Eberhard Kirchberg and the Development of Quantized Functional Analysis

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- A return to days of yore (classical convexity)

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- Operator systems come roaring back (the "Paulsen school") and a concrete description of the quantum square
- Kirchberg's theorem, Kirchberg's problem, and a torrent of new results

How I first met Eberhard.

- (1988-9?) Asked by Hirzebruch (?) to comment on Eberhard's manuscripts
- (May 29, 1989) Sent a letter of thanks to Eberhard from Institut Mittag-Leffler for copies of his manuscripts.
  "The results are *outstanding* and are certainly the best C*-algebraic structure theorems to appear in the last few years."
- (June 1989) Visited Eberhard in Heidelberg, and was awed by the stacks of his unpublished notes.

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Concrete: $1 \in V \subseteq C_{\mathbb{R}}(X)$ (or $L^{\infty}_{\mathbb{R}}(X)$)

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$$\begin{array}{ccc} W & & \\ \cup| & \overset{\exists\Phi}{\searrow} & \\ V & \overset{\varphi}{\rightarrow} & Z \end{array} \qquad \varphi, \Phi \text{ morphisms}$$

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Need: Riesz interpolation property (RIP), i.e., if one has

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then there exists an element z_0 with $c_i \leq z_0 \leq b_j$. Equivalently (RDP) $a_1 + a_2 = b_3 + b_4, a_i, b_j \geq 0 \Rightarrow \exists c_{ij} \geq 0 : a_i = \sum_j c_{ij}, b_j = \sum_i c_{ij}$.

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$V_+ \otimes W_+$ cone generated by $v \otimes w : v \in V_+, w \in W_+$

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$$(V \otimes_{max} W)_+^* = Pos(V, W^*)$$

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The classical square:

$$V_\square = Aff(\square) = \{a \in \mathbb{R}^4 : a_1 + a_2 = a_3 + a_4\}$$

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4. *$Z \otimes_{\min} V_\square = Z \otimes_{\max} V_\square$,*
5. *Z^{**} is injective.*

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Given linear $\varphi : \mathcal{V} \to \mathcal{W}$, define

$$\varphi_n : M_n(\mathcal{V}) \to M_n(\mathcal{W}) : [v_{ij}] \mapsto [\varphi(v_{i,j})]$$

Definition: $\varphi : \mathcal{V} \to \mathcal{W}$ is *completely positive* if $\varphi_n \geq 0$ for all n.

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Morphisms completely positive unital mappings $\mathcal{V} \to \mathcal{W}$.

Abstract Definition (Choi,E) $(\mathcal{V}, *, M_n(\mathcal{V})_+, I)$, $M_n(\mathcal{V})_{sa}$ is a function space, etc. $\Rightarrow$ the cones $M_n(\mathcal{V})_+$ form a matrix convex family.

Arveson-Wittstock-Hahn-Banach Theorem

Def: An operator system $\mathcal{Z}$ is *injective* if

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Theorem (Connes, et al)

A von Neumann algebra $\mathcal{Z}$ is injective $\Leftrightarrow$ $\mathcal{Z}$ is a hyperfinite von Neumann algebra.

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Key Question: Why is $\mathcal{Z} = \mathbb{M}_n$ injective?

Does $\mathcal{Z} = \mathbb{M}_n$ satisfy a matrix convex analogue of the lattice property, or of the Riesz interpolation property?

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Lemma: $\mathcal{V}$ floppy $\Rightarrow \mathcal{V}^*$ is floppy.

- We only need norms, not matrix norms on $\mathcal{V} \otimes \mathcal{W}$.

Tensor products of floppy operator systems

$\mathcal{V} \otimes \mathcal{W}$ and $\mathcal{V}^* \otimes \mathcal{W}^*$ are in duality.

Given $v \in M_n(\mathcal{V})$ and $w \in M_n(\mathcal{W})$,

$$v \times w = \sum v_{ij} \otimes w_{ij} \in \mathcal{V} \otimes \mathcal{W}$$

$$\mathcal{V}_+ \tilde{\otimes} \mathcal{W}_+ = \{v \times w : v \in M_n(V)_+, w \in M_n(W)_+\}$$

$$\begin{aligned} P_m(\mathcal{V} \otimes \mathcal{W}) &= (\mathcal{V}_+^* \tilde{\otimes} \mathcal{W}_+^*)^o \\ P_M(\mathcal{V} \otimes \mathcal{W}) &= (V_+ \tilde{\otimes} W_+)^{oo} \end{aligned}$$

Write $\mathcal{V} \otimes_m \mathcal{W}$ and $\mathcal{V} \otimes_M \mathcal{W}$ for the corresponding ordered spaces.

$$\begin{aligned} (\mathcal{V} \otimes_M \mathcal{W})^* &= \mathcal{V}^* \otimes_m \mathcal{W}^* = L_{cp}(\mathcal{V}, \mathcal{W}^*) \\ (\mathcal{V} \otimes_m \mathcal{W})^* &\simeq \mathcal{V}^* \otimes_M \mathcal{W}^* = L_{nuc}(\mathcal{V}, \mathcal{W}^*) \end{aligned}$$

A non-commutative square

Consider the matrix system

$$N_{2n} = \{v \in M_{2n} : v_{1,1} + \ldots + v_{n,n} = v_{n+1,n+1} + \ldots + v_{2n,2n}\}$$

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Theorem (Choi-Effros, Effros (1977))

If R is a von Neumann algebra, then the following are equivalent:

- *R is injective*
- *$N_{2n} \otimes_M R = N_{2n} \otimes_m R$ for all n*
- *$N_{2n} \otimes_M R_* = N_{2n} \otimes_m R_*$ for all n.*

Theorem (Choi-Effros)

Although it is finite-dimensional, N_2^* is not completely order isomorphic to a matrix system.

The categorical problems with function systems and operator systems that led to their eclipse

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- By contrast, function spaces (i.e., normed spaces) and operator spaces (quantized norm spaces) are closed under these operations, and there is a VAST literature for normed (or Banach) spaces just waiting for quantization.
- Is there a concrete realization for the non-commutative square?

*In fact they may often be regarded as "local function spaces" – used by E - Lance to characterize semidiscretness

The quantized square, quantized polytopes and Kirchberg's tensor product theorem

Theorem (The quantized square – Farenick-Paulsen (2011))

$$N_2^* \cong \mathbb{C}z + \mathbb{C}1 + \mathbb{C}z^{-1} \subseteq C(\mathbb{T}).$$

Let $\mathcal{E}_n$ be the diagonal matrices $D \in \mathbb{M}_n$ such that $\mathrm{trace}(D) = 0$ Let $w_1, \ldots, w_{n-1}$ be the generators of $\mathbb{F}_{n-1}$, and let $\mathcal{W}_n \subseteq C^*(\mathbb{F}_{n-1})$ be the operator system generated by $w_i w_j^*$.

Theorem (Hints of quantized pohyhedra – Farenick-Paulsen (2011))

$$\mathcal{E}_n^* \cong \mathcal{W}_n \subseteq C^*(\mathbb{F}_{n-1}).$$

Quotients and the Kirchberg tensor product theorem.

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See, e.g., Kavruk, Paulsen, Todorov, and Tomforde, Arkiv. Examples:

- $$\mathcal{W}_n \cong \mathbb{M}_n / \mathcal{J}_n$$

  where $\mathcal{J}_n = \{\text{diagonal}\, D : \, \text{trace} D = 0\}$
- Let $\mathcal{S}_{n-1} = span\{1, u_j, u_j^*\} \subseteq C^*(\mathbb{F}_{n-1})$, and $\mathcal{T}_n$ be the tridiagonal matrices. Then

  $$\mathcal{S}_{n-1} \cong \mathcal{T}_n / \mathcal{J}_n$$
- The fact that duals and quotients of matrix spaces lead to subspaces of $C^*(\mathbb{F}_{n-1})$ provides an elegant proof of the fundamental theorem of Kirchberg:

$$C^*(\mathbb{F}_{n-1}) \otimes_{min} B(H) = C^*(\mathbb{F}_{n-1}) \otimes_{max} B(H)$$

Operator Systems are here to stay (Other results)

Theorem (Farenick, Kavruk,Paulsen)

If $\mathcal{A}$ is a unital C^*-algebra on a Hilbert space H, then it has the WEP if and only if it has the following "relative" property: If for any $p \in \mathbb{N}$ and $A, B, C \in M_p(B(H))$ for which $A + B + C = I$ and

$$\begin{bmatrix} A & X_1 & 0 \\ X_1^* & B & X_2 \\ 0 & X_2 & C \end{bmatrix}$$

is strictly positive in $M_{3p}(B(H))$, then there also exist $\tilde{A}, \tilde{B}, \tilde{C} \in M_p(\mathcal{A})$ with the same properties.

Theorem (Kavruk)

The Smith Ward problem for matrix numerical ranges has a positive solution if and only if every three-dimensional operator system is exact.*

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The Smith Ward problem for matrix numerical ranges has a positive solution if and only if every three-dimensional operator system is exact.*

*Let $q : B(H) \rightarrow B(H)/K(H)$. Then does there exist a single $K \in K(H)$ such that $T \in B(H) \Rightarrow w_n(T + K) = w_n(q(T))$ for all n??

Operator system equivalents of the Kirchberg conjecture (= Connes conjecture) $C^*(\mathbb{F}_n) \otimes_{min} C^*(\mathbb{F}_n) = C^*(\mathbb{F}_n) \otimes_{max} C^*(\mathbb{F}_n)$???

AND LOTS MORE!! –SEE ARXIV.