The C*-algebra of a vector bundle

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The Cuntz algebra O_n

$n = 1 \Rightarrow O_n = C(\mathbb{T})$.

$n \geq 2 \Rightarrow O_n$ is simple, nuclear, purely infinite

$n \geq 2 \Rightarrow K_0(O_n) = \mathbb{Z}/(n-1), \quad K_1(O_n) = 0$

Pimsner: given a Hilbert bimodule over a C*-algebra A,
$E = {}_A E_A$
one can associate to it a C*-algebra O_E.

If $E = {}_{\mathbb{C}}\mathbb{C}^n_{\mathbb{C}}$ then $O_E = O_n$.

Cuntz-Pimsner algebra O_E

E fin. gen. projective **right** Hilbert module over C*-algebra A.
Scalar product $< \cdot, \cdot >: E \times E \to A$ is given.
Left module structure given by $A \to L(E_A)$ injective $*$-homomorphism.

$$E = {}_A E_A \Rightarrow \text{Cuntz-Pimsner algebra } O_E.$$

$$\mathcal{E} = \oplus_{n \geq 0} E^{\otimes n} = A \oplus E \oplus (E \otimes_A E) \oplus (E \otimes_A E \otimes_A E) \oplus \dots$$

The Toeplitz (or tensor) C*-algebra generated by the multiplication operators $T_\xi : \mathcal{E} \to \mathcal{E}$, $T_\xi(\eta) = \xi \otimes \eta$, where $\xi \in E$, $\eta \in \mathcal{E}$, $(T_\xi(a) = \xi a)$.

$$T_E = C^*\{T_\xi : \xi \in E\} \subset L(\mathcal{E})$$

O_E is the quotient of T_E by the ideal of "compact operators" $K(\mathcal{E})$.

$$0 \to K(\mathcal{E}) \to T_E \to O_E \to 0.$$

Examples

(1). $\alpha \in Aut(A) \Rightarrow E := A$ is A-bimodule: $a_1 \cdot \mathbf{a} \cdot a_2 = \alpha(a_1)\, \mathbf{a}\, a_2$
$O_E \cong A \rtimes_\alpha \mathbb{Z}$.

(2). $E = {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}$ is $\mathbb{C}$-bimodule $\Rightarrow$ classical Toeplitz extension

$$0 \to \mathcal{K} \to \mathcal{T} \to C(\mathbb{T}) \to 0, \quad O_E = C(\mathbb{T}).$$

Recall: $T_{\xi_1}^* T_{\xi_2} =< \xi_1, \xi_2 > \otimes I, \quad T_{\xi_1} T_{\xi_2}^* = \xi_1 < \xi_2, \cdot > \otimes I$

(3). $E = {}_{\mathbb{C}}\mathbb{C}^n_{\mathbb{C}}$ is $\mathbb{C}$-bimodule: $\Rightarrow O_E \cong O_n$ the Cuntz algebra.
Indeed if $\xi_1, ..., \xi_n$ is orthonormal basis of $\mathbb{C}^n$, then
$< \xi_i, \xi_j >= \delta_{ij} \Rightarrow T_{\xi_i}^* T_{\xi_j} = \delta_{ij} 1$
$\xi_1 < \xi_1, \cdot > + \cdots \xi_n < \xi_n, \cdot >= 1 \Rightarrow T_{\xi_1} T_{\xi_1}^* + \cdots T_{\xi_n} T_{\xi_n}^* = P_{\mathcal{E} \ominus \mathbb{C}}$.

If $E = {}_{C(X)}C(X)^n_{C(X)} \Rightarrow O_E \cong C(X) \otimes O_n$.

The C*-algebra of a vector bundle

$E \in Vect_n(X)$ complex vector bundle of rank n over compact space X

Hermitian metric on $E \Rightarrow < \cdot, \cdot >$: $E \times E \to C(X)$ scalar product.
$E \equiv \{$cont. sections in $E\}$ is a fin. gen. projective Hilbert $C(X)$-module.

$O_E :=$ Cuntz-Pimsner algebra associated to the $C(X)$-bimodule E.

The isomorphism class of E and hence of O_E is independent of the metric.
O_E locally trivial C*-bundle with fibers O_n.

A question of Cuntz:
What are the invariants of E captured by the C*-bundle O_E?

How are E and F related if $O_E \cong O_F$ as C*-bundles?

Which C*-bundles with fiber O_n can be realized as Cuntz-Pimsner algebras of vector bundles of rank n?

Invariants of O_E

Theorem (Pimsner)

$$\begin{array}{ccccc} K_0(A) & \longrightarrow & K_0(A) & \longrightarrow & K_0(O_E) \\ \uparrow & & & & \downarrow \\ K_1(O_E) & \longleftarrow & K_1(A) & \longleftarrow & K_1(A) \end{array}$$

If $E \in Vect_n(X)$, $E = {}_{C(X)}E_{C(X)}$

$$\begin{array}{ccccc} K^0(X) & \xrightarrow{1-[E]} & K^0(X) & \xrightarrow{\iota} & K_0(O_E) \\ \uparrow & & & & \downarrow \\ K_1(O_E) & \xleftarrow[\iota]{} & K^1(X) & \xleftarrow[1-[E]]{} & K^1(X) \end{array}$$

ι is induced by $C(X) \hookrightarrow O_E$.

Let $E, F \in \mathrm{Vect}(X)$. $E = {}_{C(X)}E_{C(X)}$, $F = {}_{C(X)}F_{C(X)}$.

Proposition

$O_E \cong O_F \Rightarrow (1-[E])K^0(X) = (1-[F])K^0(X)$.

Proof: Suppose that $\exists\, \Phi : O_E \cong O_F$ $C(X)$-linear isomorphism.

$$\begin{array}{ccccccc}
K^0(X) & \xrightarrow{1-[E]} & K^0(X) & \xrightarrow{\iota_E} & K_0(O_E) & \longrightarrow & K^1(X) \\
 & & \Vert & & \downarrow \Phi_* & & \\
K^0(X) & \xrightarrow{1-[F]} & K^0(X) & \xrightarrow{\iota_F} & K_0(O_F) & \longrightarrow & K^1(X)
\end{array}$$

Φ_* bijective $\Rightarrow \ker \iota_E = \ker \iota_F \Rightarrow$

$$(1-[E])K^0(X) = (1-[F])K^0(X).$$

Have seen that the ideal $(1-[E])K^0(X)$ is invariant of O_E.

$K^0(X) \cong \mathbb{Z} \oplus \widetilde{K}^0(X)$. $E \in \mathrm{Vect}_{m+1}(X) \Rightarrow [E] = (m+1) + [\widetilde{E}]$.

Define equiv. relation on $\widetilde{K}^0(X)$:

$a \sim b \Leftrightarrow a = b + mh + bh \Leftrightarrow m + a = (m+b)(1+h)$ for some $h \in \widetilde{K}^0(X)$

Let $E, F \in \mathrm{Vect}_{m+1}(X)$

Remark

$(1-[E])K^0(X) = (1-[F])K^0(X)$ if and only if $[\widetilde{E}] \sim [\widetilde{F}]$.

Conclude that the equiv. class of $[\widetilde{E}]$ in $\widetilde{K}^0(X)/\sim$ is an invariant of O_E:

$$O_E \cong O_F \Rightarrow [\widetilde{E}] \sim [\widetilde{F}].$$

How many equivalence classes are there in general?

Proposition

Let X be a finite connected CW-complex. If $\mathrm{Tor}(H^*(X,\mathbb{Z}),\mathbb{Z}/m)=0$, then $|\widetilde{K}^0(X)/\sim| = |\widetilde{K}^0(X)\otimes\mathbb{Z}/m| = |\widetilde{H}^{even}(X,\mathbb{Z}/m)|$.

Proof is involved. One uses the Atiyah-Hirzebruch spectral sequence to deduce that $\widetilde{K}^0(X)\otimes\mathbb{Z}/m \cong \widetilde{H}^{even}(X,\mathbb{Z})\otimes\mathbb{Z}/m$ and

$$\mathrm{Tor}(K^0_{2q}(X)/K^0_{2q+2}(X),\mathbb{Z}/m)=0,$$

where $K^0_q(X)=\ker\left(K^0(X)\to K^0(X_{q-1})\right)$ is the skeleton filtration of $K^0(X)$. This reduces the statement to algebra problem in filtered rings.

$\mathcal{O}_{m+1}(X) :=$ isom. classes of loc. triv. C*-bundles with fiber O_{m+1}.

Corollary

$|\mathcal{O}_{m+1}(X)| \geq |\widetilde{K}^0(X)\otimes\mathbb{Z}/m| \quad \text{if } m+1\geq \lceil d/2\rceil . \qquad d=\dim(X)$

Proof: m large $\Rightarrow$ the map $\mathrm{Vect}_{m+1}(X)\to\widetilde{K}^0(X)\to\widetilde{K}^0(X)/\sim$ is onto.

Next we'll give an upper estimate for $|\mathcal{O}_{m+1}(X)|$.

$\mathcal{O}_{m+1}(X) :=$ isom. classes of loc. triv. C*-bundles with fiber O_{m+1}.

$$\mathcal{O}_{m+1}(X) \cong [X, B\,\mathrm{Aut}(O_{m+1})]$$

Theorem

$$\pi_n(\mathrm{Aut}(O_{m+1})) : \quad 0 \quad \mathbb{Z}/m \quad 0 \quad \mathbb{Z}/m \quad 0 \quad \mathbb{Z}/m \quad 0 \cdots$$

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(This answers a question of Cuntz from Neptune Conference).
Thus the classifying space $B\,\mathrm{Aut}(O_{m+1})$ has homotopy groups:

$$0 \quad 0 \quad \mathbb{Z}/m \quad 0 \quad \mathbb{Z}/m \quad 0 \quad \mathbb{Z}/m \quad 0 \cdots$$

.

Proposition

Let X be a finite connected CW complex and let $m \geq 1$ be an integer. Then $|\mathcal{O}_{m+1}(X)| \leq |\widetilde{H}^{even}(X, \mathbb{Z}/m)|$.

Proof:

Eilenberg-Mc Lane spaces

$$[X, K(\mathbb{Z}/m, 2k)] \cong H^{2k}(X; \mathbb{Z}/m)$$

Use Postnikov decomposition. Thus $B\,\mathrm{Aut}(O_{m+1})$ is a twisted product

$$B\,\mathrm{Aut}(O_{m+1}) = K(\mathbb{Z}/m, 2) \rtimes K(\mathbb{Z}/m, 4) \rtimes K(\mathbb{Z}/m, 8) \rtimes \ldots$$

Since

$$\mathcal{O}_{m+1}(X) \cong [X, B\,\mathrm{Aut}(O_{m+1})]$$

$$|[X, B\,\mathrm{Aut}(O_{m+1})]| \leq \prod_{k \geq 2} |H^{2k}(X, \mathbb{Z}/m)| = |\widetilde{H}^{even}(X, \mathbb{Z}/m)|.$$

Theorem

If $m \geq \lceil \frac{d}{2} \rceil - 1$ *and* $\mathrm{Tor}(H^*(X,\mathbb{Z}),\mathbb{Z}/m) = 0$*, then*

(1) each element of $\mathcal{O}_{m+1}(X)$ *is isomorphic to* O_E *for some* $E \in Vect_{m+1}(X)$.

(2) $O_E \cong O_F \Leftrightarrow (1-[E])K^0(X) = (1-[F])K^0(X)$

(3) $|\mathcal{O}_{m+1}(X)| = |\widetilde{H}^{even}(X,\mathbb{Z}/m)|$.

Proof: lower estimate = upper estimate (good to be lucky)

$$|\widetilde{H}^{even}(X,\mathbb{Z}/m)| = |\widetilde{K}^0(X)/\sim| \leq |\mathcal{O}_{m+1}(X)| \leq |\widetilde{H}^{even}(X,\mathbb{Z}/m)|.$$

Computable invariants of O_E

Definition

$$p_n(x) = \ell(n)\, m^n \, \log(1 + \frac{x}{m})_{[n]} \in \mathbb{Z}[x],$$

$$p_n(x) = \ell(n) m^n (\frac{x}{m} - \frac{x^2}{2m^2} + \frac{x^3}{3m^3} + \cdots + (-1)^{n-1}\frac{x^n}{nm^n})$$

$\ell(n) = \mathrm{lcm}\{1, 2, ..., n\}$. The first five polynomials in the sequence are:

$$p_1(x) = x,$$
$$p_2(x) = 2mx - x^2,$$
$$p_3(x) = 6m^2x - 3mx^2 + 2x^3,$$
$$p_4(x) = 12m^3x - 6m^2x^2 + 4mx^3 - 3x^4,$$
$$p_5(x) = 60m^4x - 30m^3x^2 + 20m^2x^3 - 15mx^4 + 12x^5.$$

Theorem

Let X be a finite connected CW complex of dimension d and let $E, F \in Vect_{m+1}(X)$. Set $n = [d/2]$.

$$O_E \cong O_F \Rightarrow p_n([\widetilde{E}]) - p_n([\widetilde{F}]) \text{ is divisible by } m^n \text{ in } \widetilde{K}^0(X).$$

"Proof": Then $\widetilde{K}^0(X)^{n+1} = 0$. Set $a = [\widetilde{E}]$ and $b = [\widetilde{F}]$. We have seen that $O_E \cong O_F \Rightarrow (m+a) = (m+b)(1+h)$ for some $h \in \widetilde{K}^0(X)$.

$$(m+a) = (m+b)(1+h) \Rightarrow (1+\tfrac{a}{m}) = (1+\tfrac{b}{m})(1+h)$$

$$\log(1+\tfrac{a}{m}) = \log(1+\tfrac{b}{m}) + \log(1+h)$$

If $x^{n+1} = 0$, $\log(1+x/m) = x/m - x^2/2m^2 + \cdots + (-1)^{n-1}x^n/nm^n$. Multiply by $\ell(n) = \mathrm{lcm}\{1, 2, ..., n\}$ and m^n to get integral classes:

$$m^n\ell(n)\log(1+\tfrac{a}{m}) = m^n\ell(n)\log(1+\tfrac{b}{m}) + m^n(\ell(n)\log(1+h))$$

Is p_n a complete invariant?

Theorem

Suppose $\mathrm{Tor}(H^*(X,\mathbb{Z}),\mathbb{Z}/m)=0$ *and* $\gcd(m,n!)=1$*. Then:*
$O_E \cong O_F \Leftrightarrow p_n([\widetilde{E}])-p_n([\widetilde{F}])$ *is divisible by* m^n *in* $\widetilde{K}^0(X)$*.*

Both conditions are necessary.

Example

If $dim(X)\leq 9$ and m is not divisible by 2 or 3, then the image of

$$p_4(x)=12m^3[\widetilde{E}]-6m^2[\widetilde{E}]^2+4m[\widetilde{E}]^3-3[\widetilde{E}]^4$$

in $\widetilde{K}^0(X)\otimes\mathbb{Z}/m^4$ is a complete invariant of O_E.

Which characteristic classes of E are invariants of O_E?

For each $n \geq 1$ consider the polynomial $q_n \in \mathbb{Z}[x_1, ..., x_n]$:

$$\sum_{k_1+2k_2+...+nk_n=n} (-1)^{k_1+\cdots+k_n-1} m^{n-(k_1+\cdots+k_n)} \frac{n!\,(k_1+\cdots+k_n-1)!}{1!^{k_1}\cdots n!^{k_n}\,k_1!\cdots k_n!}\, x_1^{k_1}\cdots x_n^{k_n}$$

$$q_1(x_1) = x_1$$
$$q_2(x_1, x_2) = mx_2 - x_1^2$$
$$q_3(x_1, x_2, x_3) = m^2 x_3 - 3mx_1x_2 + 2x_1^3, \quad \text{etc}$$

Theorem

Let $E \in Vect_{m+1}(X)$. Then the classes $q_n(\dot{ch}_1(E), ..., \dot{ch}_n(E)) \in H^{2n}(X, \mathbb{Z}/m^n)$ are invariants of O_E.

Here ch_n are the components of the Chern character: $ch = \sum_n ch_n/n!$

By expressing ch_n in terms of Chern classes we obtain a sequence of characteristic classes of E which are invariants of O_E. For illustration, the first three classes in the sequence are:

(1) $\dot{c}_1(E) \in H^2(X; \mathbb{Z}/m)$

(2) $(m-1)\dot{c}_1(E)^2 - 2m\dot{c}_2(E) \in H^4(X; \mathbb{Z}/m^2)$

(3) $(m^2 - 3m + 2)\dot{c}_1(E)^3 - (3m^2 - 6m)\dot{c}_1(E)\dot{c}_2(E) + 3m^2\dot{c}_3(E) \in H^6(X; \mathbb{Z}/m^3)$

Corollary

Let L and L' be two line bundles over X. If $O_{m+L} \cong O_{m+L'}$, then $q_n(1, ..., 1)(c_1(L)^n - c_1(L')^n)$ is divisible by m^n in $H^{2n}(X, \mathbb{Z})$ for all $n \geq 1$.

Thus

$c_1(L) - c_1(L')$ divisible by m

$(m-1)(c_1(L)^2 - c_1(L')^2)$ divisible by m^2

$(m^2 - 3m + 2)(c_1(L)^3 - c_1(L')^3)$ divisible by m^3

etc

We have seen that $O_E \cong O_F \Leftrightarrow (1-[E])K^0(X) = (1-[F])K^0(X)$ in the absence of m-torsion. The proof by algebraic topology methods gives no geometric insight. Why should $(1-[E])K^0(X) = (1-[F])K^0(X)$ imply the isomorphism $O_E \cong O_F$?

Approach this question using parametrized KK-theory: $KK_X(A, B)$. By work of Kirchberg it is known that two separable unital C*-bundles with simple nuclear purely infinite fibers are isomorphic iff there is $\alpha \in KK_X(A, B)^{-1}$ which maps $[1_A]$ to $[1_B]$.

Perhaps the condition $(1-[E])K^0(X) = (1-[F])K^0(X)$ does assure the existence of such KK_X-equivalence α.

Theorem

Let X be a compact metrizable space and let $E, F \in \mathrm{Vect}_{m+1}(X)$ be complex vector bundles $m \geq 1$. Then $O_E \cong O_F$ as $C(X)$-algebras if and only if $(1-[E])K^0(X) = (1-[F])K^0(X)$.

"Proof" KK_X is a triangulated category (Meyer-Nest). Pimsner's methods can be used to deduce existence of a sequence of $C(X)$-algebras and KK_X-maps

$$SC(X) \hookrightarrow SO_E \to C(X) \xrightarrow{1-[E]} C(X)$$

which is an exact triangle in KK_X. The condition $(1-[E]) = (1-[F])H$ with H of virtual rank one leads to an element $\Phi_H \in KK_X(SO_E, SO_F)^{-1}$ such that the following diagram is commutative.

$$\begin{array}{ccccccc} SC(X) & \longrightarrow & SO_E & \longrightarrow & C(X) & \xrightarrow{1-[E]} & C(X) \\ \| & & \downarrow{\scriptstyle \Phi_H} & & {\scriptstyle H}\downarrow & & \| \\ SC(X) & \longrightarrow & SO_F & \longrightarrow & C(X) & \xrightarrow{1-[F]} & C(X) \end{array}$$

$$KK_X(C(X), C(X)) \cong KK(\mathbb{C}, C(X)) \cong K^0(X)$$

is an isomorphism of rings.
Unsuspend the commutative diagram

$$\begin{array}{ccc} SC(X) & \longrightarrow & SO_E \\ \| & & \downarrow \Phi_H \\ SC(X) & \longrightarrow & SO_F \end{array}$$

to obtain the existence of a KK_X-equivalence from O_E to O_F which preserves the class of [1]. One concludes that $O_E \cong O_F$ by Kirchberg's isomorphism theorem for C*-bundles.

The same argument yields:

Theorem

Let X be a compact metrizable space and let $E, F \in \mathrm{Vect}(X)$ be complex vector bundles of rank ≥ 2. Then O_E embeds as a unital $C(X)$-subalgebra of O_F if and only if $(1-[E])K^0(X) \subset (1-[F])K^0(X)$.

If $X = point$,

O_{m+1} embeds unitally in $O_{n+1} \Leftrightarrow m\mathbb{Z} \subset n\mathbb{Z} \Leftrightarrow n$ divides m.

This happens precisely when there is a morphism of pointed groups
$(K_0(O_{m+1}), [1]) \to (K_0(O_{n+1}), [1]), \quad (\mathbb{Z}/m, \bar{1}) \to (\mathbb{Z}/n, \bar{1})$.
Remark: Hard to decide when two elements in $K^0(\mathbb{CP}^n) = \mathbb{Z}[x]/(x^{n+1})$ generate same ideal.

$\mathcal{O}_{m+1}(X)$ and the image of the map $\mathrm{Vect}_{m+1}(X) \to \mathcal{O}_{m+1}(X)$ are well understood for suspensions $X = SY$. Use the universal coefficient

$$0 \to K^0(X) \otimes \mathbb{Z}/m \xrightarrow{\bar{\rho}} K^0(X, \mathbb{Z}/m) \xrightarrow{\beta} \mathrm{Tor}(K^1(X), \mathbb{Z}/m) \to 0,$$

β is the Bockstein operation and $\bar{\rho}$ is the coefficient map.

Theorem

Suppose $X = SY$. Then:

(i) *There is a bijection $\gamma : \mathcal{O}_{m+1}(X) \to K^0(X, \mathbb{Z}/m)$.*

(ii) *If $E, F \in Vect_{m+1}(X)$, then $O_E \cong O_F$ iff $[E] - [F] \in mK^0(X)$.*

(iii) *$A \in \mathcal{O}_{m+1}(X)$ is isomorphic to O_E for some $E \in Vect_{m+1}(X)$ iff $\beta(\gamma(A)) = 0$.*

(iii) assumes that $m \geq \lceil d/2 \rceil - 1$ so that $Vect_{m+1}(X) \to \widetilde{K}^0(X)$ is surjective. This simple description is possible since $\widetilde{K}^0(SY)^2 = 0$.