Noncommutative PL-Topology

Bruce Blackadar

Happy 65th, Eberhard!

Piecewise-Linear Topology

PL-Topology is the study of topological spaces via triangulations and piecewise-linear maps.

For us, "space" means "compact metrizable space."

All C*-algebras will be assumed separable.

Definition:

A *triangulation* of a space X is a homeomorphism from X to the underlying space of a (finite) simplicial complex.

Definition:

A *piecewise-linear map* between simplicial complexes is a continuous map between the underlying spaces which is linear (affine) on each subsimplex.

Triangulations are related to open covers.

If $\mathcal{U}$ is a (finite) open cover of X, the *nerve* of $\mathcal{U}$ is the simplicial complex whose vertices are the open sets in $\mathcal{U}$, with the subsimplex spanned by $U_{i_1}, \ldots, U_{i_m}$ included if and only if

$$U_{i_1} \cap \cdots \cap U_{i_m} \neq \emptyset \ .$$

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$$U_{i_1} \cap \cdots \cap U_{i_m} \neq \emptyset \ .$$

There is no natural map from X to the nerve of $\mathcal{U}$.

If $\mathcal{V}$ is a refinement of $\mathcal{U}$, there is no natural map from the nerve of $\mathcal{V}$ to the nerve of $\mathcal{U}$. However, there is a unique well-defined PL-homotopy class of piecewise-linear maps.

The nerves of fine open covers carry the essential homotopy information about X (Čech cohomology).

We want a finer object which gives actual maps.

Partitions of Unity

Definition:

A *partition of unity* on a space X is a set

$$\{f_1, \ldots, f_n\}$$

of continuous functions from X to $[0, 1]$ such that

$$\sum_{k=1}^{n} f_k(x) = 1$$

for all $x \in X$.

We will assume that each f_k takes the value 1 somewhere (nondegeneracy).

A triangulation gives a partition of unity using coordinates. More generally, any continuous function from X to a simplicial complex gives a partition of unity, provided all vertices are in the range.

Conversely, a partition of unity $\mathcal{P} = \{f_1, \ldots, f_n\}$ gives an open cover

$$\mathcal{U}_{\mathcal{P}} = \{U_1, \ldots, U_n\}$$

where

$$U_k = \{x \in X : f_k(x) > 0\} .$$

The nondegeneracy condition says that this is a minimal open cover.

There is then a continuous function $\gamma_{\mathcal{P}}$ from X to the nerve of $\mathcal{U}_{\mathcal{P}}$ defined by sending $x \in X$ to the point with coordinates

$$(f_1(x), \ldots, f_n(x)) \ .$$

There is thus a natural one-one correspondence between partitions of unity on X and *weak triangulations* of X: continuous functions from X to a simplicial complex for which all vertices are in the range.

Refinement of Partitions of Unity

Definition:

If $\mathcal{P} = \{f_1, \ldots, f_n\}$ and $\mathcal{Q} = \{g_1, \ldots, g_m\}$ are partitions of unity on X, then $\mathcal{Q}$ *refines* $\mathcal{P}$ if there are scalars α_{ij} such that

$$f_i = \sum_{j=1}^{m} \alpha_{ij} g_j$$

for all i.

The α_{ij} are necessarily in $[0, 1]$, and for each j we have

$$\sum_{i=1}^{n} \alpha_{ij} = 1 \ .$$

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If $\mathcal{Q}$ refines $\mathcal{P}$, then the open cover $\mathcal{U}_\mathcal{Q}$ refines the open cover $\mathcal{U}_\mathcal{P}$, and there is a natural PL-map $\gamma_{\mathcal{QP}}$ from the nerve of $\mathcal{U}_\mathcal{Q}$ to the nerve of $\mathcal{U}_\mathcal{P}$ defined by

$$\gamma_{\mathcal{QP}}(\lambda_1, \ldots, \lambda_m) = \left(\sum_{j=1}^{m} \alpha_{1j}\lambda_j, \ldots, \sum_{j=1}^{m} \alpha_{nj}\lambda_j \right) .$$

These maps satisfy

$$\gamma_\mathcal{P} = \gamma_{\mathcal{QP}} \circ \gamma_\mathcal{Q}$$

as maps from X to the nerve of $\mathcal{U}_\mathcal{P}$.

PL-Structures on a Space

Definition

A PL-structure on a space X is a sequence $(\mathcal{P}_n)$ of partitions of unity on X, each refining the previous one, such that the sequence of corresponding open covers $\mathcal{U}_n$ eventually refines any open cover of X.

Equivalently, $\cup\mathcal{U}_n$ is a base for the topology of X.

It is not obvious that such a PL-structure exists on a given X. There are various ways to prove this; one of the best (in my humble opinion) is using our theorem.

If $(\mathcal{P}_n)$ is a PL-structure on X, then as above we get continuous maps γ_n from X to the nerve X_n of $\mathcal{U}_n$, and for $n < m$ a PL-map γ_{mn} from X_m to X_n satisfying

$$\gamma_n = \gamma_{mn} \circ \gamma_m \ .$$

Thus an inverse system of simplicial complexes (polyhedra) is obtained, along with a map from X to the inverse limit.

Theorem:

This map from X to the inverse limit is a homeomorphism. So

$$X \cong \lim_{\leftarrow}(X_n, \gamma_{mn}) \ .$$

Note that this says a little more than that X can be written as an inverse limit of polyhedra, since under the connecting maps all vertices at each stage must be in the range.

If f is a continuous function from X to $\mathbb{R}$ or $\mathbb{C}$, then f is "approximately PL": it is a uniform limit of functions of the form $\psi \circ \gamma_n$, where $\psi : X_n \to \mathbb{C}$ is PL. [Choose n large enough that f is approximately constant on the sets of $\mathcal{U}_n$.]

The Operator Algebra Perspective

Key Observation: A partition of unity on X is just a set of positive elements of $C(X)$ of norm 1 adding to the constant function 1.

Such a set $\{f_1, \ldots, f_n\}$ defines a unital (complete) order embedding $\beta_{\mathcal{P}}$ of $\mathbb{C}^n$ into $C(X)$:

$$(\lambda_1, \ldots, \lambda_n) \mapsto \sum_{k=1}^{n} \lambda_k f_k \ .$$

Conversely, if β is a unital (complete) order embedding of $\mathbb{C}^n$ into $C(X)$, and $f_k = \beta(\mathbf{e}_k)$, then $\{f_1, \ldots, f_n\}$ is a partition of unity on X.

If $\mathcal{P} = \{f_1, \ldots, f_n\}$ is a partition of unity on X, there is also a homomorphism $\alpha_{\mathcal{P}}$ from $C(X)$ to $\mathbb{C}^n$ such that $\alpha_{\mathcal{P}} \circ \beta_{\mathcal{P}}$ is the identity on $\mathbb{C}^n$: for each k choose x_k for which $f_k(x_k) = 1$, and set

$$\alpha_{\mathcal{P}}(f) = (f(x_1), \ldots, f(x_n)) \ .$$

This homomorphism is canonical if the partition of unity gives a true triangulation (or if $\gamma_{\mathcal{P}}$ is just injective), but requires choices in general.

If $\mathcal{Q} = \{g_1, \ldots, g_m\}$ is a partition of unity refining $\mathcal{P}$, there is also a unital (complete) order embedding $\beta_{\mathcal{P}\mathcal{Q}}$ of $\mathbb{C}^n$ into $\mathbb{C}^m$ defined similarly, and

$$\beta_{\mathcal{P}} = \beta_{\mathcal{Q}} \circ \beta_{\mathcal{P}\mathcal{Q}}$$

and a homomorphism $\alpha_{\mathcal{P}\mathcal{Q}} : \mathbb{C}^m \to \mathbb{C}^n$ with $\alpha_{\mathcal{P}\mathcal{Q}} \circ \beta_{\mathcal{P}\mathcal{Q}}$ the identity on $\mathbb{C}^n$.

The $\alpha_{\mathcal{P}}$, $\alpha_{\mathcal{Q}}$, and $\alpha_{\mathcal{P}\mathcal{Q}}$ can be chosen so that

$$\alpha_{\mathcal{P}} = \alpha_{\mathcal{P}\mathcal{Q}} \circ \alpha_{\mathcal{Q}} \ .$$

Thus, if X is a compact metrizable space, one can generate a system

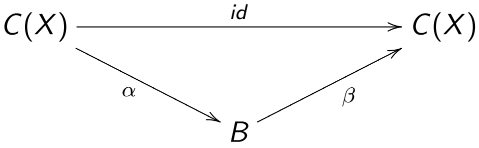
$$B_1 \to B_2 \to \cdots$$

of finite-dimensional commutative C*-algebras, where the connecting maps β_{nm} are not homomorphisms but are complete order embeddings, and compatible complete order embeddings

$$\beta_n : B_n \to C(X)$$

such that the union $\cup_n \beta_n(B_n)$ is dense in $C(X)$, and (unique!) homomorphisms $\alpha_n : C(X) \to B_n$ which are coherent and left inverses for the β_n.

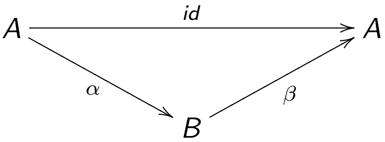
It is notationally convenient to write things "locally" by saying there are diagrams



which approximately commute in the point-norm topology, where the B is finite-dimensional and α and β are completely positive contractions, with the additional properties that
(1) α is a homomorphism.
(2) $\alpha \circ \beta$ is the identity on B. (Hence $\beta \circ \alpha$ is an idempotent map from $C(X)$ to $C(X)$.)
(3) β is a complete order embedding.

The Noncommutative Case

If A is a (separable) C*-algebra, it is natural to regard an inductive system of finite-dimensional C*-algebras and complete order embeddings into A with analogous properties to be a "PL structure" on A. Phrasing things locally, we want a set of diagrams



which approximately commute in the point-norm topology, where the B is finite-dimensional and α and β are completely positive contractions, satisfying as many of (1)–(3) as possible.

One can pass to the inductive system picture by fairly routine perturbation arguments (if A is separable).

To get started, we need A to be nuclear.

In addition, we want at least some of the following:

(1) α is a homomorphism.

(2) $\alpha \circ \beta$ is the identity on B. (Hence $\beta \circ \alpha$ is an idempotent map from $C(X)$ to $C(X)$.)

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We can only hope for (1) if A is residually finite-dimensional.

Theorem:

If A is nuclear and residually finite-dimensional, we can get (1)–(3). (Such a C*-algebra is called an *RF algebra*.)

We can, however, weaken (1) to

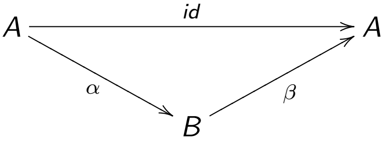
(4) α is approximately multiplicative.

Condition (4) is extremely natural: it means that not only the complete order structure but also the algebraic structure (multiplication) of A can be approximated in finite-dimensional C*-algebras.

We can only hope to get (4) if A is stably finite.

Definition:

A separable C*-algebra A is an *NF algebra* if, for any $x_1, \ldots, x_n \in A$ and $\epsilon > 0$ there is a finite-dimensional C*-algebra B and completely positive contractions $\alpha : A \to B$ and $\beta : B \to A$ such that $\|\beta \circ \alpha(x_i) - x_i\| < \epsilon$ and $\|\alpha(x_i x_j) - \alpha(x_i)\alpha(x_j)\| < \epsilon$ for all i, j.



Here are a few of the many characterizations of NF algebras:

Theorem:

Let A be a separable C*-algebra. The following are equivalent:

(i) A is an NF algebra.

(ii) A is nuclear and quasidiagonal.

(iii) A can be written as a generalized inductive limit of a sequence of finite-dimensional C*-algebras in which the connecting maps are completely positive contractions (and asymptotically multiplicative).

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$$A_1 \xrightarrow{\phi_{1,2}} A_2 \xrightarrow{\phi_{2,3}} A_3 \xrightarrow{\phi_{3,4}} \cdots \longrightarrow A$$

Such a system is called an *NF system* for A.

If $(A_n, \phi_{n,n+1})$ is an NF system for A, there is a completely positive contraction $\phi_n : A_n \to A$, and $\cup_n \phi_n(A_n)$ is dense in A. But $\phi_n(A_n)$ is not a subalgebra of A in general.

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From the quasidiagonality characterization, we obtain the fact, not obvious from the definition, that a nuclear C*-subalgebra of an NF algebra is NF.

A quotient of an NF algebra is not necessarily NF. In fact, any separable nuclear C*-algebra is a quotient of an NF algebra [if A is separable and nuclear, then the cone over A is an NF algebra by Voiculescu.]

Conditions (2) and (3) are closely related: diagrams satisfying (2) automatically satisfy (3), and diagrams satisfying (3) can be modified to diagrams satisfying (2). Diagrams satisfying (2) automatically satisfy (4).

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It turns out that we cannot always get (2) or (3) for NF algebras.

Definition:

A (separable) C*-algebra A with diagrams satisfying (2) (hence also (3) and (4)) is a *strong NF algebra.*

Here are a few of the many characterizations of strong NF algebras:

Theorem:

Let A be a separable C*-algebra. The following are equivalent:

(i) A is a strong NF algebra.

(ii) A is nuclear and has a separating family of quasidiagonal irreducible representations.

(iii) A can be written as a generalized inductive limit of a sequence of finite-dimensional C*-algebras in which the connecting maps are complete order embeddings (and asymptotically multiplicative).

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$$A_1 \xrightarrow{\phi_{1,2}} A_2 \xrightarrow{\phi_{2,3}} A_3 \xrightarrow{\phi_{3,4}} \cdots \longrightarrow A$$

Such a system is called a *strong NF system* for A.

If A is a C*-algebra and every quotient of A is an NF algebra (i.e. A is strongly quasidiagonal), then A is a strong NF algebra.

In particular, every simple NF algebra is a strong NF algebra.

Theorem:

Every strong NF algebra is an *ordinary* inductive limit of RF algebras (with injective connecting maps).

Ideal Structure of NF Algebras

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Example. Let $A_n = \mathbb{M}_n$,

$$\phi_{n,n+1}\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 \\ 0 & 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

The inductive limit A is isomorphic to $\mathbb{K} + \mathbb{C}1$. Each A_n is simple, but A is not simple.

But there is something that can be said about ideals in a generalized inductive limit. Let $A = \lim_{\rightarrow}(A_n, \phi_{n,n+1})$, and let J be an ideal in A. Since $\phi_n(A_n)$ is not a subalgebra of A, $\phi_n^{-1}(J)$ is not a subalgebra of A_n in general.

However, since ϕ_n is positive, $\phi_n^{-1}(J) \cap A_{n+}$ is a (closed) hereditary cone in A_{n+}, so its span is a hereditary C*-subalgebra J_n of A_n (not an ideal in general.) Since A_n is finite-dimensional, J_n is a corner, i.e. $J_n = p_n A_n p_n$ for a projection $p_n \in A_n$.

$\phi_{n,n+1}(p_n)$ is not a projection in A_{n+1} in general. However, p_{n+1} is a unit for $\phi_{n,n+1}(p_n)$.

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Definition:

If $\cup\phi_n(J_n)$ is dense in J, then the ideal J is *induced* from the system $(A_n, \phi_{n,n+1})$. J is an *induced ideal* of A if it is induced from some NF system for A.

It is unclear whether an ideal can be induced from one NF system but not from another.

Proposition:

In $C(X)$, every ideal is induced from any NF system.

To see why, suppose J is an ideal of $C(X)$ consisting of all functions vanishing on a closed set Y in X. A_n consists of all piecewise-linear functions on a simplicial complex X_n, with $\gamma_n : X \to X_n$ a continuous map whose range contains the vertices of X_n.

Then J_n consists of the span of the set of nonnegative piecewise-linear functions vanishing on $\gamma_n(Y)$ (an ideal in A_n in this case). Such a function also vanishes on any entire subsimplex of X_n containing a point of $\gamma_n(Y)$ in its interior. $\phi_n(J_n)$ consists of all functions vanishing on the inverse image Y_n under γ_n of these simplexes.

We have $Y \subseteq Y_n$ for each n. If ρ is a metric on X and $\epsilon > 0$, then for sufficiently large n we have Y_n contained in an ϵ-neighborhood of Y. Thus $\cap Y_n = Y$ and $\cup_n \phi_n(J_n)$ is dense in J.

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Definition:

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Suppose J is induced from the system $(A_n, \phi_{n,n+1})$, and let p_n be the projection in A_n corresponding to J_n. Define a generalized inductive system as follows: let $B_n = (1 - p_n)A_n(1 - p_n)$, and define $\psi_{n,n+1} : B_n \to B_{n+1}$ by

$$\psi_{n,n+1}(x) = (1 - p_{n+1})\phi_{n,n+1}(x)(1 - p_{n+1}) .$$

It is routine to check that this is indeed a generalized inductive system, hence an NF system, and that the generalized inductive limit is naturally isomorphic to A/J.

Using a similar (but simplified) argument, one can show:

Proposition:

Let J be an ideal in an NF algebra A. If J has a quasicentral approximate unit of projections, then J is an NF ideal.

This can also be proved using the known result that the quotient of a quasidiagonal C*-algebra by an ideal with a quasicentral approximate unit of projections is quasidiagonal.

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We say an ideal J in a C*-algebra A is *locally approximately split* if, for every $x_1, \ldots, x_n \in A/J$ and $\epsilon > 0$, there is a completely positive contraction $\sigma : A/J \to A$ such that $\|\pi \circ \sigma(x_i) - x_i\| < \epsilon$ and $\|\sigma(x_i x_j) - \sigma(x_i)\sigma(x_j)\| < \epsilon$ for all i, j, where $\pi : A \to A/J$ is the quotient map.

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Proposition:

A locally approximately split ideal in an NF algebra is an NF ideal.

There is a "converse" to the proposition. Suppose J is an NF ideal in an NF algebra A. We want to show that J is an induced ideal from some NF system for A. We outline the argument.

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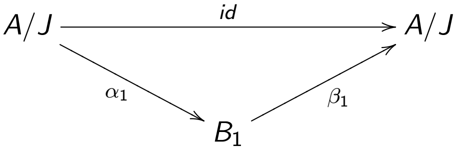
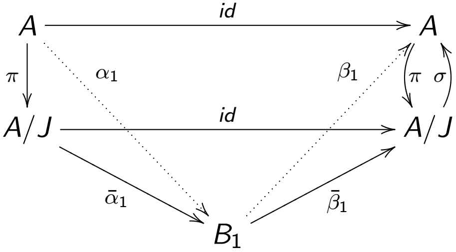
Begin with finite subsets $\{x_i\}$ of A and $\{y_j\}$ of J and an approximately commutative diagram



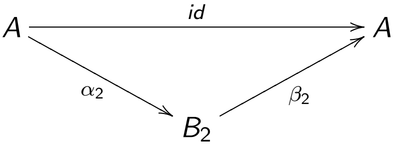
Lift the diagram to



This diagram only approximately commutes mod J.

Using a quasicentral approximate unit for J, choose $h \in J_+$, $\|h\| \leq 1$, such that h almost commutes with the elements of A and is almost a unit for the elements of J.

Choose a diagram



which is approximately commutative and approximately multiplicative on all elements defined so far.

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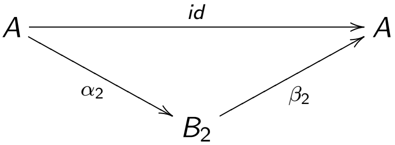

which is approximately commutative and approximately multiplicative on all elements defined so far.

Let $\alpha = \alpha_1 \oplus \alpha_2 : A \to B_1 \oplus B_2$ and $\beta : B_1 \oplus B_2 \to A$, where

$$\beta(x, y) = (1 - h)^{1/2} \beta_1(x)(1 - h)^{1/2} + h^{1/2} \beta_2(y) h^{1/2}$$

To get an NF system, set $A_1 = B_1 \oplus B_2$. At the next stage, reduce ϵ and expand $\{x_i\}$ by throwing in all the images of matrix units of $B_1 \oplus B_2$ and more elements of a dense subset of A, and expand $\{y_i\}$ by throwing in the images of the matrix units of B_2 (which lie in J) as well as more elements of a dense subset of J.

The ideal J is induced by this NF system.

So the conclusion is:

Theorem:

Let A be an NF algebra, J an ideal of A. Then J is an NF ideal if and only if J is induced by some NF system for A.

It would be nice to get a single NF system for an NF algebra A such that all NF ideals can be induced from this system. It is really only necessary to be able to do the previous construction with two NF ideals J and K (or finitely many) simultaneously.

If $J \cap K$ and $J + K$ are also NF ideals, it appears the construction can be made to work with some technical complications. It is true that $J \cap K$ is always an NF ideal, since $A/(J \cap K)$ can be embedded in $A/J \oplus A/K$ and hence is an NF algebra. But it is not obvious that $J + K$ is always NF.

The construction at least appears to work for residually NF algebras (strongly quasidiagonal nuclear C*-algebras).

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Is every stably finite separable nuclear C*-algebra an NF algebra?

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The theorem gives a potential approach to showing that a separable nuclear C*-algebra B with a faithful tracial state τ must be an NF algebra. Any separable nuclear C*-algebra is a quotient of an NF algebra, so B is a quotient of an NF algebra A, and τ may be regarded as a tracial state on A. If J is the kernel of τ, i.e.

$$J = \{x \in A : \tau(x^*x) = 0\}$$

then J is the kernel of the quotient map from A to B. So it suffices to show that the kernel of a tracial state on an NF algebra is an induced ideal.