2\pi i t})| \, dt$$

of the characteristic polynomial f of A. This hints that we should shift our perspective from matrices to polynomials in order to expand the theory to more general acting groups.

Algebraic actions

Let f be an element in the group ring $\mathbb{Z}G$. Then G acts on $\mathbb{Z}G/\mathbb{Z}Gf$ by left translation, and this gives rise to an action α_f of G by automorphisms on the compact Abelian dual group $\widehat{\mathbb{Z}G/\mathbb{Z}Gf}$. This yields a rich class of actions which has been extensively studied in the case $G = \mathbb{Z}^d$.

Example

When $f = n \cdot 1$ we obtain the Bernoulli action $G \curvearrowright \{1, \ldots, n\}^G$ with uniform weights.

Algebraic actions

Example

When $G = \mathbb{Z}$ and $f = 1 + a_1 u + a_2 u^2 + \cdots + a_{n-1} u^{n-1} + u^n$ we obtain the automorphism T_A of $\mathbb{T}^n$ defined by the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -1 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$$

Algebraic actions

Lind, Schmidt, and Ward proved that, in the case $G = \mathbb{Z}^d$, the entropy α_f with respect to the Haar measure is equal to the logarithm of the Mahler measure $M(f) := \exp \int_{\mathbb{T}^d} \log |f(t)|\, dt$ where the integral is with respect to normalized Haar measure.

Later Deninger showed that, for a larger class of amenable groups G, if f is invertible in $\ell^1(G)$ then the entropy of α_f is equal to the logarithm of the Fuglede-Kadison determinant

$$\det_{\mathcal{L}G} f := \exp \tau(\log |f|).$$

More generally:

Theorem (Li)

Suppose that G is amenable and f is invertible in $\mathcal{L}G$. Then

$$h_\mu(\alpha_f) = \log \det_{\mathcal{L}G} f.$$

Algebraic actions

Suppose now that G is residually finite. Let $\{G_i\}_{i=1}$ be a sequence of finite-index normal subgroups of G with $\bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} G_i = \{e\}$, and let $\Sigma = \{\sigma_i : G \to \mathrm{Sym}(G/G_i)\}_{i=1}^{\infty}$ be the associated sofic approximation sequence.

Theorem (Bowen)

Suppose that f is invertible in $\ell^1(G)$. Then

$$h_{\Sigma,\mu}(\alpha_f) = \log \det_{\mathcal{L}G} f.$$

The invertibility of f in $\ell^1(G)$ is important in Bowen's argument because it implies the existence of a finite generating partition.

Algebraic actions

Using the variational principle, one can show the following.

Theorem (K.-Li)

Suppose that f is invertible in $C^*(G)$. Then

$$h_\Sigma(\alpha_f) = \log \det_{\mathcal{L}G} f.$$

The entropy in this case is equal to the exponential growth rate of the number of G_i-fixed points.

Local entropy of algebraic actions

Given an action $G \curvearrowright X$ on a compact space, we say that a pair $(x_0, x_1) \in X \times X$ is an **IE-pair** if for all open sets $U_0 \ni x_0$ and $U_1 \ni x_1$ there is a $\lambda > 0$ such that for every finite set $F \subseteq G$ there is an $F' \subseteq F$ with $|F'| \geq \lambda |F|$ such that $\bigcap_{s \in F'} sU_{\sigma(s)} \neq \emptyset$ for every $\sigma : F' \to \{0, 1\}$.

Suppose that $G \curvearrowright X$ is an algebraic action. The set of all $x \in X$ such that (x, e_X) is an IE-pair forms a group, called the **IE group**. Write $\Delta(X)$ for the set of all **homoclinic** points, i.e., points $x \in X$ such that $\lim_{s \to \infty} sx = e_X$.

Local entropy of algebraic actions

Theorem (Chung-Li)

Let $G \curvearrowright X$ be an expansive algebraic action of a polycyclic-by-finite group. Then

1. $\Delta(X)$ *is dense in the IE group,*
2. $\Delta(X)$ *is nonempty iff the action has positive entropy,*
3. $\Delta(X)$ *is dense in* X *iff the action has completely positive entropy.*

Theorem (Rudolph-Schmidt)

An algebraic $\mathbb{Z}^d$*-action has completely positive entropy iff it is conjugate to a Bernoulli shift.*