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The Simplicial Lusternik-Schnirelmann Category

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Summary

In the present thesis we describe the simplicial Lusternik-Schnirelmann category and the simplicial geometric category for finite simplicial complexes. Moreover, we relate these categories with the Lusternik-Schnirelmann and geometric categories of finite T_0-spaces, showing that they behave in a similar way under specific conditions. For example, the L-S category is a homotopy invariant and the simplicial L-S category is a strong homotopy invariant, and the geometric category for finite T_0-spaces increases under elimination of beat points and its simplicial version increases under elimination of dominated vertices. With this aim, we describe the structure of finite T_0-spaces and finite simplicial complexes and we relate them through the Order Complex functor $\mathcal{K}$ and the Face Poset functor χ. Finally, we show that the L-S category of the geometric realisation provides a lower bound for the simplicial L-S category and a lower bound for the simplicial L-S category of the iterated subdivision. There will be computations of simplicial L-S categories and examples of the main theorems. However, no examples have been found for simplicial complexes, such that the value of the geometric category strictly increases under removing a dominated vertex, or such that the category of its face poset is strictly smaller than its simplicial category. Moreover, there is no criteria for defining an upper bound for the value of the simplicial L-S category.

Max Ernst, 1965

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Chapter 0

Introduction

0.1 The simplicial Lusternik-Schnirelmann category

The *simplicial Lusternik-Schnirelmann category* is the definition of Lusternik-Schnirelmann category for simplicial complexes. The Lusternik-Schnirelmann (L-S) category of a topological space X represents the least number n such that there is an open cover of X of $n+1$ subsets contractible to a point in the space X.

Originally, this concept was introduced in 1930 by L. Lusternik and L. Schnirelmann [9] in the study of manifolds. The L-S category, in fact, provides a lower bound for the number of critical points for any smooth function on a manifold and links invariants of manifolds with topological invariants. Later on, other definitions of L-S category were given. R. Fox [7] introduced the geometric category, where each subset of the cover is required to be contractible in itself, and he developed the L-S category in the field of algebraic topology. In the '50s and '60s G.W. Whitehead gave the first homotopy theoretic definition of L-S category of a space and later T. Genea gave a second one. A description of the two alternative definitions and a complete view of the results about the L-S category for topological spaces can be found in [4].

Recently, definitions of L-S category have been extended to the case of simplicial complexes. These definitions do not refer to the L-S category of the geometric realisation of the simplicial complex but they are built on the simplicial structure itself. The first simplicial version of L-S category was given in 2013 by S. Aaronson and N. A. Scoville [1]. This definition uses the notion of *simplicial collapse*, that was introduced by G.W. Whitehead in the late thirties. A simplicial collapse is the deletion from the simplicial complex of a free face, that is a simplex τ such that there is a simplex σ that is a face of τ and σ has no other cofaces. A simplicial collapse of the simplicial complex K onto the simplicial complex L is denoted by $K \searrow L$. The definition of the simplicial category given in [1] is based on the definition of geometric category given by Fox [7] and represents the minimum, among the simplicial complexes L such that $K \searrow L$, of the smallest number of collapsible subcomplexes that can cover L. However, the concept of collapsibility presents some difficulties, for example the core of a simplicial complex is not unique and a simplicial complex can collapse to two non-isomorphic minimal complexes (that are complexes without any free face). Therefore, it results difficult to understand whether a space is collapsible or not. Already in 2009, and not in relation with the simplicial L-S category, J.Barmak and E. Minian [3] developed the idea of *strong collapsibility*. An *elementary strong collapse* is a deletion of a *dominated vertex*, that is a vertex v such that its link $lk(v)$ is a simplicial cone. A strong collapse is a sequence of elementary strong collapses and it is denoted by $K \searrow\searrow L$, for two simplicial complexes K and L. This new notion is a special case of the old collapse and it satisfies some useful properties, for example the core of a simplicial complex K is unique up to isomorphisms and it is strong homotopic to K. Moreover, the concepts of *contiguity classes* of simplicial maps and *strong homotopy type* provide a simplicial analogous of homotopy classes of continuous functions between topological spaces and homotopy type. In the same paper a correspondence between finite T_0-spaces and finite simplicial complexes is established and, for example, homotopic finite spaces have a correspondent strong homotopic finite simplicial complexes (via order complex) and vice-versa (via face poset). In 2015, D. Fernández-Ternero, E. Macías-Virgós and J. A. Vilches [14] gave a new definition of simplicial L-S category and simplicial geometric category based on the concept of strong collapsibility. In this way the simplicial L-S category of a finite simplicial complex K, denoted by $scat(K)$, is defined as the least integer n such that there is a cover of K of $n+1$ subcomplexes that strong collapse to a vertex in K. With this definition the simplicial L-S category is a strong homotopy invariant. Therefore, a simplicial complex and its core have the same category. The simplicial geometric category is defined in the same way but the subcomplexes are required to be strong collapsible. It is not an homotopic invariant, but differently from the geometric category defined with simple collapsibility, the geometric category of the core of a simplicial complex is the maximum of the category in its strong homotopy classes. Moreover, in the article they relate the simplicial category to the L-S category of a finite T_0-space via order complex and face poset, deducing some new results about the category of finite spaces. In fact, simplicial L-S category defined on strong collapsibility behaves in a symmetric way to the classical L-S category in the case of a finite topological space. Since the simplicial L-S category is a new concept there is not a general overview of the results related to it and concrete examples of some theorems have not yet been found. The objective of this thesis is to provide a complete overall view of the simplicial L-S category, including also examples, some new results and computations of simplicial L-S category.

0.2 Structure of the thesis

This thesis aims to study the Lusternik-Schrinelmann category for simplicial complexes. In particular, we will present the results concerning the topic in order to give a complete idea of how the concept is structured. We will add some new results, examples and computations of the simplicial L-S category. This work is based on "*Lusternik-Schrinelmann category for simplicial complexes and finite spaces*" by D. Fernández-Ternero, E. Macías-Virgós and J. A. Vilches [14]. It is structured in three main chapter: first, we will introduce some preliminary notions, then we will define the L-S category for simplicial complexes and give the results related to it and finally, we will show some additional results and computations. In order to clarify some notions that will be used in the thesis, we include two appendices.

In the first chapter, we describe the relation between finite T_0-spaces, finite partially ordered sets (posets) and finite simplicial complexes. We will first show that the category of finite T_0-spaces and the category of finite posets are equivalent. In fact, continuous maps between finite T_0-spaces correspond to order preserving maps and in the category of posets we can define a relation between order preserving functions that is given by the order, and we show that functions in the same order component are homotopic. We will therefore consider finite T_0-spaces and finite posets as the same object. Then we show that given a finite topological space we can always associate a minimal finite T_0-space. Moreover, we will define the core of a finite T_0-space that is the minimal space obtained by removing all beat points. An important result is that the core is unique up to homeomorphism and two finite T_0-spaces are homotopy equivalent if and only if their cores are homeomorphic. We then define the category of finite simplicial complexes and the functors between the category of finite simplicial complexes and the one of finite posets. In fact, to every finite poset X we can associate its order complex and to each finite simplcial complex we can associate its face poset. Moreover, we define the relation of contiguity between simplicial maps. The notion of contiguity corresponds to that of homotopy. In fact, contiguity classes of simplicial maps are sent by the functor to homotopy classes of continuous maps and vice-versa. As in the case of finite T_0-spaces, we can define the core of a simplicial complex that is the simplicial complex obtained by removing dominated vertices. The removal of a dominated vertex is called a strong collapse and two simplicial complexes have the same strong homotopy type if there is a sequence of strong collapses and expansion from one to the other. Two simplicial complexes have the same strong homotopy type if and only if they are strong equivalent (homotopy equivalent in the sense of contiguity). As in the case of finite T_0-space, the core of a simplicial complex is unique up to isomorphism and two simplicial complexes have the same strong homotopy type if and only if their cores are isomorphic. Finally, we will present some results that show some properties that are preserved if we pass from finite posets to finite simplicial complexes and vice-versa via the order complex functor and face poset functor. For example, homotopy equivalent posets correspond to strong homotopy equivalent simplicial complexes.

In the second chapter, we will introduce the definition of L-S category and geometric category for topological spaces in general and then in the specific case of finite T_0-spaces. For example, one of the results shows the relation between the geometric category and the L-S category. In fact, the latter is always smaller than or equal to the former. Then, we define the L-S category and the geometric category for simplicial complexes. In this case the definition is given using the concept of contiguity. The categorical subcomplexes are defined as the subcomplexes such that the inclusion map is in the same contiguity class of some constant map. We will show that results that are similar to the case of finite space, hold also for simplicial complexes. Finally, we compare the L-S category and the geometric category for finite T_0-spaces with the one of finite simplicial complexes. In particular, we will show some inequalities that relate the category of a finite topological space with the one of its order complex and the category of a simplicial complex with the one of its face poset. One interesting result is that the category of the subdivision is smaller or equal than the category of the simplicial complex. In this chapter we will also present some examples of finite spaces and simplicial complexes that satisfy these results.

In the third chapter, we discuss the L-S category of the geometric realization and we show that it is always smaller or equal to the simplicial category of the simplicial complex. Moreover, it provides a lower bound for the category of the iterated subdivision $sd^n(K)$ of the simplicial complex K. We will use these results to compute the value of the simplicial category of the triangulation of the projective plane and the torus. We also compute the simplicial category of the n sphere.

Appendix A shows some results regarding the general theory of L-S category for topological spaces. In fact, we will show that it can be bounded from below by the cup length and that the dimension of the space gives an upper bound for the value of the L-S category.

Appendix B discusses the topology of the geometric realisation. We introduce the closed L-S category, that is the category defined with a closed cover of categorical subsets, and we will show that in the case of the geometric realisation of a finite simplicial complex the value of the closed L-S category coincides with the value of the classical L-S category.

0.3 Notation

We give here a list of symbols used in our thesis.
$\boldsymbol{TOP}$, $\boldsymbol{fTOP}$ and $\boldsymbol{fT_0}$ denote the category of topological spaces, finite topological spaces and the category of finite T_0-spaces
$\boldsymbol{fPOSET}$ is the category of finite partially ordered sets.
$\boldsymbol{h(fT_0)}$ and $\boldsymbol{h(fPOSET)}$ are the Homotopy category and Order category
$\boldsymbol{fSC}$ denotes the category of finite simplicial complexes
$\boldsymbol{T}$ and $\boldsymbol{P}$ represent the finite topological space functor and the poset functor
$\boldsymbol{\mathcal{K}}$ denotes the Order Complex functor and $\boldsymbol{\chi}$ denotes the Face Poset functor
$\boldsymbol{K^0}$ is the set of vertices of a simplicial complex K
$\boldsymbol{|K|}$ denotes the geometric realisation of a simplicial complex K
$\boldsymbol{sd(K)}$ represents the barycentric subdivision of a simplicial complex K
$\prec$ denotes the relation of *being covered* between elements of the Hasse diagram *Definition 1.1.4*
$\boldsymbol{U_x}$ is the intersection of all open subset containing x and it is an element of the base of the topology, if X is a finite space (finite T_0-space). If we regard X as a finite preorder (finite poset) $U_x = \{y \in X : y \leq x\}$
$\simeq$ is the relation of being homotopic functions or homotopy equivalent spaces
$\sim_{\leq}$ denotes the equivalent relation generated by the order $\leq$
$\boldsymbol{X \searrow\searrow Y}$ and $\boldsymbol{K \searrow\searrow L}$ indicate a strong collapse between finite posets and finite simplicial complexes
$\boldsymbol{\sim_c}$ denotes the relation of being contiguous simplicial maps
$\sim$ represents the relation between simplicial maps of being in the same contiguity class

$\boldsymbol{lk(v)}$, $\boldsymbol{st(v)}$ and $*$ represent the link and the star of a vertex v of a simplicial complex, and the operation of join between simplicial complexes
$\boldsymbol{cat(X)}$ and $\boldsymbol{gcat(X)}$ denote the L-S category and the geometric category for a topological space X
$\boldsymbol{scat(K)}$ and $\boldsymbol{gscat(K)}$ denote the simplicial category and the simplicial geometric category for a simplicial complex K
$\boldsymbol{cat^{cl}(X)}$ represents the closed category, that is the L-S category defined for a cover of closed categorical subsets

Chapter 1

Preliminaries

1.1 The category of finite T_0-spaces and the category of finite posets

In this section, we will define the category of finite topological spaces, that we will denote by $fTOP$, the category of finite T_0-spaces fT_0 and the category of finite partially ordered set $fPOSET$. We will show that fT_0 and $fPOSET$ are equivalent categories. The results in this sections refer to Barmak's *Chapter 1 [2]* but they are presented in the language of Category Theory. *Proposition 1.1.18* is due to the author.

Definition 1.1.1. The category $fTOP$ of finite topological spaces is defined by the class of objects $ob(fTOP)$ that is the class of all finite topological spaces (X, τ), where τ is the topology on X, and its set of morphisms $Hom(fTOP)$,that is the set of continuous functions between finite topological spaces. We denote $Hom((X, \tau), (X', \tau'))$ the set of continuous functions from (X, τ) to $(X, \tau') \in ob(fTOP)$.

We recall the definition of T_0 separation property for a topological space.

Definition 1.1.2. Let X be a topological space, X is a $T_0 - space$ if for any two points of X, there is an open neighbourhood of one that does not contain the other.

Definition 1.1.3. The category fT_0 of finite T_0-spaces is the subcategory of $fTOP$ such that the objects are the finite T_0-spaces and the morphisms are continuous maps between finite T_0-spaces.

A finite partially ordered set is a finite set on which a partial order relation is defined. Morphisms between partially order sets are *order preserving functions*, that are functions $f : X \to Y$, X and Y partially ordered sets, such that $x \leq x'$ implies $f(x) \leq f(x')$, $\forall x, x' \in X$.
On a partially ordered set X we can define *maximal elements* and *minimal elements*, *maximum*, *minimum* and *chains* that are subset $Y \subseteq X$ such that Y is totally ordered.

Definition 1.1.4. [2] The *Hasse diagram* of a finite poset X is the digraph whose vertices are the points of X and whose edges are the ordered pairs (x, y), $x, y \in X$ such that $x < y$ and there is no z such that $x < z < y$. If the segment with vertices x and y is an edge of the *Hasse diagram* we say that x *covers* y, in the literature this relation is denoted by $x \prec y$.

The next example shows a representation of an Hasse diagram, we will not write arrows in the edges but the relation is given by the position of the elements in the diagram.

Example 1.1.5. *Let X be a finite ordered set $X = \{a, b, c, d\}$ with the order $\leq$ defined by $a \leq b$, $a \leq c \leq d$. The Hasse diagram of X is showed in the following figure. Here a is a minimum, b and d are maximal elements, $a \prec b$, $a \prec c$ and $c \prec d$.*

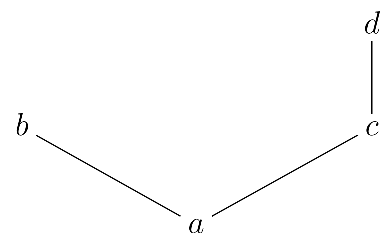


Figure 1.1.1: The Hasse diagram associated to the set X.

Definition 1.1.6. The category $fPOSET$ is the category that has as class of objects $ob(fPOSET)$ the class of finite partially order sets and as set of morphisms $Hom(fPOSET)$ the set of order preserving maps between partially ordered sets.

Remark 1.1.7. *The categories $fTOP$, fT_0 and $fPOSET$ are well defined because we can define a composition of maps $Hom(A, B) \times Hom(B, C) \to Hom(A, C)$ for all A, B, $C \in ob(fTOP)$, $ob(fT_0)$ or $ob(fPOSET)$. This composition is associative and there are identity functions in all these sets of morphisms.*

We remind some definitions from Category Theory that we will use in order to show that the categories fT_0 and $fPOSET$ are equivalent. More details about Category Theory ca be found in [12].

Definition 1.1.8. Two categories C and D are *equivalent* if and only if there is a functor $F : C \to D$ and a functor $G : D \to G$ and two natural isomorphisms $GF \cong Id_C$ and $FG \cong Id_D$.

Definition 1.1.9. Let F and $G : C \to D$ be covariant functors, a *natural transformation* $\tau : F \to G$ is a family of morphisms in D, $\tau = \{\tau_X : F(X) \to G(X)\}_{X \in ob(C)}$ that make the following diagram commute for all morphisms $f : X \to X'$ in $Hom(C)$:

$$\begin{array}{ccc} F(X) & \stackrel{\tau_X}{\to} & G(X) \\ \downarrow_{F(f)} & & \downarrow_{G(f)} \\ F(X') & \stackrel{\tau_{X'}}{\to} & G(X') \end{array}$$

A natural transformation is called a *natural isomorphisms* if every morphism τ_X is an isomorphism.

Now we want to define functors from fT_0 to $fPOSET$ and vice-versa from $fPOSET$ to fT_0. With this aim we will show that given a finite topological space we can define a partial order on it and given a finite partially ordered set we can define a T_0 topology on it.

Lemma 1.1.10. *Let (X, τ) be a finite T_0-space, then there is a partial order relation $\leq$ on X.*

Proof. Let U_x be the intersection of all open sets containing x, it is open because X is finite. We can define the relation $\leq$ in this way: $x \leq y$ if and only if $U_x \subseteq U_y$. This relation is reflexive: $x \leq x$ because $U_x \subseteq U_x$ and transitive $x \leq y$ and $y \leq z$ implies $U_x \subseteq U_y$ and $U_y \subseteq U_z$, so $U_x \subseteq U_z$. The antisymmetry is given by the T_0 separation property, in fact $x \leq y$ and $y \leq x$ implies that $U_x \subseteq U_y$ and $U_y \subseteq U_x$ so $U_x = U_y$. $\cap W_x = \cap W_y$ where W_x and W_y are open subsets of $U_x = U_y$ containing respectively x and y. So every open set that contains x contains also y and every open set that contains y contains also x, but since X is a $T_0 - space$, if $x \neq y$ there is an open neighbourhood of x that does not contain y, so the only case is that $x = y$.

□

Lemma 1.1.11. *Let $(X, \leq)$ be a finite partially order set, then we can define a T_0 topology on it.*

Proof. Consider the set $U_x = \{y \in X : y \leq x\}$. We want to show that $\{U_x\}_{x \in X}$ is a base of open sets that induce a T_0-topology. In fact, for every $x \in X$ we have that $x \in U_x$ and if $x \in U_y$ and $x \in U_z$ so $x \leq y$ and $x \leq z$ then $U_x = \{w \in X : w \leq x\} \subseteq U_y \cap U_z$ by the transitivity of the relation. The T_0 property is verified by the antisymmetry of the relation, in fact for any two distinct points x, y we have that $x < y$ or $y < x$ or x and y are not in relation, because if $x \leq y$ and $y \leq x$ then $x = y$. In the first case $x < y$ so U_x is the open neighbourhood of x that does not contain y, in the second case $y < x$ so the open neighbourhood is U_y and in the third case is either U_x or U_y.

□

Remark 1.1.12. *The previous two proofs show that given a finite topological space (X, τ) we can define a preorder relation $\leq$ on X, that is a symmetric and transitive relation, and given a finite preordered set $(X, \leq)$ we can define a topology τ on X. In fact, the T_0 separation property is equivalent to antisymmetry of the relation $\leq$. That means that in general finite topological spaces can be considered as preordered sets and vice-versa. In the specific, finite T_0-spaces can be seen as finite posets.*

Definition 1.1.13. Let $(X, \leq)$ be a finite poset. We define the base of the T_0-topology τ on X as the collection of sets $\{U_x\}_{x \in X}$, where $U_x = \{y \in X : y \leq x\}$. We will call U_x *basic open set*. Moreover, U_x is the intersection of all open subset of X containing x.

Lemma 1.1.14. *A function $f : X \to Y$, X and Y finite spaces, is continuous if and only if it is order preserving.*

Proof. If f is continuous the pre-image of an open set is open, so $f^{-1}(U_y)$ is open for all $y \in Y$. Suppose that $x \leq x'$, we have that $f^{-1}(U_{f(x')})$ is open and $x' \in f^{-1}(U_{f(x')})$ so $x \in f^{-1}(U_{f(x')})$. Therefore $f(x) \in (U_{f(x')})$ that implies $f(x) \leq f(x')$. Suppose now that f is order preserving. We want to show that the pre-image of every open set U_y $y \in Y$ that form a basis for the topology of Y, is open. Let $x \in f^{-1}(U_y)$ and $x' \leq x$ so $f(x') \leq f(x) \leq y$ because f is order preserving. Therefore $x' \in f^{-1}(U_y)$ that means that $f^{-1}(U_y) = U_z$ for some $z \in X$ that is an open set of the base of the topology on X. So $f^{-1}(U_y)$ is open and therefore f is continuous. □

Definition 1.1.15. The *poset functor* $P : fT_0 \to fPOSET$ is a functor such that for all $(X, \tau) \in ob(fT_0)$ $P((X, \tau)) = (X, \leq)$ where $\leq$ is the partial order defined in *lemma 1.1.10* and for all morphisms $f \in Hom((X, \tau), (X', \tau'))$ $P(f) = f \in Hom(P(X, \tau), P(X', \tau'))$ for all $(X, \tau), (X', \tau') \in ob(fT_0)$. These morphisms are order preserving by *lemma 1.1.14*.

Definition 1.1.16. The *finite topological space functor* $T : fPOSET \to fT_0$ is a functor such that for all $(X, \leq) \in ob(fPOSET)$ $T((X, \leq)) = (X, \tau)$ with the T_0 topology τ defined in *lemma 1.1.11*. For all morphisms $f \in Hom((X, \leq), (X', \leq'))$, $(X, \leq), (X', \leq') \in ob(fPOSET)$ $P(f) = f \in Hom(T(X, \tau), T(X', \tau'))$, f is continuous by *lemma 1.1.14*.

Remark 1.1.17. *The functors P and T are well defined. In fact, $P(X, \tau) \in ob(fPOSET)$ by lemma 1.1.10, $T(X, \leq) \in ob(fT_0)$ by lemma 1.1.11, by lemma 1.1.14 we have that $f : X \to X' \in Hom(fPOSET)$ if and only if $f : X \to X' \in Hom(fT_0)$ and the composition of function is preserved. Moreover $P(id_X) = id_{P(X)}$ for all $X \in ob(fT_0)$and $T(X) = id_{T(X)}$ for all $X \in ob(fPOSET)$.*

Proposition 1.1.18. *The categories $fPOSET$ and fT_0 are equivalent.*

Proof. Consider the functors $P : fT_0 \to fPOSET$ and $T : fPOSET \to fT_0$. We want to show that there are natural isomorphisms $TP \cong Id_{fT_0}$ and $PT \cong Id_{fPOSET}$. Consider the diagram for X and X' in $ob(fT_0)$ and $f : X \to X'$ in $Hom(fT_0)$:

$$\begin{array}{ccc} TP(X) & \overset{\tau_X}{\to} & Id_{fT_0}(X) \\ \downarrow_{TP(f)} & & \downarrow_{Id_{fT_0}(f)} \\ TP(X') & \overset{\tau_{X'}}{\to} & Id_{fT_0}(X') \end{array}$$

The base of the topology τ is given by the set $\overline{U_x}$ that are the intersection of all open subsets that contain x, and the order in $P(X,\tau)$ is defined by $y \leq x$ if and only if $\overline{U_y} \subseteq \overline{U_x}$ by *lemma 1.1.10*. Moreover the base of the topology in $TP(X,\tau)$ is given by $U_x = \{y \in X : y \leq x\}$ by *lemma 1.1.11*. Therefore, we have that $y \in U_x$ if and only if $y \leq x$ if and only if $\overline{U_y} \subseteq \overline{U_x}$, that is if and only if $y \in \overline{U_x}$. Then we have that $U_x = \overline{U_x}$ and so $(X,\tau) = TP(X,\tau)$. Therefore, by *lemma 1.1.14* the previous diagram is equivalent to the following one:

$$\begin{array}{ccc} X & \overset{\tau_X}{\to} & X \\ \downarrow_{f} & & \downarrow_{f} \\ X' & \overset{\tau_{X'}}{\to} & X' \end{array}$$

This diagram is clearly commutative and τ_X is the identity morphism so it is an isomorphism for all $X \in ob(fT_0)$. The proof that $PT \cong Id_{fPOSET}$ is analogous. In fact $PT(X,\leq) = (X,\leq)$ because the topology τ in $T(X,\leq)$ is given by the base $U_x = \{y \in X : y \leq x\}_{x \in X}$ and the relation $\leq'$ in $PT(X,\leq)$ is defined by $y \leq x$ if and only if $\overline{U_y} \subseteq \overline{U_x}$, where $\overline{U_x}$ is the intersection of the open sets that contain x. Now $\overline{U_x} = U_x$ therefore $y \leq x$ if and only if $y \in U_x$ if and only if $U_y \subseteq U_x$ if and only if $y \leq' x$. □

1.2 Homotopy Category and Order Category

In this section we will define the Homotopy Category for finite T_0-spaces and the Order Category that is its analogous for finite partially ordered sets and we will prove that they are equivalent categories. The following results refer to Barmak's

Chapter 1 [2], with the exception of *Lemma 1.2.6*, *Proposition 1.2.13* and the definition of *Order Category* that are due to the author. They are proved in general for finite topological spaces and preordered sets (that are sets with a symmetric and transitive relation defined on it) but we will later use the results in the specific case of finite T_0-spaces and partially ordered sets.
We give first some basics definitions. We say that two maps between topological spaces $f : X \to Y$ and $g : X \to Y$ are *homotopic* if there is a continuous function $H : X \times I \to Y$ where $I = [0, 1]$, such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ and we denote it as $f \simeq g$.

Definition 1.2.1. The Homotopy Category of finite T_0 spaces, $h(fT_0)$ is the category that has as objects all finite T_0-spaces and as morphisms the equivalence classes of homotopic morphisms between them. The morphisms f and g of finite T_0-spaces are in the same equivalent class, that we denote as $[f]$, if and only if they are homotopic $f \simeq g$.

Remark 1.2.2. *The composition of functions is defined for all classes* $[f]$, $[g] \in Hom(h(fT_0))$ *by* $[f][g] = [fg]$. *It is well defined because* $f \simeq f'$ *and* $g \simeq g'$ *implies* $fg \simeq f'g'$, *it is associative and the identity morphism is the class* $[id_{fT_0}]$.

Lemma 1.2.3. *(Lemma 1.2.3 [2]) Let X be a finite topological space and x and y two comparable points, then there is a path from x to y in X.*

Proof. We want to show that there is a continuous function $\alpha : I \to X$, $I = [0, 1]$ such that $\alpha(0) = x$ and $\alpha(1) = y$. We assume without loss of generality that $x \leq y$ and we define α as $\alpha(t) = x$ for $t \neq 1$ and $\alpha(1) = y$. Let $U \subseteq X$ be a open subset, then $\alpha^{-1}(U)$ equal $\emptyset$, I or $[0, 1)$ because we suppose that $x \leq y$ so an open set that contain y also contains x. These sets are open in I, so α is continuous. □

Definition 1.2.4. A *fence* of a finite preorder set $(X, \leq)$ is a sequence of elements $z_0, ..., z_n$ such that any two consecutive elements are comparable $z_0 \leq z_1 \geq z_2 \leq ... \leq z_n$.

On a preorder set X we can define an equivalence relation that we will call *equivalence relation generated by the order.*

Definition 1.2.5. Let X be a preorder set. The *equivalence relation generated by the order* $\sim_{\leq}$ is defined for all $x, y \in X$ by $x \sim_{\leq} y$ if there is a *fence* between x

and y. And we call the equivalence classes the *order components* of X. If there is only one order component we call X *order connected.*

Lemma 1.2.6. *The connected components of a finite topological space X are the order components, that are the equivalence classes of the equivalence relation generated by the order.*

Proof. Suppose without loss of generality that X has one connected component and suppose by contradiction that there are n order components, $n > 1$. So there is no fence between the elements in the different components. If for each component we take the union $U = \cup U_x$ where x is a maximal element in the component, we obtain n disjoint open sets such that the union is X and this is a contradiction.
Suppose now that X has one order component and more then one connected components. So there are at least two points x and y that are contained in different components. But there is a fence between x and y, therefore by *Lemma 1.2.3* there is a sequence of paths from x to y. The two points are in the same path component so they are in the same component. □

Corollary 1.2.7. *(Proposition 1.2.4 [2]) Let X be a finite topological space. The following are equivalent:*

- *X is a path connected topological space*
- *X is a connected topological space*
- *X is an order connected preordered set*

Proof. X is path connected implies that X is connected. X is connected if and only if it is order connected by *Lemma 1.2.6*. Finally X is order connected implies that X is path connected by *Lemma 1.2.3*. □

We define the relation $\leq$ between continuous functions between finite spaces as $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in X$. As in the previous case, the equivalence relation generated by the order, that we denote $\sim_{\leq}$ is defined by $f \sim_{\leq} g$ if and only if there is a fence of functions from f to g $f = h_0 \leq h_1 \geq h_2 \leq ... \leq h_n = g$.

Definition 1.2.8. The Order category for finite partially ordered sets $h(fPOSET)$ is defined by a class of objects that is the class of finite partially ordered sets and the set of morphisms that is the set of classes of morphisms, denoted by $[f]_{\leq}$ generated by the relation $\sim_{\leq}$.

Remark 1.2.9. *The composition of morphisms is defined for all classes $[f]$, $[g] \in Hom(h(fPOSET))$ by $[f][g] = [fg]$. It is well defined because $f \sim_{\leq} f'$ and $g \sim_{\leq} g'$ implies $fg \sim_{\leq} f'g'$. In fact, if there is a fence between f and f' $f(x) = h_0(x) \leq h_1(x) \geq h_2(x) \leq ... \leq h_n(x) = f'(x)$ for all $x \in X$, and we consider fg we have the fence $f(g(x)) = h_0(g(x)) \leq h_1(g(x)) \geq h_2(g(x)) \leq ... \leq h_n(g(x)) = f'(g(x))$. Then we consider the fence between g and g' $g = h_0 \leq h_1 \geq h_2 \leq ... \leq h_n = g'$ and we combine the two, since the morphisms are order preserving, we get a fence $f(g) = h_0(g) \leq h_1(g) \geq h_2(g) \leq ... \leq h_n(g)) = f'(g) \leq f'(h_1) \geq f'(h_2) \leq ... \leq f'(h_n) = f'(g')$ from fg to $f'g'$. The composition is associative and the identity morphism is the class $[id_{fPOSET}]$.*

Proposition 1.2.10. *(Corollary 1.2.6 [2]) Let f and g be two functions between finite topological spaces X and Y, $f : X \to Y$ and $g : X \to Y$. $f \simeq g$ if and only if $f \sim_{\leq} g$.*
Moreover, Let $A \subseteq X$ then $f \simeq g$ rel A if and only if $f \sim_{\leq} g$ and the fence $f = h_0 \leq h_1 \geq h_2 \leq ... \leq h_n = g$ given by the relation satisfies $h_i|_A = f|_A$ for all $0 \leq i \leq n$.

Proof. Consider the set Y^X of functions from X to Y. Since X and Y are finite set Y^X is also finite. We can define a preorder on it given by $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in X$. This relation is a preorder because the relation on Y is a preorder. Moreover, we can define on Y^X the topology τ induced by the functor T (we consider here the generalisation of T to the case of finite preorder. The topology is constructed in the same way but is not in general T_0). Therefore, f and g are homotopic if and only if there is a homotopy function between them, that mean that there is a path between them in Y^X. There is a path if an only if f and g are in the same connected component. By *Corollary 1.2.7* this holds if and only if they are in the same order component. □

Corollary 1.2.11. *A finite space X with a maximum or minimum is contractible.*

Proof. If X has a maximum or minimum the identity map id_X is comparable to the constant map c_m, where m is the maximum or minimum. □

Corollary 1.2.12. *The open sets $\{U_x\}_{x \in X}$ of the base for the topology induce by the preorder are contractible.*

Proof. x is a maximum for U_x so by *Corollary 1.2.11* it is contractible. □

Proposition 1.2.13. *The categories $h(fT_0)$ and $h(fPOSET)$ are equivalent.*

Proof. Consider the functors $P : fT_0 \to fPOSET$ and $T : fPOSET \to fT_0$ defined in section 1.1. By Proposition 1.2.10 we know that for all morphisms $f,\ g : X \to X'$ in $Hom(fT_0)$ $f \simeq g$ if and only if $f \simeq_{\leq} g$ $f,\ g : X \to X'$ in $Hom(fPOSET)$. So the functor P sends equivalence classes of homotopic functions to order classes of functions and T sends order classes of functions to equivalence classes of homotopic functions. Therefore we can define $P : h(fT_0) \to h(fPOSET)$ and $T : h(fPOSET) \to h(fT_0)$. We want to show that there are natural isomorphisms $TP \cong id_{h(fT_0)}$ and $PT \cong id_{h(fPOSET)}$. Consider the diagram for X and X' in $ob(h(fT_0))$ and $[f] : X \to X'$ in $Hom(h(fT_0))$:

$$\begin{array}{ccc} TP(X) & \overset{\tau_X}{\to} & Id_{fT_0}(X) \\ \downarrow_{TP([f])} & & \downarrow_{Id_{fT_0}([f])} \\ TP(X') & \overset{\tau_{X'}}{\to} & Id_{fT_0}(X') \end{array}$$

By the same argument in *Proposition 1.1.18* we have that $TP(X,\tau) = (X,\tau)$, therefore the previous diagram is equivalent to:

$$\begin{array}{ccc} X & \overset{\tau_X}{\to} & X \\ \downarrow_{[f]} & & \downarrow_{[f]} \\ X' & \overset{\tau_{X'}}{\to} & X' \end{array}$$

This diagram is clearly commutative and τ_X is an isomorphism for all $X \in ob(h(fT_0))$. The proof that $PT \cong Id_{h(fPOSET)}$ is analogous. □

1.3 Finite spaces and minimal finite posets

We showed in the previous sections that the categories of fT_0 and $fPOSET$ are equivalent, as well as the categories of $h(fT_0)$ and $h(fPOSET)$. Finite topological spaces and finite preordered sets are the same object and finite T_0-spaces correspond to finite posets. Therefore from now on we will consider finite sets $(X, \tau, \leq)$ with the topology τ, with basis $\{U_x\}_{x \in X}$, and the corresponding order relation $\leq$ defined in *Definition 1.1.13*. In this section we will show that we can associate to every finite topological space a minimal (in the sense of number of points) T_0-space, called the core of X.
Proposition 1.3.1 and *Corollary 1.3.3* refer to *Proposition 1.3.1* and *Remark 1.3.2* in Barmak's book [2].

Proposition 1.3.1. *Any finite topological space has the same homotopy type of a finite $T_0 - space$.*

Proof. Let (X, τ) be a finite topological space. We want to show that X is homotopic equivalent to a finite T_0-space that we denote as X'. We take X' to be the space X quotient by the equivalent relation given by $x \sim y$ if and only if $x \leq y$ and $y \leq x$ (or equivalently $U_x = U_y$). Let $q : X \to X'$ be the quotient map and consider a section $s : X' \to X$, so $qs(x) = id_{X'}$. s is continuous because q is continuous and $qs(x) = id_{X'}$ is order preserving so by *lemma 1.1.14* it is continuous. $sq \leq id_X$ because for all $x \in X$ $sq(x) = s([x])$ where $[x] = \{y \in X : x \leq y \text{ and } y \leq x\}$, so $s([x]) = y \leq x$. Therefore $sq(x) \leq id_X(x)$, for all $x \in X$. There is a fence between sq and id_X so $sq \sim_{\leq} id_X$, by *Proposition 1.2.10* sq and id_X are homotopic maps. Therefore $X \simeq X'$.
Now we want to show that X' is a finite $T_0 - space$. In fact, if $[x] \leq [y]$ then $q(x) \leq q(y)$, since s is continuous $x \leq sq(x) \leq sq(y) \leq y$ ($sq(x) \leq id_X(x)$, but also $sq(x) \geq id_X(x)$ by definition of $[x]$). If also $[y] \leq [x]$, we have that $y \leq x$, so by definition of the quotient space $[x] = [y]$. The relation on X' is antisymmetric so X' is a T_0-space. □

We recall the definition of *retraction* and *strong deformation retraction*, that will be used in the following results.

Definition 1.3.2. Let X be a topological space and $A \subseteq X$ a subset. A continuous map $r : X \to A$ is called a *retraction* of A if $ri = id_A$, where $i : A \hookrightarrow X$ is the inclusion map. Moreover, a continuous map $F : X \times I \to X$, where $I = [0,1]$ is a *deformation retraction* of X onto A if $F(x,0) = x$, $F(x,1) \in A$ and $F(a,1) = a$ for all $x \in X$ and $a \in A$. Equivalently A is a *deformation retract* of X if there is a retraction r such that $ir \simeq id_X$. A *strong deformation retraction* is a deformation retract $F : X \times I \to X$ such that $F(a,t) = a$ for all $t \in I$ and for all $a \in A$. Equivalently, A is a *strong deformation retract* of X if there is a retraction r such that $ir \simeq id_X$ rel A.

Corollary 1.3.3. *X' is a strong deformation retract of X.*

Proof. Consider the section s and the quotient map q defined in *Proposition 1.3.1*. As showed in the proof we have that $qs(x) = id_{X'}$ and $sq(x) \leq id_X(x)$. Moreover, $sq(x)$ and $id_X(x)$ coincide on X' so by *Proposition 1.2.10* $sq \simeq id_X$ rel X'. So X' is a strong deformation retract of X. □

Proposition 1.3.1 implies that in order to study finite topological spaces up to homotopy we can reduce ourself to finite T_0-space. In fact, every topological space X is homotopic equivalent to a finite T_0-space called the *core* of X, that is obtained from X' by eliminating *beat points* (beat points are also called linear and colinear points in [13] or up-beat, down-beat points in [2]).

Definition 1.3.4. (Definition 5.3 [14]) Let X be a finite T_0 topological space. A point x in X is called *beat point* if there exists an other point $x' \neq x$ such that:

- for all $y \in X$, if $x < y$ then $x' \leq y$
- for all $y \in X$, if $y < x$ then $y \leq x'$
- x and x' are comparable

Remark 1.3.5. *Equivalently, $x \in X$ is a beat point if and only if it covers exactly one point or it is covered exactly by one point, if and only if $U'_x = U_x \smallsetminus x$ has a maximum or $F'_x = F_x \smallsetminus x$ has a minimum, where $F_x = \{y \in X : y \geq x\}$.*

Example 1.3.6. *The picture represents a beat point $x \leq x'$.*

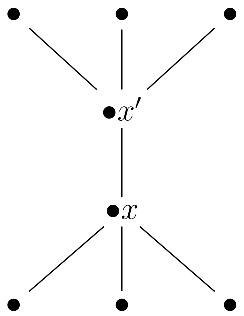


Figure 1.3.1: An example of beat point from [14], page 12.

Proposition 1.3.7. *(Proposition 1.3.4 [2]) Let X be a finite T_0-space and let $x \in X$ be a beat point. Then $X \smallsetminus \{x\}$ is a strong deformation retract of X.*

Proof. Define the map $r : X \to X \smallsetminus \{x\}$ such that $r(z) = z$ for all $z \in X$ such that $z \neq x$ and $r(x) = x'$, where x' is the point described in the *Definition 1.3.4*. r is order preserving, so continuous and it is a retraction, in fact $r(z) \in X \smallsetminus \{x\}$ for all $z \in X$ and $r(y) = y$ for all $y \in X \smallsetminus \{x\}$. Let $i : X \smallsetminus \{x\} \to X$ be the inclusion map, so $ri = id_{X \smallsetminus \{x\}}$. Moreover, x ad x' are comparable, so $x \leq x'$ or $x' \leq x$. Therefore we have that $ir \leq id_X$ or $ir \geq id_X$ and $id_X|_{X \smallsetminus \{x\}} = ir|_{X \smallsetminus \{x\}}$, by *Proposition 1.2.10* $ir \simeq id_X$ rel $X \smallsetminus \{x\}$. □

Definition 1.3.8. Let X be a finite T_0 topological space and $Y \subseteq X$ obtained by removing beat points from X. We say that Y is a *strong collapse* of X and we denote it by $X \searrow\searrow Y$. Moreover, we have by *Proposition 1.3.7* that $Y \simeq X$.

Lemma 1.3.9. *(Lemma 2.2.2 [2]) Let X be a finite T_0 topological space and $Y \subseteq X$ such that all beat points of X are in Y. Let $f : X \to X$ then $f \simeq id_X$ rel Y if and only if $f = id_X$.*

Proof. Since $f \simeq id_X$ rel Y then by *Proposition 1.2.10* we can suppose without loss of generality that $f \leq id_X$ or $f \geq id_X$. Suppose $f \leq id_X$ and define $U'_x = U_x \smallsetminus x$.

If $x \in Y$ then $f(x) = x$. Define the set $A = \{y \in X : f(y) \neq y\}$ and suppose by contradiction that $A \neq \emptyset$. Consider a minimal element x of A, then $f|_{U'_x} = id_{U'_x}$ and $f(x) \neq x$. Then $f(x) \in U'_x$ and for every $y < x$, $y = f(y) \leq f(x)$. So $f(x)$ is a maximum of U'_x. So x is a beat point because it is covered by just one point. Then $x \in Y$ and so $f(x) = x$ but this is contradiction because $x \in A$. Therefore, $A = \emptyset$ and $f = id_X$.

□

Proposition 1.3.10. *(Corollary 2.2.5 [2]) Let X be a finite T_0 topological space and $Y \subseteq X$. $X \searrow\searrow Y$ if and only if Y is a strong deformation retract of X.*

Proof. If $X \searrow\searrow Y$ then Y is obtained by removing beat points, by *Proposition 1.3.7* follows that Y is obtained from X by performing a sequence of strong deformation retractions, so Y is a strong deformation retract of X.
Suppose now that Y is a strong deformation retract of X. Let Z be the set $Z \subseteq X$ such that $X \searrow\searrow Z$ by removing all beat points of X that are not in Y. We claim that Y and Z are homeomorphic. In fact, since $Y \subseteq Z$ and Y and Z are strong deformation retracts of X there are functions $f : Y \to Z$ and $g : Z \to Y$ such that $gf \simeq id_Y$ rel Y and $fg \simeq id_Z$ rel Y. By *Lemma 1.3.9*, Y and Z are homeomorphic so $X \searrow\searrow Z = Y$. □

Proposition 1.3.1 allows us to associate to each finite topological space X a homotopic finite T_0-space X'. In addition, we can reduce X' to a *minimal finite space* by eliminating its beats points. The *minimal finite space* is the smallest (in terms of cardinality) finite T_0-space that has the same homotopy type of X.

Definition 1.3.11. A finite T_0-space is a *minimal finite space* if it has no beat points.

Definition 1.3.12. A *core* of a finite topological space X is a deformation retract of X that is a minimal finite space.

Remark 1.3.13. *Every finite topological space has a core.*

Proposition 1.3.14. *(Theorem 1.3.6 [2]) Let X be a minimal finite space. A map $f : X \to X$ is homotopic to the identity if and only if $f = id_X$.*

Proof. It is a special case of *Proposition 1.3.9.* □

Proposition 1.3.15. *Classification Theorem.(Corollary 1.3.7 [2]) A homotopy equivalence between minimal finite spaces is a homeomorphism. In particular the core of a finite space is unique up to homeomorphism and two finite spaces are homotopy equivalent if and only if they have homeomorphic cores.*

Proof. Let X and Y be homotopy equivalent minimal finite space and $f : X \to Y$ and $g : Y \to X$ the homotopic maps. By *Proposition 1.3.14* $f \circ g = id_Y$ and $f \circ g = id_X$ so X and Y are homeomorphic. If X' and X'' are two cores of X they are homotopy equivalent to X so they are homotopy equivalent minimal spaces and therefore homeomorphic. Two space X ad Y are homotopy equivalent if and only if their cores are homotopy equivalent if and only if they have homeomorphic cores. □

1.4 The Face Poset and the Order Complex functors

In this section we will introduce the category of finite simplicial complexes. We will define the functor that associates to each finite poset a finite simplicial complex and the one that associates to every finite simplicial complex a finite poset. We will then describe the concept of contiguity for simplicial maps, that is the analogous of homotopy for functions between simplicial complexes. The results in this section refer to Barmak [2], *Proposition 1.4.18* and *Proposition 1.4.19* are presented in the language of category theory, the definitions of Order Complex functor and Face poset functor are due to the author.

Definition 1.4.1. An *abstract simplicial complex* K is a non empty set K^0 of $n > 0$ vertices and sets K^i $0 \leq i \leq n$ of subsets of K^0 of cardinality $(i+1)$ (not necessarily all subsets), called *simplices*, such that any subset of cardinality $(j+1)$ of a simplex in K^i is a simplex in K^j. A simplex σ of cardinality $(n+1)$ is called *n-simplex.*

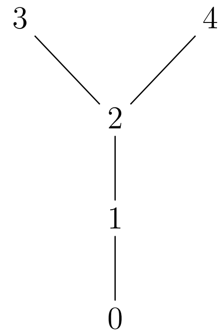
Definition 1.4.2. Let K and L be two simplicial complexes. A *simplicial map* is a function $\phi : K \to L$ such that for all simplices $\sigma \subseteq K$, $\phi(\sigma)$ is a simplex in L.

A simplicial map ϕ is determined by the map $\phi' : K^0 \to L^0$ and the image of a simplex $\sigma = \{x_0, ..., x_i\}$, x_0,...,$x_i \in K^0$ is given by $\phi(\sigma) = \{\phi'(x_0), ..., \phi'(x_i)\}$.

Definition 1.4.3. The category fSC of finite simplicial complexes is the category that has as class of objects all finite simplicial complexes (that are simplicial complexes with a finite number of vertices) and as set of morphisms the set of simplicial maps between finite simplicial complexes.

Definition 1.4.4. (Definition 1.4.4 [2]) Let $(X, \leq)$ be a finite partially ordered set. The *order complex* $\mathcal{K}(X)$ associated with X is the simplicial complex whose set of vertices K^0 is X and whose simplices are given by the finite non-empty chains given by the order relation $\leq$ on X.

Example 1.4.5. *Let $X = \{0, 1, 2, 3, 4\}$ be the following poset.*



The order complex $\mathcal{K}(X)$ associated to X is

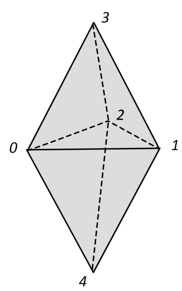


Figure 1.4.1: The order complex $\mathcal{K}(X)$ associated to the finite poset X

It is the simplicial complex with sets of n-simplices given by $K^0 = X$,
$K^1 = \{[01], [02], [03], [04], [12], [13], [14], [23], [24]\}$,
$K^2 = \{[012], [023], [013], [024], [123], [124], [014]\}$ *and* $K^3 = \{[0123], [0124]\}$.

Definition 1.4.6. The *Order Complex functor* $\mathcal{K} : fPOSET \to fSC$ is a functor such that for all $X \in ob(fPOSET)$ $\mathcal{K}(X)$ is the order complex associated to X, and for all order preserving maps $f : X \to Y \in Hom(fPOSET)$, $X, Y \in ob(fPOSET)$, $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$ is defined as $\mathcal{K}(f)(x) = f(x)$ for all $x \in X$.

Remark 1.4.7. *The map $\mathcal{K}(f)$ is a simplicial map because it is defined by the function $f : X \to Y$ on the sets of vertices of $\mathcal{K}(X)$ and $\mathcal{K}(Y)$.*

Definition 1.4.8. (Definition 1.4.10 [2]) Let K be a finite simplicial complex. The *face poset* associated to K, $\chi(K)$ is the poset whose elements are the simplices of K ordered by inclusion.

Example 1.4.9. *Let K be the simplicial complex in Example 1.4.5. The face poset associated to K, $\chi(K)$ is:*

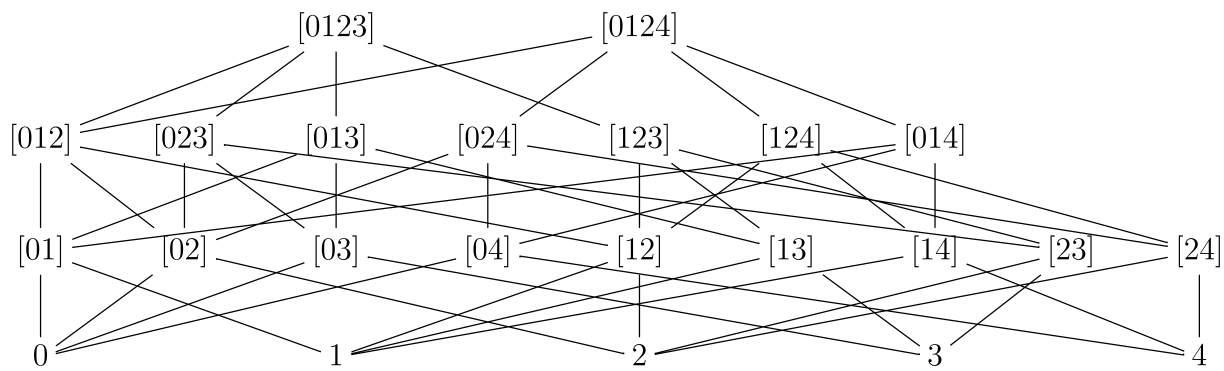


Figure 1.4.2: The face poset $\chi(K)$ associated to the simplicial complex K.

Definition 1.4.10. The *Face Poset functor* $\chi : fSC \to fPOSET$ is a functor such that for all $K \in ob(fSC)$, $\chi(K)$ is the face poset associated to the simplicial complex K, and for all simplicial maps $\psi : K \to L \in Hom(fSC)$, $\chi(\psi) : \chi(K) \to \chi(L)$ defined by $\chi(\psi)(\sigma) = \psi(\sigma)$ for all simplex $\sigma \in K$.

Remark 1.4.11. *$\chi(\psi)$ is order preserving. I fact, $\sigma' \leq \sigma$ in $\chi(K)$ implies $\sigma' = \{x_0, ..., x_n\} \subseteq \sigma = \{x_0, ..., x_n, ..., x_k\}$ in K. So $\psi(\sigma') = \{\psi'(x_0), ..., \psi'(x_n)\} \subseteq \psi(\sigma) = \{\psi(x_0), ..., \psi(x_n), ..., \psi(x_k)\}$ that means $\chi(\psi') \leq \chi(\psi)$.*

From now on we will work in the category fSC. We define the concept of contiguous simplicial maps and contiguity classes and describe the relationship between homotopic maps between finite T_0-spaces and contiguous maps between finite simplicial complexes.

Definition 1.4.12. Let ϕ, $\psi : K \to L$ be two simplicial maps between two simplicial complexes K and L. ϕ and ψ are said to be *contiguous* if for every simplex $\sigma \in K$ $\phi(\sigma) \cup \psi(\sigma)$ is a simplex in L. We denote this relation by $\phi \sim_c \psi$.

Remark 1.4.13. *The relation $\sim_c$ is reflexive and symmetric but in general not transitive.*

Definition 1.4.14. Let ϕ, $\psi : K \to L$ be two simplicial maps between two simplicial complexes K and L. ϕ and ψ are in the same *contiguity class* if there is a sequence of maps $\phi_0 ... \phi_n$ such that $\phi = \phi_0 \sim_c \phi_1 \sim_c ... \sim_c \phi_n = \psi$, $\phi_i : K \to L$ for all $0 \leq i \leq n$. Being in the same contiguity class is an equivalence relation and we write it $\phi \sim \psi$.

Definition 1.4.15. The *Contiguity category* $h(fSC)$ is the category that has as class of objects $ob(h(fSC))$ the class of all finite simplicial complexes and as set of morphisms the contiguity classes $[f]_\sim$ of morphisms between simplicial complexes $f : K \to L$, K, $L \in ob(h(fSC)$.

Remark 1.4.16. *Let $f : K \to L$ and $g : N \to K$ be two simplicial maps. The composition of morphisms is defined for all classes $[f]_\sim \in Hom(h(fSC))$ by $[f]_\sim [g]_\sim = [fg]_\sim$. It is well defined because $f \sim f'$ and $g \sim g'$ implies $f = h_0 \sim_c h_1 \sim_c ... \sim_c h_n = f'$ and $g = k_0 \sim_c k_1 \sim_c ... \sim_c k_n = g'$. So we have that $g(\sigma)$ is a simplex in K for all $\sigma' in N$. Therefore $f(g) = h_0(g) \sim_c h_1(g) \sim_c ... \sim_c h_n(g) = f'(g)$, that means that $fg \sim_c f'g$. Moreover,$f(g) = h_0(g) \sim_c h_1(g) \sim_c$*

... $\sim_c h_n(g) = f'(g) = f'(k_0) \sim_c f'(k_1) \sim_c ... \sim_c f'(k_n) = f'g'$ because $k_i(\sigma')$ is a simplex in K for all $\sigma' \in N$, so $fg \sim_c f'g'$. It is associative and the identity morphism is the class $[id_{fSC}]$.

As proved in *Section 1.1* the category of finite partially ordered sets and the category of finite T_0-spaces are equivalent. Therefore in the next theorems finite T_0-spaces and finite partially ordered sets are considered as the same objects.

Lemma 1.4.17. *(2.1.1 [2]) Let $f, g : X \to Y$ be homotopic maps between T_0-spaces. Then there is a sequence of functions $f = f_0, ..., f_n = g$ such that for every i, $0 \leq i \leq n$ there is a point $x_i \in X$ such that:*

- *f_i and f_{i+1} coincide in $X \smallsetminus x_i$*
- *$f_i \prec f_{i+1}$ or $f_i \succ f_{i+1}$, that is f_i cover f_{i+1} or f_i is covered by f_{i+1}*

Proof. We can assume without loss of generality that $f = f_0 \leq g$ by *Proposition 1.2.10*. Let $A = \{x \in X : f(x) \neq g(x)\}$, if $A = \emptyset$ then $f = g$ and the theorem holds. Suppose $A \neq \emptyset$, let x_0 be a maximal element of A. Take $y \in Y$ such that $f(x_0) \prec y \leq g(x_0)$. Define $f_1 : X \to Y$ by $f_1|_{X \smallsetminus x_0} = f|_{X \smallsetminus x_0}$ and $f_1(x_0) = y$. f_1 is continuous because f is and if $x' \leq x_0$ then $f_1(x') = f(x') \leq f(x_0) \leq y$ and if $x' \geq x_0$ then x' is not in A therefore $f_1(x') = f(x') = g(x') \geq g(x_0) \geq y = f_1(x_0)$. We define in the same way, by induction f_{i+1}. The process ends because X and Y are finite sets. □

Proposition 1.4.18. *(Proposition 2.1.2 [2]) Let $f, g : X \to Y$ be homotopic maps between T_0-spaces. Then the simplicial maps $\mathcal{K}(f)$, $\mathcal{K}(g) : \mathcal{K}(X) \to \mathcal{K}(Y)$ lie in the same contiguity class $\mathcal{K}(f) \sim \mathcal{K}(g)$. That is, the functor $\mathcal{K}$ sends homotopic maps to maps in the same contiguity class.*

Proof. By the previous lemma we can assume without loss of generality that $f(x) = g(x)$ for all $x \in X \smallsetminus x'$ and $f(x') \prec g(x')$. Therefore if C is a chain in X, $f(C) \cup g(C)$ is a chain C' in Y. The map $C \hookrightarrow X$ induces a map $\mathcal{K}(C) \hookrightarrow \mathcal{K}(X)$ and C correspond to the simplex in $\mathcal{K}(X)$, $\sigma = \mathcal{K}(C)$ and in the same way C' corresponds to a simplex $\mathcal{K}(C')$ in $\mathcal{K}(Y)$. Therefore $\mathcal{K}(f(C) \cup g(C)) = \mathcal{K}(f(C)) \cup \mathcal{K}(g(C)) = f(C) \cup g(C)$ is a simplex in $K(Y)$. So we have that $\mathcal{K}(f) \sim \mathcal{K}(g)$. □

Proposition 1.4.19. *(Proposition 2.1.3 [2]) Let ϕ and $\psi : K \to L$ be two simplicial maps that lies in the same contiguity class $\phi \sim \psi$, then $\chi(\phi) \simeq \chi(\psi)$. That is, the functor χ sends maps in the same contiguity class in homotopic maps.*

Proof. ϕ and $\psi : K \to L$ be two simplicial maps that lies in the same contiguity class, we can suppose without loss of generality that they are contiguous, so that for all simplices $\sigma \in K$ we have that $\phi(\sigma) \cup \psi(\sigma)$ is a simplex in L. We define the function $f : \chi(K) \to \chi(L)$ as $f(\sigma) = \phi(\sigma) \cup \psi(\sigma)$ for all $\sigma \in \chi(K)$. So we have that the induced functions satisfies $\chi(\phi) \leq f \geq \chi(\psi)$ so $\chi(\phi) \sim_{\leq} \chi(\psi)$, then by *Proposition 1.2.10* $\chi(\phi) \simeq \chi(\psi)$. □

Remark 1.4.20. *The functors $\mathcal{K} : fPOSET \to fSC$ and $\chi : fSC \to fPOSET$ can be restricted to the functors $\mathcal{K} : h(fPOSET) \to h(fSC)$ and $\chi : h(fSC) \to h(fPOSET)$ by Proposition 1.4.18 and Proposition 1.4.19.*

Remark 1.4.21. *The functors $\mathcal{K} : fPOSET \to fSC$ and $\chi : fSC \to fPOSET$ do not provide an equivalence between the category $fPOSET$ and fSC or $h(fPOSET)$ and $h(fSC)$. In fact, given a simplicial complex K, $\mathcal{K}(\chi(K))$ is in general non isomorphic to K and a partially ordered set X is in general non isomorphic to $\chi(\mathcal{K}(X))$. Examples 1.4.5 and 1.4.9 show a partially ordered set X, its order complex $\mathcal{K}(X)$ and $\chi(\mathcal{K}(X))$. X and $\chi(\mathcal{K}(X))$ are clearly non isomorphic because they have different number of points. We can define $\mathcal{K}(\chi(K))$ and $\chi(\mathcal{K}(X))$ as follows.*

Definition 1.4.22. Given a finite simplicial complex K, the *barycentric subdivision* of K, denoted as $sd(K)$, is defined by $sd(K) = \mathcal{K}(\chi(K))$.

Definition 1.4.23. Given a finite partially ordered set X, the *subdivision* of X, is defined by $sd(X) = \chi(\mathcal{K}(X))$.

1.5 Strong homotopy type

In this section we will define the notion of strong homotopy equivalence and strong homotopy type. We will show that two simplicial complexes have the same strong homotopy type if and only if they are strong homotopy equivalent. In this section we will work in the category fSC, therefore we will consider just finite simplicial complexes. The results refer to Barmak's *Chaper 5* in [2].

Definition 1.5.1. A simplicial map $\phi : K \to L$ is a *strong equivalence* if there is a map $\psi : L \to K$ such that $\psi\phi \sim id_K$ and $\phi\psi \sim id_L$. In this case K and L are *strong equivalent* simplicial complexes and the relation is denoted by $K \sim L$.

Definition 1.5.2. A vertex v of a simplicial complex K is *dominated* by another vertex $v' \neq v$ of K if every maximal simplex (in the sense of inclusion) that contains v also contains v'.

Definition 1.5.3. Let K be a simplicial complex and v a vertex in K. The *deletion* of v, denoted by $K \smallsetminus v$, is the full subcomplex of K spanned by the vertices different from v.

Proposition 1.5.4. *(Proposition 5.1.8 [2]) Let K be a simplicial complex and v a vertex in K dominated by the vertex v' in K. Then the inclusion $i : K \smallsetminus v \hookrightarrow K$ is a strong equivalence. The retraction $r : K \to K \smallsetminus v$ defined by $r|_{K \smallsetminus v} = id_{K \smallsetminus v}$ and $r(v) = v'$ is its contiguity inverse, that is $ir \sim id_K$ and $ri \sim id_{K \smallsetminus v}$. In particular, $K \sim K \smallsetminus v$.*

Proof. We want to show that $ir \sim id_K$ and $ri \sim id_{K \smallsetminus v}$. Let $\sigma \in K$ be a simplex that contains v and $\sigma' \supseteq \sigma$ a maximal simplex, so $v' \subseteq \sigma'$. $r(\sigma) = \sigma \cup \{v'\} \smallsetminus \{v\} \subseteq \sigma'$ so it is a simplex in $K \smallsetminus v$. $ir(\sigma) \cup id_K(\sigma) = \sigma \cup \{v'\} \subseteq \sigma'$ so it is a simplex in K and so $ir \sim id_K$. Let now $\sigma \in K \smallsetminus v$ then $ri(\sigma) \cup id_{K \smallsetminus v}(\sigma) = \sigma$ that is a simplex in $K \smallsetminus v$, so $ri \sim id_{K \smallsetminus v}$. Therefore $K \sim K \smallsetminus v$. □

Definition 1.5.5. The retraction $r : K \to K \smallsetminus v$ is called *elementary strong collapse* from K to $K \smallsetminus v$ and it is denoted by $K \searrow\!\!\searrow K \smallsetminus v$

Definition 1.5.6. A *strong collapse* is a finite sequence of elementary strong collapses. The inverse of a strong collapse is called *strong expansion.*

Definition 1.5.7. Two simplicial complexes K and L have the same *Strong homotopy type* if there is a sequence of strong collapses and strong expansions that transform K into L.

Example 1.5.8. *The picture shows a simplicial complex K with a vertex v dominated by the vertex v'. The second simplicial complex $L = K \setminus \{v\}$ represents an elementary strong collapse of K. K and L have the same strong homotopy type.*

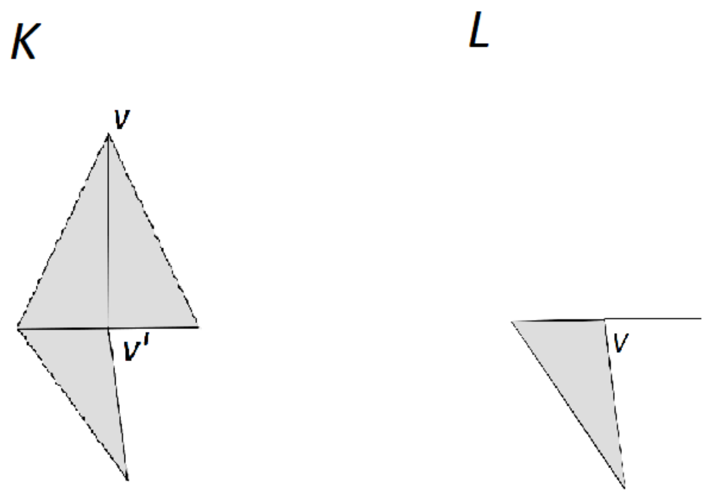


Figure 1.5.1: A strong collapse: $K \searrow\searrow K \setminus v$.

Definition 1.5.9. A finite simplicial complex is a *minimal complex* if it has no dominated points.

Proposition 1.5.10. *(Proposition 5.1.6 [2]) Let K be a minimal complex and $\phi : K \to K$ such that $\phi \sim id_K$. Then $\phi = id_K$*

Proof. We can assume without loss of generality that $\sigma \sim_c id_K$. Let v in K and σ the maximal simplex that contains v. By contiguity $\phi(\sigma) \cup \sigma$ is a simplex in K and since $v \in \phi(\sigma) \cup \sigma$ by the maximality of σ we have that $\phi(\sigma) \cup \sigma = \sigma$. Moreover $\phi(v) \in \phi(\sigma) \cup \sigma = \sigma$, so every maximal simplex that contains v contains also $\phi(v)$ that means that v is dominated by $\phi(v)$. But K is a minimal complex so $\phi(v) = v$ for all $v \in V$. □

Corollary 1.5.11. *A strong equivalence between minimal complexes is an isomorphism.*

Proof. Let K and L be simplicial complexes, $\phi : K \to L$ and $\psi : L \to K$ such that $\psi\phi \sim id_K$ and $\phi\psi \sim id_L$. Then $\psi\phi = id_K$ and $\phi\psi = id_L$. □

Definition 1.5.12. The *core* of a simplicial complex K is a minimal subcomplex K_0 such that $K \searrow\searrow K_0$

Theorem 1.5.13. *(Proposition 5.1.10 [2]) Every simplicial complex has a core that is unique up to isomorphisms. Two simplicial complexes have the same strong homotopy type if and only if their cores are isomorphic.*

Proof. Suppose that K has two cores K_0 and K_0' then they have the same strong homotopy type of K since they are obtained from K by removing dominated points. By *Proposition 1.5.4* $K_0 \sim K_0'$ and since they are minimal complex *Corollary 1.5.11* they are isomorphic. If K and L have the same strong homotopy type then their cores K_0 and L_0 do. Then K_0 and L_0 are isomorphic. On the other hand if K_0 and L_0 are isomorphic, by *Remark 5.1.2* in [2] they have the same strong homotopy type, therefore there is a sequence of strong collapses and expansion between them. Therefore there is a sequence of strong expansion and collapses between K and L, that is K and L have the same strong homotopy type. □

Corollary 1.5.14. *Let K and L be two simplicial complexes. K and L have the same strong homotopy type if and only if they are strong homotopy equivalent, $K \sim L$.*

Proof. By *Theorem 1.5.13*, K and L have the same strong homotopy type if and only if their cores K_0 and L_0 are strong homotopy equivalent $K_0 \sim L_0$ if and only if $K \sim L$. □

Definition 1.5.15. A complex is said to be *strong collapsible* if it strong collapses to a point or equivalently if it has the strong homotopy type of a point.

Example 1.5.16. *The simplicial complex K, showed in the picture, is an example of a non strong collapsible simplicial complex whose realisation $|K|$ is contractible. K in fact is a minimal space because it has no dominated points.*

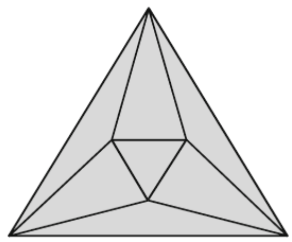


Figure 1.5.2: A non strong collapsible simplicial complex whose realisation $|K|$ is contractible, [2] page 76.

1.6 Homotopy type of finite T_0-spaces and strong homotopy type of finite simplicial complexes

In this section we want to describe the behaviour of the functors $\mathcal{K}$ and χ defined in *Section 1.4*. In particular, we want to see which properties of finite T_0-spaces correspond to specific properties of finite simplicial complexes and vice-versa if we apply the two functors. At the end of the section there will be a table that summarises the results. *Theorem 1.6.1* and *Theorem 1.6.2* refer to *Theorem 5.2.1* in Barmak's book [2] but they are presented in the language of category theory, *Theorem 1.6.5* refers to *Theorem 5.2.2* in [2] but the proof is due to the author. *Theorem 1.6.7* refers to *Theorem 5.2.5* in [2] but the proof differs and it uses the definition of dominated point given in *Section 1.5*. *Theorem 1.6.9* and *Remark 1.6.10* are due to the author. *Theorem 1.6.16* and *Theorem 1.6.17* correspond to *Theorem 5.2.6* and *Corollary 5.2.7* in [2].

Theorem 1.6.1. *If two finite T_0-spaces are homotopic equivalent, then their order complexes have the same strong homtopy type.*

Proof. Let X and Y be two homotopic equivalent finite T_0-spaces, and $f : X \to Y$ and $g : Y \to X$ continuous functions such that $fg \simeq id_Y$ and $gf \simeq id_X$. The Order complex functor $\mathcal{K}$ induces the maps $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$ and $\mathcal{K}(g) : \mathcal{K}(Y) \to \mathcal{K}(X)$ where $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are the order complexes associated to X and Y. By *Proposition 1.4.18* we have that $\mathcal{K}(fg) \sim id_{\mathcal{K}(Y)}$ and $\mathcal{K}(gf) \sim id_{\mathcal{K}(X)}$, that is $\mathcal{K}(f)\mathcal{K}(g) \sim id_{\mathcal{K}(Y)}$ and $\mathcal{K}(g)\mathcal{K}(f) \sim id_{\mathcal{K}(X)}$ so $\mathcal{K}(X) \sim \mathcal{K}(Y)$. □

Theorem 1.6.2. *If two finite simplicial complexes have the same strong homotopy type then the associated face posets are homotopy equivalent finite T_0-spaces.*

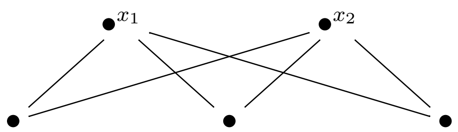
Proof. Let K and L be finite simplicial complexes such that they have the same strong homotopy type so there are the simplicial maps $\phi : K \to L$ to $\psi : L \to K$ and $\psi\phi \sim id_K$ and $\phi\psi \sim id_L$. We obtain the maps induced by the poset functor $\chi(\phi) : \chi(K) \to \chi(L)$, $\chi(\psi) : \chi(L) \to \chi(K)$ and by *Proposition 1.4.19* we have that $\chi(\psi)\chi(\phi) \simeq id_{\chi(K)}$ and $\chi(\phi)\chi(\psi) \simeq id_{\chi(L)}$ therefore $\chi(K) \simeq \chi(L)$. □

The implication in the previous theorem holds also in the other direction, as it is proved in *Corollary 9.2.2* in [2].

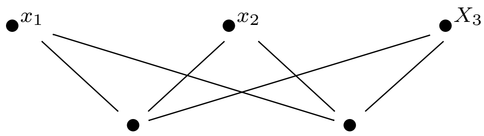
Theorem 1.6.3. *(Corollary 9.2.2 [2]) Let K and L be finite simplicial complexes and $\chi(K) \simeq \chi(L)$, then $K \sim L$.*

Remark 1.6.4. *The same theorem does not hold for the functor $\mathcal{K}$. If $\mathcal{K}(X) \sim \mathcal{K}(Y)$ then in general X and Y are not homotopic equivalent. An example is the posets X and X^{op} showed in the following picture:*

X



X^{op}



X and X^{op} are not homotopic equivalent (we will show that they have different category that is an homotopy invariant) but the associated order complexes are isomorphic, $\mathcal{K}(X) = \mathcal{K}(Y)$.

Theorem 1.6.5. *Let X be a finite T_0-space and let $Y \subseteq X$. If $X \searrow\searrow Y$ then $\mathcal{K}(X) \searrow\searrow \mathcal{K}(Y)$.*
Let K be a finite simplicial complex and $L \subseteq K$. If $K \searrow\searrow L$ then $\chi(K) \searrow\searrow \chi(L)$.

Proof. We can suppose without loss of generality that $Y = X \setminus x'$ where x' is a beat point. Let y be the unique point in X such that $x' \prec y$ or $y \prec x'$. Let $X \searrow\searrow Y$ so by *Proposition 1.3.7* there is a retraction $r : X \to Y$ such that $ir \simeq id_X$ rel Y and $ri = id_Y$. There are functions $\mathcal{K}(r) : \mathcal{K}(X) \to \mathcal{K}(Y)$ and $\mathcal{K}(i) : \mathcal{K}(Y) \to \mathcal{K}(X)$ induced by the functor $\mathcal{K}$, such that $\mathcal{K}(i)\mathcal{K}(r) \sim id_{\mathcal{K}(X)}$ and $\mathcal{K}(r)\mathcal{K}(i) \sim id_{\mathcal{K}(Y)}$. Now $\mathcal{K}(r) : \mathcal{K}(X) \to \mathcal{K}(Y)$ is defined by $\mathcal{K}(r)(x) = r(x)$ for all $x \in X$, that is $\mathcal{K}(r)(x) = x$ for all $x \neq x'$ and $\mathcal{K}(r)(x') = y$. Since $x' \prec y$ or $y \prec x'$, all maximal chains in X that contain x' contains y that means that all maximal simplices in the order complex $\mathcal{K}(X)$ that contain x' contain also y. So $x' \in K(X)$ is dominated by $y \in \mathcal{K}(X)$ and $\mathcal{K}(r)$ is an elementary strong collapse. Therefore $\mathcal{K}(X) \searrow\searrow \mathcal{K}(Y)$.
Let suppose now that K is a finite simplicial complex, $L \subseteq K$ and $K \searrow\searrow L$. We can suppose without loss of generality that L is obtained from K by an elementary strong collapse, so by eliminating the point v dominated by v'. Therefore there is a retraction defined in *Proposition 1.5.4* $r : K \to L$ such that $ir \sim id_K$ and $ri = id_L$. We have induced maps $\chi(r) : \chi(K) \to \chi(L)$ and $\chi(ir) \simeq id_{\chi(K)}$ and $\chi(ri) = id_{\chi(L)}$. Now $\chi(r)(\sigma) = r(\sigma)$ for all $\sigma \in \chi(K)$, therefore $\chi(i)\chi(r) \simeq id_{\chi(K)}$ rel $\chi(K \setminus v)$ and $\chi(r)\chi(i) = id_{\chi(L)}$. So $\chi(r)$ is a strong deformation retract, therefore by *Proposition 1.3.10* $\chi(K) \searrow\searrow \chi(L)$.

□

Remark 1.6.6. *The previous proof shows that the functor $\mathcal{K}$ sends beat points to dominated points. On the other hand, the functor χ does not send, in general, dominated points to beat points. Consider, for example the simplicial complex in the following figure and its face poset. The vertex 1 is dominated, for example by the vertex 2 but in the face poset 1 is clearly not a beat point.*

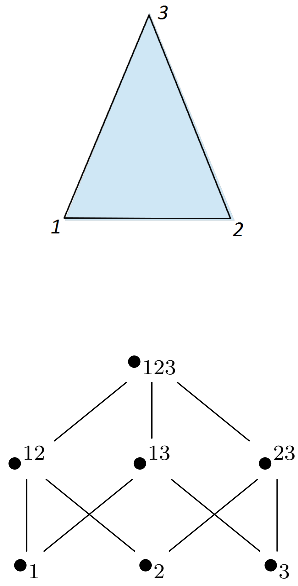


Figure 1.6.1: The functor χ does not send dominated vertices to beat points.

Theorem 1.6.7. *Let X be a finite T_0-space. Then X is a minimal finite space if and only if $\mathcal{K}(X)$ is a minimal simplicial complex.*

Proof. Suppose that X is not a minimal space, so it has a beat point x, and $X \searrow\searrow X \smallsetminus x$. Therefore by *Theorem 1.6.5* we have that $\mathcal{K}(X) \searrow\searrow \mathcal{K}(X \smallsetminus x)$. Therefore $\mathcal{K}(X)$ is not a minimal complex.
Suppose now that $\mathcal{K}(X)$ is not minimal so there is a dominated vertex $v \in \mathcal{K}(X)$. If v is dominated by $v' \in K$ all maximal subcomplexes that contain v also contain v'. But maximal subcomplexes are induced by the functor $\mathcal{K}$ from maximal chains in X, therefore all maximal chains that contain v also contain v'. Suppose that $v < v'$, we have that all element that are comparable with v are also comparable with v'. We define $V = \{x \in X : x > v$ and all elements comparable with x are comparable with v$\}$ and we call x' the minimum of V. Then v is the maximum of $U_{x'} \smallsetminus x'$ therefore $x' \in X$ is a beat point. So X is not a minimal space. □

Remark 1.6.8. *The functor $\mathcal{K}$ sends minimal finite spaces in minimal simplicial complexes.*

Theorem 1.6.9. *Let K be a finite simplicial complex. K is a minimal simplicial complex if $\chi(K)$ is a minimal space.*

Proof. Suppose that K is not a minimal simplicial complex then there is a dominated vertex v of K and $K \searrow\searrow K \smallsetminus v$. Therefore by *Theorem 1.6.5* we have that $\chi(K) \searrow\searrow \chi(K \smallsetminus v)$. □

Remark 1.6.10. *The other implication does not hold. If K is a minimal simplicial complex in general $\chi(K)$ is not a minimal space. An example of that is provided by the simplicial complex in Example 1.5.16. K is a minimal complex but its face poset has bit points that are the points that correspond to the 1-simplices in the external triangle. Each 1-simplex is contained in just one 2-simplex therefore it is a point in the face poset that is covered by just one other point, so it is a beat point.*

We recall the definitions of link and star of a vertex of a simplicial complex and the operation of join of two simplicial complexes that will be needed in the following theorems.

Definition 1.6.11. Let K be a simplicial complex and v a vertex of K, the *star* of v is the subcomplex

$$st(v) = \{\sigma \in K : \sigma \cup \{v\} \in K\}$$

Definition 1.6.12. Let K be a simplicial complex and v a vertex of K, the *link* of v is the subcomplex of $st(v)$ of simplices that do not contain v

$$lk(v) = \{\sigma \in K : \sigma \cup \{v\} \in K, v \notin \sigma\}$$

Definition 1.6.13. Let K and L be two simplicial complexes, the *simplicial join*, $K * L$ is the simplicial complex given by

$$K * L = K \cup L \cup \{\sigma \cup \tau : \sigma \in K, \tau \in L\}$$

The *simplicial cone*, vK with base K and v a vertex not in K is the simplicial join $K * v$.

Remark 1.6.14. *In the literature, for example in [2] an equivalent definition of elementary strong collapse is given. We say that there is an elementary strong collapse from K to $K \searrow v$ if $lk(v)$ is a simplicial cone $v'L$, in this case we say that v is dominated by v'.*

Lemma 1.6.15. *Let K be a finite simplicial complex and v a vertex of K. v is dominated by v', $v \neq v'$ if and only if $lk(v)$ is a simplicial cone $v'L$.*

Proof. Suppose that v is dominated by v', then all maximal simplices that contain v also contain v'. The maximal simplices that contain v are the simplices in $st(v) = vlk(v)$, therefore all maximal simplices in $lk(v)$ contain v', therefore $lk(v)$ is a simplicial cone $v'L$ for some simplicial complex L. On the other hand, if $lk(v)$ is a simplicial cone $v'L$ all maximal simplices contain v'. If we consider then $st(v) = vlk(v)$ we have that all maximal simplices that contain v also contain v'. □

Theorem 1.6.16. *(Theorem 5.2.6 [2]) Let K be a finite simplicial complex. Then K is strong collapsible if and only if $sd(K)$ is strong collapsible.*

Proof. If $K \searrow\searrow *$ then $\chi(K) \searrow\searrow *$ then $sd(K) = K(\chi(K)) \searrow\searrow *$ by *Theorem 1.6.5.*

Suppose now that $sb(K) \searrow\searrow *$ and suppose that L is the core of K therefore $K \searrow\searrow L$ implies by *Theorem 1.6.5* that $sd(K) \searrow\searrow sd(L)$. Now $sd(K)$ is strong collapsible therefore $sd(L) \searrow\searrow *$ and $sd(L) = L_0 \searrow\searrow ... \searrow\searrow L_n = *$ is a sequences of elementary strong collapses from $sd(L)$ to a point. We want to show by induction that $L_i \subseteq sd(L)$ contains as vertices all the barycenters of all 0-simplices and of all maximal simplices of L.

This is clearly true for $L_0 = sd(L)$, now we suppose that $L_i \subseteq sd(L)$ contains as vertices all the barycenters of all 0-simplices and of all maximal simplices of L and we want to show that this is true also for L_{i+1}.

Let $\sigma = [v_0, ..., v_k]$ be a maximal simplex of L. Suppose that $b(\sigma)$ is a vertex of L_i, we want to show that it is a vertex of L_{i+1}. First we claim that $lk_{L_i}(b(\sigma))$ is not a cone. If σ is a 0-simplex the link is empty, so we can suppose that σ is not a 0-simplex. Now, $b(v_j)b(\sigma)$ is a simplex in $sd(L)$, $b(v_j)$ is in L_i by the inductive hypothesis and L_i is a full subcomplex of $sd(L)$ therefore $v_j \in lk_{L_i}(b(\sigma))$

for all $0 \leq j \leq k$. If $lk_{L_i}(b(\sigma))$ is a cone then there is a simplex σ' in L such that $b(\sigma') \in lk_{L_i}(b(\sigma))$ and $b(\sigma')b(v_j) \in lk_{L_i}(b(\sigma))$ for all $0 \leq j \leq k$. Since σ is a maximal simplex then $\sigma' \subseteq \sigma$ and $v_j \in \sigma'$ for all $0 \leq j \leq k$, therefore $\sigma \subseteq \sigma'$ but this is a contradiction. Therefore $b(\sigma)$ is not a dominated vertex therefore $b(\sigma)$ is a vertex of L_{i+1}.

Let v be a vertex of L. Suppose that v is a vertex in L_i. If v is maximal then $lk_{L_i}(b(v))$ is the empty set. Suppose that v is not a maximal simplex and σ_0,..., σ_k the maximal simplices of L that contain v, $b(\sigma_j)$ for all $0 \leq j \leq k$ are vertices of L_i by inductive hypothesis and since L_i is a full subcomplex of $sd(L)$ we have that $b(\sigma_j) \in lk_{L_i}(b(v))$. Suppose that $lk_{L_i}(b(v))$ is a cone. Then there is a simplex σ in L such that $b(\sigma) \in lk_{L_i}(b(v))$ and $b(\sigma)b(\sigma_j) \in lk_{L_i}(b(v))$ for all $0 \leq j \leq k$. In particular $v \subseteq \sigma$ and $\sigma \subseteq \sigma_j$ for all $0 \leq j \leq k$. Let $v' \in \sigma$ $v' \neq v$, then v' is contained in every maximal simplex that contain v. Therefore there is a dominated vertex and this contradicts the minimality of L. Therefore $b(v)$ is not dominated in L_i and so $b(v)$ is a vertex in L_{i+1}.

By induction all L_i contains as vertices all the barycenters of all 0-simplices and all maximal simplices of L. We supposed that $L_n = *$ and L_n contains as vertices all the barycenters of all vertices of L therefore $L = *$. Since L is the core of K we can conclude that K is strong collapsible. □

Theorem 1.6.17. *X is a contractible finite T_0-space if and only if $X' = \chi(K(X))$ is contractible.*

Proof. If X is a contractible finite T_0-spaces then $X \searrow\searrow *$ so by *Theorem 1.6.5* $K(X) \searrow\searrow *$ and $\chi(K(X)) \searrow\searrow *$.

Suppose that $\chi(K(X)) \searrow\searrow *$. Let Y be the core of X then $X \searrow\searrow Y$ implies by *Theorem 1.6.5* $X' \searrow\searrow Y'$ and $Y' \searrow\searrow *$. Now $\mathcal{K}(Y') = \mathcal{K}(\chi(\mathcal{K}(Y))) = sd(\mathcal{K}(Y))$ that is strong collapsible, therefore by *Theorem 1.6.16* $\mathcal{K}(Y)$ is strong collapsible. Therefore by *Theorem 1.6.7* since Y is a minimal space $\mathcal{K}(Y)$ is a minimal simplicial complex therefore $\mathcal{K}(Y) = *$ that implies $Y = *$ so X is contractible. □

Corollary 1.6.18. *Let X be a finite T_0-space. X is contractible if and only if $\mathcal{K}(X)$ is strong collapsible. Let K be a finite simplicial complex. Then K is strong collapsible if and only if $\chi(K)$ is contractible.*

We can summarize the results in this chapter as follows:

The functor $\mathcal{K}$:

- $X \simeq Y \Rightarrow \mathcal{K}(X) \sim \mathcal{K}(Y)$
- $X \searrow\searrow Y \Rightarrow \mathcal{K}(X) \searrow\searrow \mathcal{K}(Y)$
- $x \in X$ is a beat point $\Rightarrow \mathcal{K}(\{x\})$ is a dominated vertex in $\mathcal{K}(X)$
- X is a minimal finite space $\Leftrightarrow \mathcal{K}(X)$ is a minimal finite simplicial complex
- X is contractible $\Leftrightarrow \mathcal{K}(X)$ is strong collapsible

The functor χ

- $K \sim L \Leftrightarrow \chi(K) \simeq \chi(L)$
- $K \searrow\searrow L \Rightarrow \chi(K) \searrow\searrow \chi(L)$
- In general χ doesn't send dominated vertices to beat points
- K is a minimal finite simplicial complex $\Leftarrow \chi(K)$ is a minimal finite space
- K is strong collapsible $\Leftrightarrow \chi(K)$ is contractible

- K is strong collapsible if and only if $sd(K)$ is strong collapsible
- X is contractible if and only if $sd(X)$ is contractible

Chapter 2

The simplicial Lusternik-Schnirelmann category

In this chapter we will introduce the simplicial L-S category. First we define the L-S category in the classical way for topological spaces, then we will restrict to the case of finite T_0-topological spaces showing some results from [14]. Following [14] in *Section* 3 and 4 we define the simplicial L-S category and geometric category using the concept of contiguity. Finally we study how the L-S category for finite T_0-spaces and the simplicial category are related if we apply the functors $\mathcal{K}$ and χ.

2.1 L-S category for topological spaces

The L-S category of a topological space X represents the minimal number of open subsets contractible in the space that can cover X. The definitions and results refer to [4], the proofs of *Proposition 2.1.3*, *Proposition 2.1.8* and *Example 2.1.5* are due to the author. *Example 2.1.9* can be found in [14].

Definition 2.1.1. Let X be a topological space. A subset $U \subseteq X$ is called *categorical* if U can be contracted to a point x in X. Equivalently, $U \subseteq X$ is *categorical* if the inclusion map $i_U : U \hookrightarrow X$ is homotopic to some constant map c_x, $x \in X$.

Definition 2.1.2. Let X be a topological space. The *Lusternik-Schnirelmann category*, $cat(X)$ is the least integer $n \geq 0$ such that there is a cover of X of $n+1$ categorical open subsets. We write $cat(X) = \infty$ if this cover does not exist.

Proposition 2.1.3. *Let X be a topological space. The L-S category $cat(X)$ is an homotopy invariant, that is if $X \simeq Y$ for some topological space Y then $cat(X) = cat(Y)$.*

Proof. Since $X \simeq Y$ we have two continuous maps $f : X \to Y$ and $g : Y \to X$ such that $gf \simeq id_X$ and $fg \simeq id_Y$. Suppose that $\{U_i\}_{i\geq 0}$ is a cover of Y of open categorical subsets, then $i_{U_i} \simeq c_{y_i}$, $y_i \in Y$. Then consider the open set $f^{-1}(U) \subseteq X$ where U is an element of the cover, $U = U_i$ for some $i \geq 0$. We want to show that $f^{-1}(U)$ is categorical, that is $i_{f^{-1}(U)} \simeq c_x$, $x \in X$. We define $f' : f^{-1}(U) \to U$ to be the restriction $f_{|f^{-1}(U)}$, since $gf \simeq id_X$ we have that $gfi_{f^{-1}(U)} \simeq id_X i_{f^{-1}(U)}$. Moreover $fi_{f^{-1}(U)} = i_U f'$ so $gi_U f' \simeq i_{f^{-1}(U)}$ and $gi_U f' \simeq gc_y f'$, the last map is constant so we have that $f^{-1}(U)$ is a categorical subset of X and $\{f^{-1}(U_i)\}_{i\geq 0}$ is a categorical cover of X, therefore $cat(X) \leq cat(Y)$. In an analogous way we can prove that if $\{U_j\}_{j\geq 0}$ of X is a categorical cover of X then $\{g^{-1}(U_j)\}_{j\geq 0}$ is a categorical cover of Y, therefore $cat(Y) \leq cat(X)$. We can conclude that $cat(X) = cat(Y)$. □

We can consider the minimal cover of open subsets of X that are contractible in a point x in the subset. So we can define in a similar way the notion of geometric category.

Definition 2.1.4. Let X be a topological space. A subset $U \subseteq X$ is called *contractible in itself* if U can be contracted to a point x in U. Equivalently, $U \subseteq X$ is *contractible in itself* if the identity map on U $id_U : U \to X$ is homotopic to some constant map c_x, $x \in U$.

Example 2.1.5. *The subset $U \subseteq S^2$, $U \simeq S^1$ is an example of a subset of S^2 that is contractible in S^2 but it is not contractible in itself. In fact id_U is clearly non homotopic to any constant map c_x such that $x \in U$ but $i_U \simeq c_N$ as is shown in the picture.*

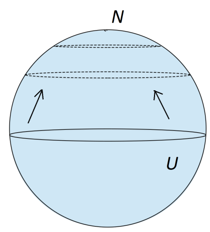


Figure 2.1.1: A subset of S^2 that is contractible in the sphere but not in itself.

Definition 2.1.6. Let X be a topological space. The *geometric category*, $gcat(X)$, is the least integer $n \geq 0$ such that there is a cover of X of $n+1$ open subsets that are contractible in themselves.

Remark 2.1.7. *The geometric category is not an homotopic invariant. A classical example, due to Fox, is showed in Proposition 3.11 in [4]: let J be the wedge of a sphere and two circles and J' is obtained from the sphere by identifying three distinct points, then $J \simeq J'$ but $gcat(J) < gcat(J')$. An other example is Example 2.2.7 that shows a topological space X and its homotopic core X_0 such that $gcat(X) = 1$ while $gcat(X_0) = 2$.*

Proposition 2.1.8. *Let X be a topological space. Then $cat(X) \leq gcat(X)$.*

Proof. Let $U \subseteq X$ be an open subset that is contractible in itself and so $id_U \simeq c_x$, $x \in U$. Consider the inclusion $i : U \hookrightarrow X$, we have that $i_U = i_U id_U \simeq i_U c_x = c_x$, so U is a categorical subset. Therefore if $\{U_i\}_{i \geq 0}$ is a cover of X of open subset contractible in themselves, $\{U_i\}_{i \geq 0}$ is a cover of X of categorical subsets. □

Example 2.1.9. *The following topological space X provides an example of a space for which Proposition 2.1.8 holds with a strict inequality.*

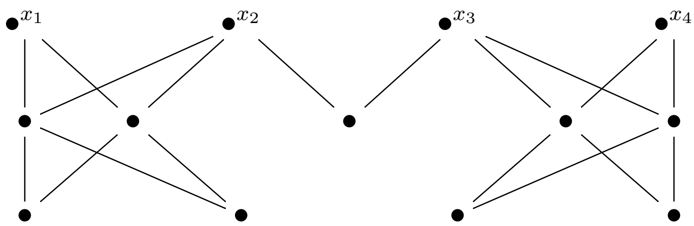


Figure 2.1.2: An example of a finite poset X such that $cat(X) < gcat(X)$, [14] page 15.

In fact, $\{U_{x_1} \cup U_{x_4}, U_{x_2} \cup U_{x_3}\}$ provides a cover of categorical subsets and X is non contractible so $cat(X) = 1$. However $\{U_{x_1} \cup U_{x_4}, U_{x_2} \cup U_{x_3}\}$ is not a cover of subsets contractible in themselves because $U_{x_1} \cup U_{x_4}$ doesn't contract in itself and there is no cover given by two subsets contractible in themselves because $U_{x_2} \cup U_{x_3}$ is the only union of basic open subsets that contract in one of its points. We can conclude that $\{U_{x_1}, U_{x_4}, U_{x_2} \cup U_{x_3}\}$ is a cover of X and so $gcat(X) = 2$.

Remark 2.1.10. *A categorical subset may not be connected. For example $\{U_{x_1} \cup U_{x_4}\}$, in the previous example is categorical. The two connected components can be contracted in themselves to two different points, in this case x_1 and x_4 and there is a path in X but not in $\{U_{x_1} \cup U_{x_4}\}$ that connect two points, therefore we can contract one point on the other.*

2.2 L-S category and geometric category for finite T_0-spaces

We will now consider the concept of L-S category, defined in the previous section, in the special case of finite T_0-spaces. Therefore we will study the L-S category and geometric category for finite partially ordered set. The following results refer to *Chaper 5* in [14].

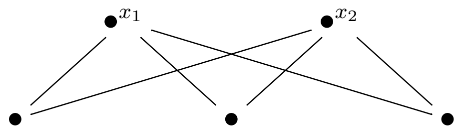
The following result gives an upper bound for the categories of finite partially ordered sets.

Proposition 2.2.1. *Let X be a finite partially ordered set. Let $M(X)$ be the number of maximal elements of X. Then $cat(X) \leq gcat(X) < M(X)$.*

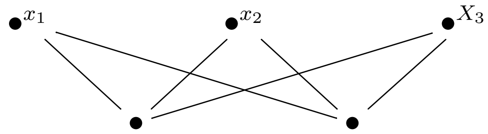
Proof. Let $x \in X$ be a maximal element. Then we have that U_x is contractible by *Corollary 1.2.12*, so $id_{U_x} \simeq c_x$. Therefore $\{U_x\}$, where x is a maximal element of X, is a cover of X of subsets contractible in itself. □

Remark 2.2.2. *(Proposition 5.1 [14]) If X has a unique maximal or minimal element then X is contractible. In fact the identity map is homotopic to the constant map c_{max} in the first case and c_{min} in the second.*

Example 2.2.3. *(Example 5.2 [14]) Consider the following finite poset X:*



The number of maximal elements is $M(X) = 2$ and X is clearly not contractible therefore we have that $0 < cat(X) < M(X) = 2$. Then $cat(X) = 1$. Consider now the opposite poset X^{op}, that is the set X with order given by the reverse order of the poset X.



X^{op} has three maximal points $M(X^{op}) = 3 > cat(X^{op})$, moreover the union of two open basic sets $U_{x_i} \cup U_{x_j}$, $i \neq j$ and $i, j = 1, ..., 3$ is not contractible, therefore the smallest categorical cover is given by $U_{x_1}, U_{x_2}, U_{x_3}$ and therefore $cat(X^{op}) = 2$.

Remark 2.2.4. *The category of a poset and its opposite may be not the same, as the previous example shows.*

Now we want to consider the geometric category of a finite T_0-space and its behaviour under elimination of a beat point (*Definition 1.3.4*). In particular the following theorem shows that the geometric category increase when a beat point is eliminated.

Theorem 2.2.5. *(Proposition 5.6 [14]) Let X be a finite T_0-space and x a beat point of X. Then $gcat(X) \leq gcat(X \smallsetminus x)$.*

Proof. We want to show that a cover of $X \smallsetminus x$ of open subsets contractible in themselves is a cover of X. Let U_0,...,U_n be such a cover of $X \smallsetminus x$. Let x be the beat point and x' the point associated to x defined in *Definition 1.3.4*. We want to define a cover U'_0,...,U'_n of X by taking for each U_i $0 \leq i \leq n$ the following subsets:

- If x is a maximal element of X, then $x' \leq x$ and there is some U_i that contains x' then $U'_i = U_i \cup \{x\}$ and for the other U_j, $j \neq i$ we take $U'_j = U_j$.
- If x is not a maximal element of X then if there is $y \in U_i$ for some $0 \leq i \leq n$ such that $x < y$, then we take $U'_i = U_i \cup \{x\}$. For the other U_j, $j \neq i$ we take $U'_j = U_j$.

Now we want to check that every U'_i $0 \leq i \leq n$ is open. We will show that every U'_i is the union of some basic open subset U_z for some $z \in X$ (the basic open subsets of X are the open subsets that form a base for the topology on X defined in *Section 1.1*). Consider U_i of the cover of $X \smallsetminus x$, since it is open it is the union of some basic open sets $\{U_{z_{i,j}}\}_{0 \leq j \leq m}$ for some $z_{i,j} \in X \smallsetminus x$. In the first case if U_i contains x' then there is a basic open set $U_{z_{i,k}}$, $k \leq m$ that contains x', therefore since $U'_i = U_i \cup \{x\}$ and x is a maximum U'_i is the union of $\{U_{z_{i,j}}\}_{j \in \mathbb{N}}$ where $U_{z_{i,k}}$ is substituted with U_x, that is a basic subset of X. Therefore U'_i is open. For the other U_j, $j \neq i$ we take $U'_j = U_j$ since U_j are union of basic open subset that do not contain x then are U'_j. In the second case we have that x is not a maximal element and so there is $y \in U_i$ for some $0 \leq i \leq n$ such that $x < y$. Again U_i is the union of some basic open set $\{U_{z_{i,j}}\}_{0 \leq j \leq m}$ for some $z_{i,j} \in X \smallsetminus x$ and there is a basic open set $U_{z_{i,k}}$, $k \leq m$ that contains y. Now $U_{z_{i,k}} \subseteq X$ contains x since $x < y$ then U'_i is the union of basic open sets $\{U_{z_j}\}_{0 \leq j \leq m}$ in X. For the other sets U_j the argument is the same as in the previous case. Now we want to check that U'_i $0 \leq i \leq n$ is contractible in itself. We supposed that U_i are contractible in itself, therefore that $id_{U_i} \simeq c$ for some constant map $c : U_i \to U_i$, if and only if there is a fence of function from id_{U_i} to c, $id_{U_i} = \phi_0 \leq \phi_1 \geq ... \leq \phi_n = c$. In the first case we can define $\phi'_k : U'_i \to U'_i$ by $\phi'_k(z) = \phi_k(z)$ if $z \neq x$ and $\phi'_k(x) = \phi_k(x')$.

The maps ϕ'_k are order preserving because x is a beat point and therefore they are continuous. Moreover, $\phi_k \leq \phi_{k+1}$ implies $\phi'_k \leq \phi'_{k+1}$ (the same for $\geq$). Therefore we have that there is a fence between ϕ'_0 and ϕ'_n, ϕ'_n is a constant map because ϕ_n is constant and ϕ'_0 is homotopic to the identity id_{U_i} because it is comparable with it by definition of ϕ'_0. We have that U'_0,...,U'_n is a cover of X of open subsets contractible in themselves therefore $gcat(X) \leq n$. □

By eliminating all the beat points of a finite T_0-space X we obtain the core X_0 of X, that is unique up to homeomorphism. Moreover, we showed that a space Y such that $X \searrow\searrow Y$ is homotopic equivalent to X. Therefore we have the following result.

Corollary 2.2.6. *(Corollario 5.8 [14]) Let X a finite T_0-space and X_0 the core of X. The geometric category $gcat(X_0)$ equals the maximum of the geometric categories in its homotopy class.*

$$gcat(X_0) = max\{gcat(Y) : Y \simeq X\}$$

Example 2.2.7. *The following picture shows an example of a finite topological space for which the inequality of Theorem 2.2.5 is strict. The example, provided by J. Barmak and G. Minian, is presented in [14].*
Consider the following finite topological space X.

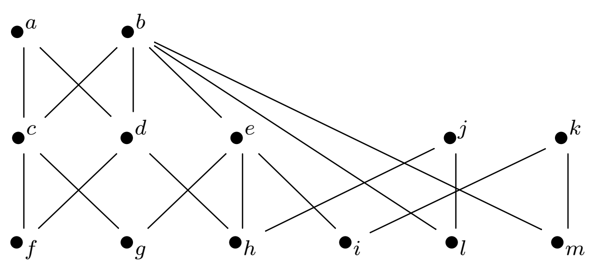
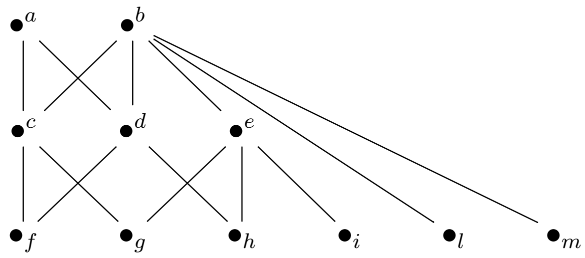


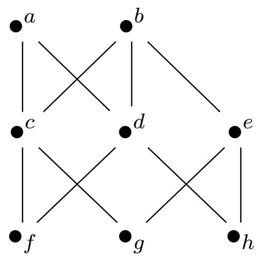
Figure 2.2.1: An example of a finite poset X such that $gcat(X_0) > gcat(X)$, [14] page 14.

The space X is not collapsible therefore $cat(X) \geq 1$. Moreover we can find a cover $\mathcal{U}$ of subsets collapsible in themselves given by $\mathcal{U} = \{U_a \cup U_b, U_e \cup U_j \cup U_k\}$.

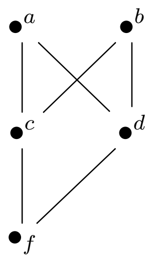
$U_a \cup U_b$, showed in the picture, is collapsible in itself, in fact



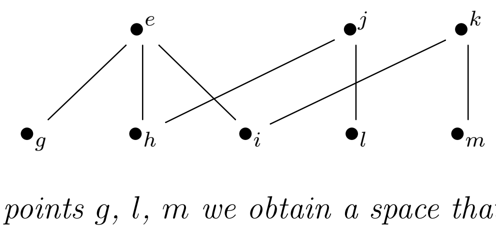
removing the beat points i, l and m we obtain



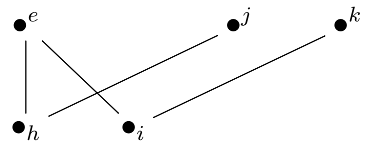
If we remove the beat point e and then the beat points h and g we obtain a finite space with a minimum f that is collapsible to f by Remark 2.2.2



The other open set of the cover is $U_e \cup U_j \cup U_k$



Removing the beat points g, l, m we obtain a space that collapse to e



We can conclude that $1 \leq cat(X) \leq gcat(X) = 1$*, then* $cat(X) = gcat(X) = 1$*. On the other hand, we can consider the core* X_0 *of* X *showed in the following picture*

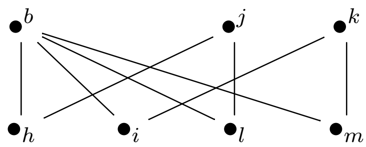


Figure 2.2.2: The core X_0 of the finite poset X such that $gcat(X_0) > gcat(X)$, [14] page 15.

A cover of X_0 *is given by* $\mathcal{U} = \{U_b, U_j, U_k\}$ *therefore* $gcat(X_0) \leq 2$*. On the other hand it is not possible to cover* X_0 *with less then three subsets collapsible in themselves because an open subset is a union of basic open subsets but unions of two of* U_b*,* U_j *or* U_k *give a non collapsible subset. Therefore there are not covers composed by two collapsible subset, and* $gcat(X_0) = 2$*. We can conclude that* $2 = gcat(X_0) > gcat(X) = 1$*. (Note that* $2 = gcat(X_0) > cat(X_0) = 1$ *and* $U_b, U_l \cup U_k$ *is a cover of categorical subsets for* X_0*).*

2.3 The simplicial L-S category

We will define now the L-S category for finite simplicial complexes, namely the *simplicial L-S category*. With this aim we will define a categorical cover for simplicial complexes using the concept of contiguity, defined in *section 0.3*. We will work in the category fSC, therefore we will consider just finite simplicial complexes. The results in this section can be found in *Chapter 3* in [14] with the exception of *Remark 2.3.2*, the proof of *Corollary 2.3.4* and the proof of *Lemma 2.3.5* that is due to the author.

Definition 2.3.1. Let K be a simplicial complex. A subcomplex $U \subseteq K$ is said to be *categorical* if there exists a vertex v in K such that the inclusion $i : U \hookrightarrow K$ and the constant map $c_v : U \to K$ are in the same contiguity class, $i \sim c_v$.

Remark 2.3.2. *A categorical subcomplex may not be connected. Consider the following simplicial complex and the subcomplex U given by the simplices $[0, 1]$ and $[2, 3]$. The inclusion i_U is contiguous to the map h_1 defined by $h_1(0) = 0$, $h_1(1) = 0$,*

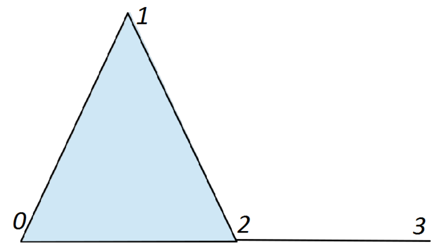


Figure 2.3.1: A categorical subcomplex, $[0, 1] \cup [2, 3]$, that is not connected.

$h_1(2) = 2$ and $h_1(3) = 2$. h_1 is contiguous to h_2 defined by $h_2(0) = 0$, $h_2(1) = 2$, $h_2(2) = 2$ and $h_2(3) = 0$ and finally h_2 is contiguous to the constant map c_0. Therefore we have a fence of maps $i_U = h_0 \sim_c h_1 \sim_c h_2 \sim_c h_3 = c_0$, so $i_U \sim c_0$ and U is a categorical subcomplex.

Definition 2.3.3. Let K be a simplicial complex. The *simplicial L-S category*, $scat(K)$ is the least integer $m \geq 0$ such that K can be covered by $m+1$ categorical subcomplexes.

Corollary 2.3.4. *Let K be a simplicial complex. $scat(K) = 0$ if and only if it has the same strong homotopy type of a point.*

Proof. If $scat(K) = 0$ then K is covered by one open categorical subcomplex, so K is categorical and $i_K = id_K \sim c_v$ for some vertex v in K. Therefore, $id_K \sim c_v = i_v c_v$ and $c_v i_v = id_v$, where $i_v : v \hookrightarrow K$ is the inclusion map. So $K \sim v$, that is K has the same strong homotopy type of a point. On the other hand if K has the same strong homotopy type of a point $i_v c_v \sim id_K$ and $c_v i_v \sim id_v$, so $i_v c_v = c_v \sim id_K$ that is $i_K \sim c_v$ that implies that K is categorical. ☐

We showed in the previous section that the L-S category is an homotopy invariant. We will show that also in the case of simplicial complexes the simplicial L-S category is a strong homotopy invariant. We first prove a lemma that will be used in order to prove this result.

Lemma 2.3.5. *(Lemma 3.5 [14]) Let K, L, N be three simplicial complexes and f, $g : K \to L$ be two contiguous maps, $f \sim_c g$. Let $i : N \to K$ (resp. $r : L \to N$) be another simplicial map. Then $fi \sim_c gi$ (resp. $rf \sim_c rg$).*

Proof. If $f \sim_c g$ then for all σ in K $f(\sigma) \cup g(\sigma)$ is a simplex in L. Let σ be a simplex in N then $i(\sigma) = \sigma$ is a simplex in K because i is a simplicial map. Therefore $fi(\sigma) \cup gi(\sigma) = f(\sigma) \cup g(\sigma)$ is a simplex in L. We can conclude that $fi \sim_c gi$. The proof in the case of $r : L \to N$ is analogous. $\square$

Proposition 2.3.6. *(Proposition 3.7 [14]) Let K, L be two simplicial complexes and $f : K \to L$, $g : L \to K$ simplicial maps such that $gf \sim id_K$. Then $scat(K) \leq scat(L)$.*

Proof. Let $U \subseteq L$ be a categorical subcomplex, so $i_U \sim c_v$ for some vertex v of L. Therefore we have a sequence of contiguous functions $\phi_i : U \to L$

$$i_U = \phi_0 \sim_c \phi_1 \sim_c ... \sim_c \phi_n = c_v$$

Consider the subcomplex $f^{-1}(U) \subseteq K$, we want to show that it is a categorical subcomplex. Since $gf \sim id_K$ there is a sequence of maps $\psi_i : K \to K$

$$id_K = \psi_0 \sim_c \psi_1 \sim_c ... \sim_c \psi_n = gf$$

We define $f' = f|_{f^{-1}(U)} : f^{-1}(U) \to U$ and $j : f^{-1}(U) \hookrightarrow K$ the inclusion map. Then

$$j = id_K j = \psi_0 j \sim_c \psi_1 j \sim_c ... \sim_c \psi_n j = gfj$$

by *lemma 2.3.5.* Since $fj = i_U f'$ we obtain

$$gfj = gi_U f' = g\phi_0 f' \sim_c g\phi_1 f' \sim_c ... \sim_c g\phi_n f' = gc_v f'$$

But $gc_v f' : f^{-1}(U) \to g(U)$ is the constant map $c_{g(v)}$ so we obtain that $j \sim c_{g(v)}$, that implies that $f^{-1}(U) \subseteq K$ is categorical. Finally, if $k = scat(L)$ and $\{U_0, ..., U_k\}$ is a categorical cover of L then $\{f^{-1}(U_0), ..., f^{-1}(U_k)\}$ is a categorical cover of K, therefore we can conclude that $scat(K) \leq k$. $\square$

Theorem 2.3.7. *(Theorem 3.4 [14]) Let $K \sim L$ be two strong equivalent simplicial complexes. Then $scat(K) = scat(L)$.*

Proof. Suppose that $K \sim L$, so there are two simplicial maps $f : K \to L$, $g : L \to K$ such that $gf \sim id_K$ and $fg \sim id_L$. By *Proposition 2.3.6*, $gf \sim id_K$ implies that $scat(K) \leq scat(L)$ and $fg \sim id_L$ implies that $scat(K) \geq scat(L)$, therefore $scat(K) = scat(L)$. □

We proved in section 1.5 that every simplicial complex K has a core K_0 that is obtained by removing dominated points from K. Moreover, we showed that $K \sim K_0$ therefore we have the following result.

Corollary 2.3.8. *Let K_0 be the core of the simplicial complex K. Then $scat(K) = scat(K_0)$.*

Proof. It follows directly from *Theorem 2.3.7.* □

2.4 The simplicial geometric category

We want to define the concept of geometric category for finite simplicial complexes using the notion of strong collapsibility that we introduced in *Definition 1.5.15.* The following results are a reformulation of *Chapter 4* in [14]. *Proposition 2.4.5* and *Remark 2.4.6* are due to the author.
We recall the fact that a simplicial complex is strong collapsible if it is strong equivalent to a point.

Remark 2.4.1. *A simplicial complex K is strong collapsible if and only if the identity map is in the contiguity class of some constant map $c_v : K \to K$, $v \in K$, $id_K \sim c_v$. This is a direct consequence of Proposition 1.5.4.*

Definition 2.4.2. The *simplicial geometric category*, $gscat(K)$ of a simplicial complex K is the least integer $m \geq 0$ such that K can be covered by $m + 1$ strongly collapsible subcomplexes. That is, there exists a cover $\{U_0, ..., U_m\}$ of K such that $U_i \sim v$, $v \in U_i$ for all $0 \leq i \leq m$.

Remark 2.4.3. *A strongly collapsible subcomplex must be connected.*

Proposition 2.4.4. *(Proposition 4.2 [14]) Let K be a simplicial complex. Then $scat(K) \leq gscat(K)$.*

Proof. We need to show that a strongly collapsible subcomplex is categorical. Let $U \subseteq K$ be a strongly collapsible subcomplex, then $id_U \sim c_v$ for some $v \in U$. Then by *lemma 2.3.5* we have that $i_U = i_U id_U \sim i_U c_v = c_v$, where $i_U : U \hookrightarrow K$ is the inclusion map. So U is a categorical subset, therefore a cover of K of strongly collapsible subcomplexes is a categorical cover. Therefore $scat(K) \leq gscat(K)$. □

Proposition 2.4.5. *Let K be a simplicial complex and $M(K)$ the number of its maximal simplices, that are the maximal simplices in the order given by the inclusion relation. Then we have that $scat(K) \leq gscat(K) < M(K)$.*

Proof. Let σ be a maximal simplex, it is a strong collapsible subcomplex of K because all its points are dominated in σ. Consider the set $\{\sigma_i\}_{0 \leq i \leq n}$, where σ_i is a maximal simplex for all $0 \leq i \leq n$. It is a cover of K because all simplices are maximal and it is a cover of strongly collapsible subcomplexes because the simplices are strongly collapsible. Combining with *Proposition 2.4.4* we obtain the result. □

Remark 2.4.6. *The star of a vertex in K, $st(v)$ is a strongly collapsible subcomplex of K because all vertices in $st(v)$ are dominated by v. Therefore $\{st(v)\}_{v \in K^0}$ provides a cover of categorical subsets and so we have that $scat(K) \leq gscat(K) < \sharp v$, where $\sharp v$ is the number of vertices of K.*

The simplicial L-S category and the simplicial geometric category are invariant under simplicial isomorphisms. Moreover, we proved that the simplicial L-S category is strong homotopy invariant. On the other hand, the geometric category for simplicial complexes, as well as the geometric category for topological spaces, is not a (strong) homotopy invariant. The next theorem shows that strong collapses increase the geometric category.

Theorem 2.4.7. *(Theorem 4.3 [14]) Let K and L be two simplicial complexes such that L is a strong collapse of K then $gscat(K) \leq gscat(L)$.*

Proof. We assume without loss of generality that L is an elementary strong collapse, so $L = K \setminus v$, v is a dominated vertex in K. We have that $i : L \hookrightarrow K$

in the inclusion and $r : K \to L$ the retraction defined in *lemma 1.5.4* such that $ri = id_L$ and $ir \sim id_K$. Suppose that V is a strong collapsible subcomplex of L, so $id_V \sim c_w$, where $c_w : V \to V$ is the constant map. Therefore there is a sequence of maps $\phi_i : V \to V$, $0 \leq i \leq n$, such that

$$id_V = \phi_0 \sim ... \sim \phi_n = c_w$$

We denote by $r' = r^{-1}(V) \to V$ the restriction of r to $r^{-1}(V)$ and $i' : V \to r^{-1}(V)$ the inclusion, that is well defined because $ri = id_V$. By *lemma 2.3.5* $\phi_i \sim_c \phi_{i+1}$ implies that $i'\phi_i r' \sim_c i'\phi_{i+1}r'$ and $i'\phi_n r' = i'c_w r' = c_{i(w)}$ that is a constant map and $i'\phi_0 r' = i'id_V r' = i'r'$. Moreover, we have that $i'r' \sim_c id_{r^{-1}(V)}$ because if σ is a simplex in $r^{-1}(V) \subseteq K$ then $\sigma \cup (ir)(\sigma)$ is a simplex in K contained in $r^{-1}(V)$ because $ri = id_V$. But $ir(\sigma) = i'r'(\sigma)$ then $\sigma \cup (i'r')(\sigma)$ is a simplex in $r^{-1}(V)$. Therefore we have that $id_{r^{-1}(V)} \sim c_w$, so $r^{-1}(V)$ is strongly collapsible. Now, if $m = gscat(L)$ and $\{V_i\}_{0 \leq i \leq m}$ a cover of strongly collapsible subcomplexes of L. Then $\{r^{-1}(V_i)\}_{0 \leq i \leq m}$ is a cover of K of strongly collapsible subcomplexes. Then we have that $gscat(K) \leq gscat(L)$. □

Remark 2.4.8. *No example of simplicial complex where the inequality of the previous theorem is strict has been found. However we have an example of finite space X such that $gcat(X) < gcat(X \setminus x)$ where x is a beat point. This, as the authors of [14] write, lead to think that the inequality is not an equality.*

We proved in *Theorem 1.5.13*, that the core K_0 of a simplicial complex K is obtained by removing dominated points and it is unique up to isomorphisms. Therefore we have the following result.

Corollary 2.4.9. *(Corollary 4.4 [14]) Let K_0 be the core of a simplicial complex K. The geometric category $gscat(K_0)$ is the maximum value of $gscat(L)$ where L is strongly equivalent to K,*

$$gscat(K_0) = max\{gscat(L) : L \sim K\}$$

2.5 Relations between categories

In this section we want to compare the Lusternik-Schnirelmann category for finite posets (finite T_0-spaces) with the simplicial L-S category, and the geometric category for finite posets with the simplicial geometric category. In particular, we want to study the behaviour of the value of the categories when we apply the functors $\mathcal{K}$ or χ. This section is a reformulation of *Chapter 6* in [14] in the language of category theory. *Proposition 2.5.1*, *Proposition 2.5.2*, *Remark 2.5.11* and *Example 2.5.14* are due to the author.

Proposition 2.5.1. *The order complex functor $\mathcal{K}$ sends categorical subsets to categorical subcomplexes and the face poset functor χ sends categorical subcomplexes to categorical subsets.*

Proof. Suppose that X is a finite partially ordered set, and $U \subseteq X$ is a categorical subset, so $i_U \simeq c_x$ for some $x \in X$. Then we have the induced functions
$\mathcal{K}(i_U) : \mathcal{K}(U) \hookrightarrow \mathcal{K}(X)$ and $\mathcal{K}(c_x) : \mathcal{K}(U) \to \mathcal{K}(X)$ such that $\mathcal{K}(i_U) \sim \mathcal{K}(c_x)$.
$\mathcal{K}(i_U)(x') = i_U(x')$ and $\mathcal{K}(c_x)(x') = x$ for all x' in the vertex set of $\mathcal{K}(X)$, therefore $\mathcal{K}(i_U) = i_{\mathcal{K}(U)}$ and $\mathcal{K}(c_x) = c_x$, that implies that $i_{\mathcal{K}(U)} \sim c_x$, so $\mathcal{K}(U)$ is a categorical subcomplex. Suppose now that K is a finite simplicial complex and $U \subseteq K$ a categorical subcomplex. Then we have $i_U \sim c_v$ for some vertex v in K. We have the maps induced by the functor χ, $\chi(i_U) : \chi(U) \hookrightarrow \chi(X)$ and $\chi(c_x) : \mathcal{K}(U) \to \chi(X)$ such that $\chi(i_U) \simeq \chi(c_v)$, $\chi(i_U)(\sigma) = i_U(\sigma)$ and $\chi(c_v)(\sigma) = v$ for all simplices σ in $\chi(U)$ so $\chi(i_U) = i_{\chi(U)}$ and $\chi(c_v) = c_v$. $i_{\chi(U)} \simeq c_v$, so $\chi(U)$ is a categorical subset. □

Proposition 2.5.2. *The order complex functor $\mathcal{K}$ sends subsets contractible in themselves to strongly collapsible subcomplexes and the face poset functor χ sends strongly collapsible subcomplexes to subsets contractible in themselves.*

Proof. Suppose that X is a finite partially ordered set, and $U \subseteq X$ is a subset contractible in itself, so $id_U \simeq c_x$ for some $x \in U$. Then we have the induced maps $\mathcal{K}(id_U) \sim \mathcal{K}(c_x)$. Now, $\mathcal{K}(id_U) = id_{\mathcal{K}(U)}$ and $\mathcal{K}(c_x)(x') = x$ for all $x' \in U$, therefore $\mathcal{K}(c_x) = c_x$, that implies that $id_{\mathcal{K}(U)} \sim c_x$, so $\mathcal{K}(U)$ is a strong collapsible subcomplex. Suppose now that K is a finite simplicial complex and $U \subseteq K$ a strong collapsible subcomplex. Then we have $id_U \sim c_v$ for some vertex v of U. We have the maps induced by the functor χ, $\chi(id_U) \simeq \chi(c_v)$, where $\chi(id_U) = id_{K(U)}$ and $\chi(c_v)(\sigma) = v$ for all σ in $\chi(U)$ so $\chi(c_v) = c_v$. $id_{\chi(U)} \simeq c_v$, so $\chi(U)$ is a subset collapsible in itself. □

Theorem 2.5.3. *(Proposition 6.1 and 6.2 in [14]) Let X be a finite partially ordered set. Then $scat(\mathcal{K}(X)) \leq cat(X)$ and $gscat(\mathcal{K}(X)) \leq gcat(X)$.*

Proof. Let $U_0, ..., U_n$ be a categorical open cover of X, therefore $X = \cup_{0 \leq i \leq n} U_i$. Then we have the induced sets $\mathcal{K}(U_0), ..., \mathcal{K}(U_n)$ such that $\mathcal{K}(X) = \cup_{0 \leq i \leq n} \mathcal{K}(U_i)$. By *Proposition 2.5.1* this is a categorical cover. Therefore $scat(\mathcal{K}(X)) \leq cat(X)$. In the second case if we suppose that $U_0, ..., U_n$ is a cover of X of subset contractible in itself then we have that $\mathcal{K}(U_0), ..., \mathcal{K}(U_n)$ is a cover of $\mathcal{K}(X)$ of strongly collapsible subcomplexes by *Corollary 1.6.18*. □

Example 2.5.4. *This example had been suggested to the author by J. Barmak. The Example 2.2.3 provides a finite poset X^{op} such that $scat(\mathcal{K}(X^{op})) < cat(X^{op})$. In fact, we saw that $cat(X^{op}) = 2$ and we can show that $scat(\mathcal{K}(X^{op}) < 2$. $\mathcal{K}(X^{op})$ is the following simplicial complex*

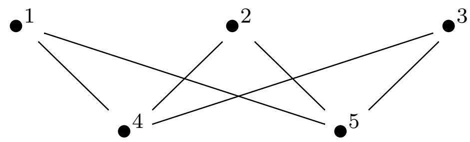


Figure 2.5.1: An order complex $\mathcal{K}(X^{op})$ such that $scat(\mathcal{K}(X^{op})) < cat(X^{op})$.

$[1, 4] \cup [2, 4] \cup [3, 4]$, $[1, 5] \cup [2, 5] \cup [3, 5]$ *is a categorical cover and $\mathcal{K}(X^{op})$ is not strongly collapsible therefore $scat(\mathcal{K}(X^{op})) = 1$.*

Example 2.5.5. *(Example 6.3 [14]) Consider the finite poset X showed in Example 2.1.9 and let $\mathcal{K}(X)$ be the order complex associated to X.*
In this case the inequality of the previous theorem is strict, $gscat(\mathcal{K}(X)) < gcat(X)$. In fact as we showed in Example 2.1.9 $gcat(X) = 2$ while $gscat(\mathcal{K}(X)) = 1$ because $\mathcal{K}(X)$ is not strong collapsible and it is covered by two strong collapsible subcomplexes. The next picture shows the order complex $\mathcal{K}(X)$.

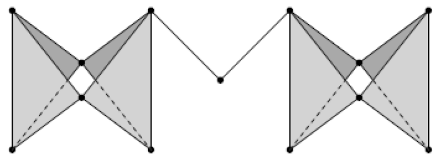


Figure 2.5.2: An order complex $\mathcal{K}(X)$ such that $gscat(\mathcal{K}(X)) < gcat(X)$, [14] page 16.

The following subcomplexes provide a cover of $\mathcal{K}(X)$.

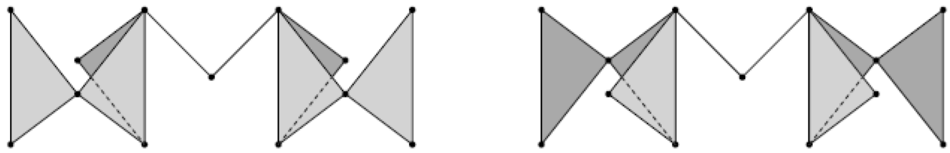


Figure 2.5.3: The cover of strong collapsible subcomplexes of $\mathcal{K}(X)$, [14] page 16.

An analogous result holds for the functor χ.

Theorem 2.5.6. *(Proposition 6.4 and 6.5 [14]) Let K be a finite simplicial complex. Then $cat(\chi(K)) \leq scat(K)$ and $gcat(\chi(K)) \leq gscat(K)$.*

Proof. Let $U_0, ..., U_n$ be a categorical cover of K, therefore $K = \cup_{0 \leq i \leq n} U_i$. Then we have the induced sets $\chi(U_0), ..., \chi(U_n)$ such that $\chi(K) = \cup_{0 \leq i \leq n} \chi(U_i)$. By *Proposition 2.5.1* this is a categorical cover. Therefore $cat(\chi(K)) \leq scat(K)$. In the second case if we suppose that $U_0, ..., U_n$ is a cover of X of strongly collapsible subcomplexes then we have that $\chi(U_0), ..., \chi(U_n)$ is a cover of $\chi(K)$ of subset contractible in itself by *Corollary 1.6.18.* □

Remark 2.5.7. *There is no examples where the two inequalities are strict.*

Corollary 2.5.8. *Let X be a finite partially ordered set. Then $cat(X) = 0$ if and only if $scat(\mathcal{K}(X)) = 0$*

Proof. The result follows directly from *Corollary 1.6.18.* □

Corollary 2.5.9. *Let K be a finite simplicial complex. Then $scat(K) = 0$ if and only if $cat(\chi(K)) = 0$.*

Proof. The result follows directly from *Corollary 1.6.18.* □

Corollary 2.5.10. *(Corollary 6.7 [14]) Let K be a finite simplicial complex and $sd(K)$ its barycentric subdivision. Then $scat(sd(K)) \leq scat(K)$*

Proof. The barycentric subdivision $sd(K)$ is $\mathcal{K}(\chi(K))$. By *Theorem 2.5.3* and *Theorem 2.5.6* we have that $cat(\chi(K)) \leq K$ and $scat(\mathcal{K}(\chi(K))) \leq cat(\chi(K))$, that implies $scat(\mathcal{K}(\chi(K))) = scat(sd(K)) \leq scat(K)$. □

Remark 2.5.11. *If K is a simplicial complex such that $scat(K) = 0$ then $scat(sd(K)) = 0$. Moreover, by Theorem 1.6.16 we have that if K is a simplicial complex, K is not strongly collapsible if and only if $sd(K)$ is not strongly collapsible. Therefore if $scat(K) = 1$ then the simplicial category of the subdivision doesn't decrease, therefore $scat(sd(K)) = 1$, otherwise $sd(K)$ would be strongly collapsible and this is a contradiction. Therefore in case of simplicial category equal to 0 or 1 the inequality of Corollary 2.5.10 is an equality.*

Remark 2.5.12. *Notice that the last result is not contradictory because it is possible that $sd(K)$ and K have not the same strong homotopy type as showed in the following example.*

Example 2.5.13. *Consider K to be the boundary of a 2-simplex and $sd(K)$ its barycentric subdivision.*

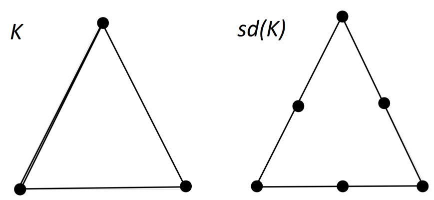


Figure 2.5.4: A simplicial complex K that has not the same strong homotopy type of its subdivision $sd(K)$.

They are both minimal complexes because they have no dominated points but they are not isomorphic therefore by corollary 1.5.11 they don't have the same strong homotopy type.

Example 2.5.14. *This example has been suggested to the author by J.Barmak. It shows a simplicial complex that satisfies Corollary 2.5.10 with a strict inequality. Let K be the complete graph K_5 considered as a 1-dimensional simplicial complex.*

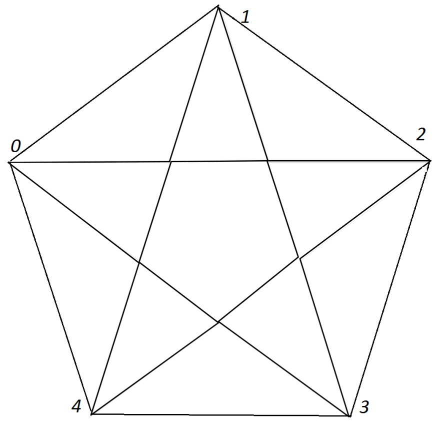


Figure 2.5.5: The complete graph K_5, an example of simplicial complex K such that $scat(sd(K)) < scat(K)$.

A cover of categorical subcomplexes is given, for example, by $\{[0,1] \cup [0,2] \cup [0,3] \cup [0,4], [1,4] \cup [1,2] \cup [2,3], [1,3] \cup [3,4] \cup [4,2]\}$*, therefore* $scat(K_5) \leq 2$*. Moreover, there is no cover of two categorical subsets. In fact, if we suppose that we can cover* K_5 *with two categorical subcomplexes then one of the two subcomplexes has to contain at least* 5 *edges. That means that, denoting* v_i*,* $i = 0, ..., 4$ *the vertices of* K_5*, if we choose the edges in order to have a categorical subcomplex (that has no "loops") we have the following cases:*

- *if four* 1*-simplices contain the same vertex* v_i*, then we have that the subcomplex is made by the simplices* $[v_i, v_j]$*,* $[v_i, v_k], [v_i, v_h], [v_i, v_m], [v_p, v_q]$ *where the vertex* v_j, v_h, v_k, v_m *are the other* 4 *vertices and* v_p, v_q *can be any two* *different vertices, let's say* $v_p = v_m$ *and* $v_q = v_h$. *Then we have a loop* $[v_i, v_m], [v_m, v_h], [v_h, v_i]$ *and so the subset is not categorical.*

- *if three* 1*-simplices contain the same vertex* v_i, *then we have* $[v_i, v_j]$, $[v_i, v_k]$,$[v_i, v_h]$, $[v_n, v_m]$,$[v_p, v_q]$ *where* v_n *or* v_m *is different than* v_i, v_j, v_k, v_h. *Suppose that* v_n *is different than the other four vertices and* $v_m = v_j$. *Then* v_p *and* v_q *have to be different from* v_i, *if* $v_p = v_j, v_k, v_h$ *or* v_n *then we obtain a loop for all value of* v_q, *so the subset cannot be categorical.*

- *the subcomplexes with at most two* 1*-simplices containing the same vertex* v_i, *that are made by at most five* 1*-simplices, are constituted by simplices with the following structure:* $[v_i, v_j]$, $[v_j, v_k]$,$[v_k, v_h]$,$[v_h, v_m]$,$[v_m, v_i]$ *and they are not categorical.*

- *we cannot have a subcomplex with at least five simplices such that every* 1*-simplex contains one different vertex.*

We can conclude that there is no categorical subcomplex with at least five vertices, therefore there is no cover of K_5 *given by two categorical subcomplexes. Therefore* $cat(K) = 2$.
On the other hand the barycentric subdivision of K, $sd(K)$ *has a cover of two categorical subcomplexes given, for example, by the two subcomplexes* U_0, *marked in red, and* U_1, *marked in black, that are showed in the following picture.*

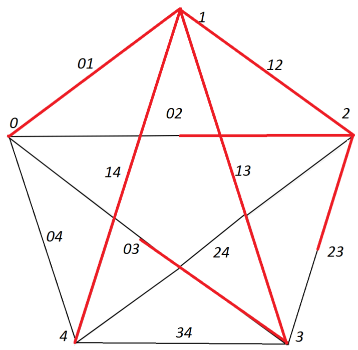


Figure 2.5.6: A cover of two categorical subsets of $sd(K_5)$.

Since $sd(K_5)$ is not strongly collapsible we can conclude that
$1 = scat(sd(K_5)) < scat(K_5) = 2.$

Chapter 3

Additional results

3.1 A lower bound for the simplicial category of the subdivision: the L-S category of the geometric realization

In this section we want to give some results about the L-S category of the geometric realization of a simplicial complex. In particular we will show that for every finite simplicial complex K, $cat(|K|) \leq scat(sd(K)) \leq scat(K)$. In addition, the value of the L-S category of the geometric realisation of a simplicial complex provides a lower bound for the value of the simplicial category of the iterated subdivision. The first results about the geometric realisation lead to *Proposition 3.1.6* that shows that maps in the same contiguity class correspond to homotopic maps in the geometric realisation. They are needed to define the *geometric realisation functor* and they can be found in *Appendix* in [2]. On the other hand, the main theorems *Theorem 3.1.8*, *Theorem 3.1.10*, *Corollary 3.1.11* and *Example 3.1.12* are due to the author.
We will first recall some basic definitions regarding the geometric realisation, this definitions refer to [11].

Definition 3.1.1. Let K be a simplicial complex and K^0 its set of vertices. The *geometric realization* of K, $|K|$ is a topological space defined as the set of functions $\alpha : K^0 \to [0,1]$ such that if $\alpha \in |K|$ then the set $supp(\alpha) = \{v \in K^0 : \alpha(v) \neq 0\}$

is a simplex in K and for all $\alpha \in |K|$, $\sum_{v \in K^0} \alpha(v) = 1$.

$$|K| = \{\alpha : K^0 \to [0,1] : supp(\alpha) \in K, \sum_{v \in K^0} \alpha(v) = 1\}$$

Therefore $|K|$ can be regarded as a subset of the vector space $\mathbb{R}[K^0] = \{f : K^0 \to \mathbb{R} : supp(f) \text{ finite}\}$, with basis the functions $v : K^0 \to \mathbb{R}$ defined for all $v' \in K^0$ by $v(v') = 1$ if $v = v'$ and $v(v') = 0$ otherwise. Now, given a simplex in K, $\sigma = \{v_0, ..., v_n\}$ where $v_i \in K^0$ we can define the realisation of σ as

$$|\sigma| = \{\sum_{v \in \sigma} \alpha_v v | \alpha_v \geq 0, \sum_{v \in \sigma} \alpha_v = 1\}$$

where v is a vertex in K and v denote also an element of base for the vector space $\mathbb{R}[K^0]$, $\alpha_v \in \mathbb{R}$. The realisation of K, $|K|$ can be regarded as the union of the realisation of the simplices σ in K

$$|K| = \cup |\sigma|$$

Therefore an element of the realisation is a function $\alpha \in |K|$ defined for all $v' \in K^0$ by $\alpha(v') = \sum_{v' \in K^0} \alpha_v v(v')$.
We can define on $|K|$ two different topologies, one induced by the metric on $|K|$ given for all α, $\beta \in |K|$ by

$$d(\alpha, \beta) = (\sum_{v \in K^0} (\alpha_v - \beta_v)^2)^{\frac{1}{2}}$$

the other is the coherent topology given by the cover $\{|\sigma| : \sigma \in K\}$. In general we will consider $|K|$ equipped with the coherent topology, but in the specific case of finite simplicial complex the two coincide. The topologies of $|K|$ are described more in detail in *Appendix B*.
Simplicial maps between simplicial complexes induce continuous maps between realisations. In fact, given a simplicial map $\phi : K \to L$ we have a linear map between the vector spaces $|\phi| : \mathbb{R}[K^0] \to \mathbb{R}[L^0]$ that takes cells $|\sigma|$ of $|K|$ to cells $|\tau|$ of $|L|$, $|\phi|(|\sigma|) = |\phi|(\sum_{v \in \sigma} \alpha_v v) = \sum_{v \in \sigma} \alpha_v \phi(v) \subseteq |\tau|$. $|\phi|$ can be restricted to map $|\phi| : |K| = \cup |\sigma| \to |L| = \cup |\tau|$.

Definition 3.1.2. The *geometric realisation functor*, $|-|$ from the category of finite simplicial complexes fSC to the category of topological spaces TOP is the functor that associate to each simplicial complex K its geometric realization $|K|$, as defined in *Definition 3.1.1* and to each simplicial map $\phi : K \to L$ the continuous map $|\phi| : |K| \to |L|$ defined for all $\alpha \in |K|$ by $|\phi|(\alpha) = |\phi|(\sum_{v \in K^0} \alpha_v v) = \sum_{v \in K^0} \alpha_v \phi(v)$.

Remark 3.1.3. *The functor* $|-|$ *is well defined. In fact, for all* $|\sigma|$ *in* $|K|$, $|id_K|(\sum_{v\in\sigma} \alpha_v v) = \sum_{v\in\sigma} \alpha_v (id_K(v)) = \sum_{v\in\sigma} \alpha_v v = id_{|K|}(\sum_{v\in\sigma} \alpha_v v)$. *Moreover,* $|\psi\phi|(\sum_{v\in\sigma} \alpha_v v) = \sum_{v\in\sigma} \alpha_v (\psi\phi(v)) = |\psi|(\sum_{v\in\sigma} \alpha_v (\phi(v))) = |\psi||\phi|(\sum_{v\in\sigma} \alpha_v v)$.

We want to show that the functor $|-|$ sends contiguous maps to homotopic maps. In order to prove this result we will use the following two lemmas.

Lemma 3.1.4 (Lemma A1.1 [2]). *Let* K *be a simplicial complex and* F *a compact subset of* $|K|$. *Then there is a finite subcomplex* L *of* K *such that* $F \subseteq |L|$.

Proof. Consider a point x in $F\cap < \sigma >$ for all $< \sigma >$, where $< \sigma >$ is the open set contained in $|\sigma|$ whose supports is exactly σ, see *Appendix B, Definition B.1.2*. Let D be the set of these points and consider a subset $A \subseteq D$. The intersection of A and a closed simplex $|\sigma|$ is finite so it is closed, therefore A is closed in $|K|$ by the definition of the coherent topology. Therefore D is discrete and it is compact because it is a closed subset of F, so it is finite. Then F intersects finitely many open simplices $< \sigma >$. Let L be the subcomplex generated by the simplices σ such that $< \sigma >$ intersect F, L is a finite subcomplex L of K such that $F \subseteq |L|$. □

Lemma 3.1.5 (Proposition A1.2 [2]). *Let* K *and* L *be two simplicial complexes and* f,$g : |K| \to |L|$ *two continuous maps such that for all* $x \in |K|$ *there is a* σ *in* L *with* $f(x)$,$g(x)$ *in* $|\sigma|$. *Then* f *and* g *are homotopic.*

Proof. We define the map $H : |K| \times I \to |L|$, $I = [0, 1]$, by
$H(x, t) = tg(x) + (1 - t)f(x)$. The map is well defined because $f(x)$,$g(x)$ lie in $|\sigma|$. We want to show that H is continuous in $|K| \times I$, it is enough to show that it is continuous in $|\sigma'| \times I$ for every simplex σ' of K because the topology on $|K|$ is the coherent topology. Now, f and g are continuous maps and $|\sigma'|$ is compact

therefore $f(|\sigma'|)$ and $g(|\sigma'|)$ are compact. By *Lemma 3.1.4* $f(|\sigma'|)$ is contained in the realisation of a finite subcomplex L_1 of L and $g(|\sigma'|)$ in the realisation of a finite subcomplex L_2 of L. Therefore $H(|\sigma'| \times I)$ is contained in the realisation of a finite subcomplex M of L, that is the subcomplex spanned by the vertices of L_1 and L_2. We want to show that $H|_{|\sigma'|\times I} : |\sigma'| \times I \to |M|$ is continuous. Since $|\sigma'|$ and $|M|$ have the metric topology we have that

$d(H(x,t), H(y,s)) \leq d(H(x,t), H(x,s)) + d(H(x,s), H(y,s)) = (1)$

$= d(tg(x)+(1-t)f(x), sg(x)+(1-s)f(x)) + d(sg(x)+(1-s)f(x), sg(y)+(1-s)f(y))$

where the first inequality is provided by the triangular inequality, and

$(1) \leq 2|t-s| + d(f(x), f(y)) + d(g(x), g(y))$

because for all $x \in |K|$, $g(x) = \sum_{v' \in L^0} g(x)_v v(v')$ and $g(x)_v$ are smaller than 1 for all v in $|L|$ and $\sum g_v = 1$ and the same holds for $f(x)$. Since f and g are continuous, we have that H is continuous. □

Proposition 3.1.6 (Corollary A1.3 [2])**.** *Let K and L be two simplicial complexes and ϕ, $\psi : K \to L$ two simplicial map that lie in the same contiguity class then the induced maps $|\phi|$, $|\psi| : |K| \to |L|$ are homotopic. That is, the functor $|-|$ sends maps in the same contiguity class in homotopic maps.*

Proof. If ϕ, $\psi : K \to L$ lie in the same contiguity class then for all simplex σ in K we have that $\phi(\sigma) \cup \psi(\sigma)$ is a simplex σ' in L. We have the maps induced by the functor $|-|$ $|\phi|$, $|\psi| : |K| \to |L|$. Therefore for all $x \in |K|$ there is a σ such that $x \in |\sigma|$ and we have that $|\phi|(\sigma) \cup |\psi|(\sigma)$ lies in $|\sigma'| = |\phi(\sigma) \cup \psi(\sigma)|$ so $|\phi|(x)$, $|\psi|(x) \in |\sigma'|$. By *Proposition 3.1.6* we have that $|\phi|$ and $|\psi|$ are homotopic maps. □

Definition 3.1.7. The *geometric realisation functor*, $|-|$ from the contiguity category $h(fSC)$ to the homotopy category $h(TOP)$ is the functor that associate to each finite simplicial complex K its geometric realization $|K|$, as defined in *Definition 3.1.1* and to each contiguity class of simplicial maps $[\phi]_\sim : K \to L$ the class homotopic continuous maps $[\phi] : |K| \to |L|$ defined for all $\alpha \in |K|$ by $[\phi](\sum_{v \in K^0} \alpha(v)) = \sum_{v \in K^0} \alpha([\phi]_\sim(v))$.

From now on we will work in the category of finite simplicial complexes fSC, therefore even though we will refer to simplicial complexes we will consider just finite simplicial complexes.

Theorem 3.1.8. *Let K be a simplicial complex. Then $cat(|K|) \leq scat(K)$.*

Proof. Suppose that $\{U_j\}_{0\leq j\leq n}$ is a cover of K of categorical subcomplexes, then we have that for all $0 \leq j \leq n$, $i_{U_j} \sim c_v$ for a vertex v in K. Therefore, for all $0 \leq j \leq n$, we have maps induced by the functor $|-|$,$i_{U_j} : |U_j| \hookrightarrow |K|$ and $c_v : |U_j| \to |v|$, $|i_{U_j}| \simeq |c_v|$. We have that $|i_{U_j}| = i_{|U_j|}$ and $|c_v| = c_{|v|}$ is a constant map because for all $\alpha \in |U_j|$, $|c_v|(\sum_{v'\in K^0} \alpha_{v'} v') = \sum_{v'\in K^0} \alpha_{v'}(c_v(v')) = \sum_{v'\in K^0} \alpha_{v'} v = 1v = |v|$ and $|v| \in |K|$, then $i_{|U_j|} \simeq c_{|v|}$. Therefore $|U_j|$ are categorical subsets of $|K|$ for all $0 \leq j \leq n$ and $\{|U_j|\}_{0\leq j\leq n}$ is a cover of $|K|$ because $|K| = \cup|\sigma|$, $\sigma \in K$ and $\{U_j\}_{0\leq j\leq n}$ is a cover of K. $\{|U_j|\}_{0\leq j\leq n}$ is a closed cover because the realisation of a subcomplex of K is closed in $|K|$ (*Proposition B.1.1*) but since $|K|$ is a normal ANR, by *Proposition B.2.6* we can consider closed covers as well as open covers. Therefore $cat(|K|) \leq n$. More details about the L-S category defined with a closed categorical cover can be found in *Appendix B*. □

We want to prove that $|sd(K)|$ and $|K|$ are homeomorphic. We defined the barycentric subdivision of a simplicial complex K as the simplicial complex $sd(K)$ given by $sd(K) = \mathcal{K}(\chi(K))$. Therefore $sd(K)$ is the simplicial complex whose set of vertices is the set of simplices of K and the simplices of $sd(K)$ are given by the finite chains of simplices of K ordered by the relation of inclusion.

Lemma 3.1.9. *Let K be a simplicial complex. Then $|sd(K)|$ and $|K|$ are homeomorphic. Moreover, $cat(|sd(K)|) = cat(|K|)$.*

Proof. We can define the map $s : |sd(K)| \to |K|$ defined for all $x \in |sd(K)|$, $x = \sum_{v\in sd(K)^0} \alpha_v v = \sum_{\sigma\in K} \alpha_\sigma \sigma$ by $s(x) = \sum_{\sigma\in K} \alpha_\sigma b(\sigma)$ where $b(\sigma) = \sum_{v\in\sigma} \frac{v}{\sharp v} \in |K|$ gives the barycentre of the simplex σ in K. The function s is an homeomorphism: it is a linear map because it preserves convex combinations; it is injective because $s(x) = s(y)$ implies that $\sum_{\sigma\in K} \alpha_\sigma b(\sigma) = \sum_{\sigma\in K} \beta_\sigma b(\sigma)$ that implies $\alpha_\sigma = \beta_\sigma$ for all $\sigma \in K$ therefore $x = y$; it is surjective because for all $x' \in |K|$, $x' = \sum_{v\in K^0} \alpha_v v$ we have an element of $|sd(K)|$, $x = \sum_{\sigma\in K} \alpha_\sigma \sigma = \sum_{v\in K^0} \alpha_v v$ where $\alpha_\sigma = 0$ for σ not in K^0. Then $s(x) = x'$ because $b(v) = v$. Therefore $|sd(K)|$ and $|K|$ are homeomorphic topological spaces and clearly $cat(|sd(K)|) = cat(|K|)$. □

Theorem 3.1.10. *Let K be a simplicial complex. Then*

$$cat(|K|) \leq scat(sd(K)) \leq scat(K)$$

.

Proof. We know by *Corollary 2.5.10* that $scat(sd(K)) \leq scat(K)$. Moreover since $cat(|sd(K)|) = cat(|K|)$ by *Lemma 3.1.9*, we have by *Theorem 3.1.8* that $cat(|K|) = cat(|sd(K)|) \leq scat(sd(K)) \leq scat(K)$. □

Corollary 3.1.11. *Let K be a simplicial complex. Then*

$$cat(|K|) \leq min\{scat(sd^n(K)) : n \in \mathbb{N}\}$$

.

Proof. *Corollary 2.5.10* shows that $scat(sd(K)) \leq scat(K)$, if we apply the same theorem to $sd(K)$ we obtain $scat(sd^2(K)) \leq scat(sd(K))$ and so on for the other iterated subdivisions $sd^n(K)$. Therefore we have that

$$scat(sd^n(K)) \leq ... \leq scat(sd(K)) \leq scat(K)$$

and by *Theorem 3.1.8*

$$cat(|K|) = cat(|sd^n(K)|) \leq scat(sd^n(K)) \leq ... \leq scat(sd(K)) \leq scat(K)$$

. Therefore $cat(|K|) \leq scat(sd^n(K))$ for $n \in \mathbb{N}$. □

Example 3.1.12. *This example shows a simplicial complex for which the inequality of the previous corollary is strict. Consider the simplicial complex K showed in the picture.*

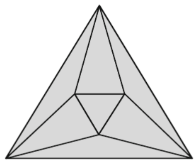


Figure 3.1.1: A simplicial complex K such that $cat(|K|) < min\{scat(sd^n) : n \in \mathbb{N}\}$.

K is not strong collapsible in fact it doesn't contain dominated vertices. Moreover, we have a cover of strong collapsible subcomplexes given by the subcomplexes

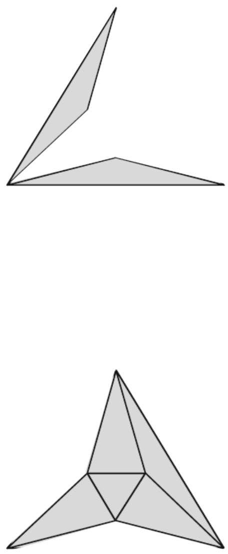


Figure 3.1.2: A cover of strong collapsible subcomplexes of K.

Therefore $scat(K) = 1$. By Proposition 1.6.16 we have that if K is non strong collapsible then $sd(K)$ is not strong collapsible and if we apply again the result we have that $sd^n(K)$ is not strong collapsible. Therefore by Corollary 2.5.10 $scat(sd^n(K)) = 1$. On the other hand $|K|$ is contractible because it is homeomorphic to a disc therefore $cat(|K|) = 0$, that implies $cat(|K|) < min\{scat(sd^n) : n \in \mathbb{N}\}$.

3.2 Computation of the simplicial L-S category: the sphere S^n

In this section we compute the value of the simplicial L-S category of the simplicial complex $S^n = D([n+1]_+) \smallsetminus \{[n+1]_+\}$, where $D([n+1]_+)$ is the simplicial complex whose simplices are all the subset of the standard $n+1$-simplex $[n+1]_+ = [0, ..., n+1]$. We will first compute the simplicial category for $n = 1, 2$ and then the general case $n \in \mathbb{N}$. The results in this section are due to the author.

We want to compute the value of $scat(S^1)$ of the sphere $S^1 = D([2]) \smallsetminus [2]$, showed in the picture.

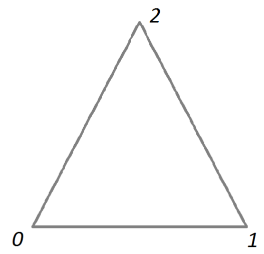


Figure 3.2.1: S^1

Consider the following two subcomplexes, we will denote U_0 the first and U_1 the second:

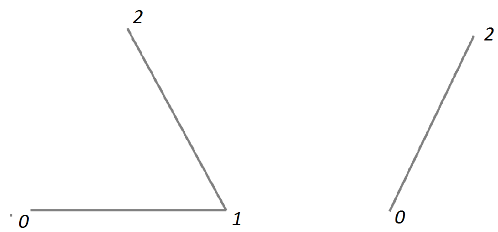


Figure 3.2.2: A cover of S^1 of two categorical subcomplexes.

Remark 3.2.1. *The simplicial complex S^1 is a minimal complex since it has no dominated vertices.*

Proposition 3.2.2. *The family of subcomplexes $\{U_0, U_1\}$ is a cover of S^1 of categorical subcomplexes.*

Proof. $\{U_0, U_1\}$ is clearly a cover of S^1. U_0 and U_1 are strongly collapsible simplicial complexes, in fact in U_0 the point 2 is dominated by 1 therefore $U_0 \searrow\searrow \{0, 1\}$ and $\{0, 1\} \searrow\searrow \{0\}$. $U_1 \searrow\searrow \{0\}$ because 1 is dominated by 0. Therefore $\{U_0, U_1\}$ is a cover of S^1 of strongly collapsible subcomplexes therefore it is a cover of categorical subcomplexes. □

Corollary 3.2.3. $scat(S^1) = 1$.

Proof. By the previous proposition we know that $scat(S^1) \leq 1$. Moreover $0 < scat(S^1)$ because S^1 is a minimal simplicial complex and therefore it doesn't strongly collapse to a single vertex. We can conclude that $scat(S^1) = 1$. □

Remark 3.2.4. *We can as well consider the face poset of S^1, showed in the following picture and compute its L-S category $cat(\chi(S^1))$. Now $\{U_{01} \cup U_{12}; U_{02}\}$ is a cat-*

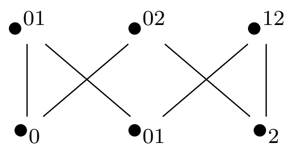


Figure 3.2.3: $\chi(S^1)$.

egorical cover of $\chi(S^1)$ since it is composed by unions of basic open sets contractible to a point. Therefore $cat(\chi(S^1)) \leq 1$, moreover $cat(\chi(S^1)) > 0$ because $\chi(S^1)$ is not contractible. By Theorem 2.5.6 we have that $1 = cat(\chi(S^1)) \leq scat(S^1) \leq 1$.

In the case of the sphere $S^2 = D([3]_+ \smallsetminus [3]_+)$, we can find a cover given by the subcomplexes $U_0 = \{[1,2,3]\}$, $U_1 = \{[0,1,2][0,2,3][0,1,3]\}$, showed in the picture:

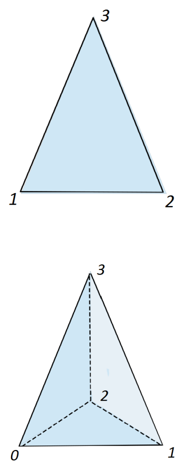


Figure 3.2.4: A cover of S^2 of two categorical subcomplexes.

Proposition 3.2.5. $scat(S^2) = 1$.

Proof. U_0 and U_1 is clearly a cover of S^2. In U_0 all vertices are dominated by another, therefore $[1,2,3] \searrow\searrow [1,2] \searrow\searrow [1]$, so these subcomplex is strongly collapsible. In U_1 all vertices are dominated by the vertex 0, therefore $U_1 \searrow\searrow [0,1,3] \searrow\searrow [0,1] \searrow\searrow [0]$, here we first delate the vertex 2, then the vertex 3 and then 1. The two subset are strongly collapsible therefore they are categorical, moreover the simplicial complex S^2 is minimal because it has no dominated point therefore it is not strongly collapsible to one point. So, $0 < scat(S_2) \leq 1$ that implies $scat(S_2) = 1$. □

We can consider now the general case $S^n = D([n+1]_+) \smallsetminus \{[n+1]_+\}$. We can find a cover of this simplicial complex by taking the union of its maximal simplices. The maximal simplices are given by all the possible n-uple of the vertices in the set of vertices of S^n that is $\{0, ..., n+1\}$. We define the cover by $U_0 = [1, ...n+1]$ and $U_1 = \cup[0, i_1, .., i_{n+1}]$ where $i_j \in \{1, ..., n+1\}$, $j \in \{1, ..., n+1\}$.

Proposition 3.2.6. *The cover of S^n given by U_0 and U_1 is a categorical cover.*

Proof. We will prove that U_0 and U_1 strong collapse to a point. With this aim, we will show that every vertex of the two subcomplexes is dominated by an other point, therefore we can perform a sequence of strong collapses by removing dominated points. As described in *Definition 1.5.2*, a vertex v is dominated by another vertex v' if and only if all maximal simplices that contain v also contain v'. Consider first $U_0 = [1, ..., n+1]$, here all vertices are dominated by an other therefore we can eliminate the vertices one by one making U_0 strongly collapse to the vertex 1, for instance. Now in the subcomplex U_1 all vertices are dominated by the vertex 0 because all maximal simplices contain 0. Therefore if $i_j \neq 0$ is a vertex of U_1 $j \in \{1, ..., n+1\}$, we eliminate v_j by defining $v_j = 0$. Therefore we obtain an other simplicial complex given by $U_1 = \cup[0, i_1, ..., \overline{i_j}, ..., i_{n+1}]$, where $\overline{i_j}$ means that i_j does not appear in the list of vertices. Again, all vertices $i_k \neq 0$ $k \in \{1, ..., \overline{j}, ..., n+1\}$ are dominated by the vertex 0 therefore we can repeat the procedure operating a series of strong collapses until $U_1 = [0]$. We showed that U_0 and U_1 are strongly collapsible subcomplexes therefore they are categorical subcomplexes. □

Proposition 3.2.7. $scat(S^n) = 1$.

Proof. By the previous proposition we know that $scat(S^n) \leq 1$. Moreover we know that the geometric realisation of S^n, $|S^n|$ is the n-sphere and it is not contractible, therefore $cat(|S^n|) > 0$. Moreover by *Example A.1.5* we know that $cat(|S^n|) = 1$ and by *Theorem 3.1.8* that $1 = cat(|S^n|) \leq scat(S^n) \leq 1$. We can conclude that $scat(S^n) = 1$. □

Remark 3.2.8. *The two subcomplexes U_0 and U_1 that cover S^n are strong collapsible. Therefore we have that $cat(S^n) = gcat(S^n) = 1$. Moreover by Corollary 3.1.11 we know that $1 = cat(|S^n|) = scat(sd^m(S^n))$ for all $m \in \mathbb{N}$.*

Remark 3.2.9. *Proposition 3.2.6 provides an algorithm for reducing a simplicial complex to its core. Let K be a simplicial complex and $K^0 = \{v_0, ..., v_n\}$ its set of vertices. Consider now the maximal simplices (maximal element of the relation of inclusion), that are m-uple of vertices of K $[v_i, ..., v_j]$ for some $m \leq n$ (the maximal simplices may contain a different number of vertices). Now, if given a vertex v_j we have that for every maximal m-uple that contains v_j it contains also a vertex v_i,* *then we define $v_i = v_j$ and we reduce the m-uple to a $m-1$-uple. In other words we consider the maximal m-uples which contain one vertex and if they all contain an other vertex we eliminate the first of the two, this operation correspond to eliminate dominated vertices. For example we can consider the simplicial complex K given by the maximal simplices: $[0,3,5]$, $[0,2,3]$, $[0,1]$, $[1,2]$, $[1,4]$. Now 3, for example, is dominated by 0 because it appears in the maximal simplices $[0,3,5]$, $[0,2,3]$. Therefore we set $3 = 0$ and $K \searrow\searrow K'$ where K' is given by $[0,5]$, $[0,2]$, $[0,1]$ $[1,2]$ and $[1,4]$. Now 5 is dominated by 0 because the only maximal element in which 5 appears is $[0,5]$ therefore we set $5 = 0$, in the same way 4 is dominated by 1. Therefore $K' \searrow\searrow K''$ where K'' is given by $[0,2]$, $[0,1]$ $[1,2]$. K'' is a minimal complex because there is no dominated vertex, in fact all the simplices where a vertex appear contain different vertices. K'' is the core of K.*

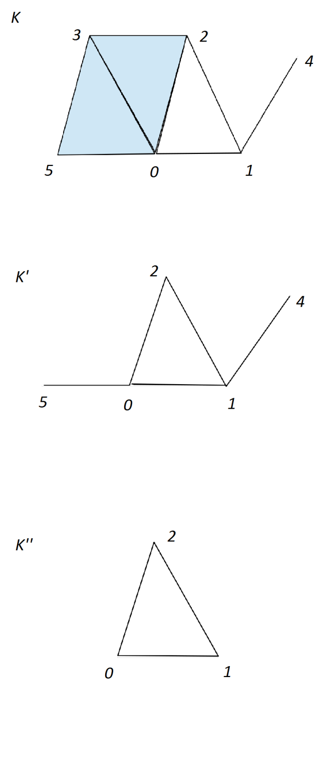


Figure 3.2.5: A sequence of strong collapses from a simplicial complex to its core.

3.3 Computation of the simplicial L-S category: the real projective plane $\mathbb{R}P^2$

In this section we will compute the value of the simplicial L-S category of a triangulation of the real projective plane $\mathbb{R}P^2$. Whit this aim we will use the results from *Section 3.1*. The triangulation can be found in [1] and the results are due to the author.

Consider the real projective plane $\mathbb{R}P^2$. We know from *Example A.2.3* in *Appendix A* that $cat(\mathbb{R}P^2) = 2$. Moreover if we consider the following triangulation K of $\mathbb{R}P^2$

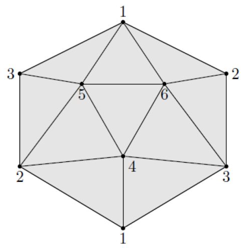


Figure 3.3.1: A triangulation of $\mathbb{R}P^2$, [1] page 7.

By *Theorem 3.1.8* we have that $cat(\mathbb{R}P^2) = scat(|K|) = 2 \leq scat(K)$. Consider now the following categorical cover of K.

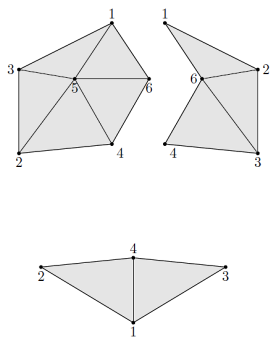


Figure 3.3.2: A categorical cover of $\mathbb{R}P^2$ of three subcomplexes, [1] page 7.

Therefore we have that $cat(\mathbb{R}P^2) = cat(|K|) = scat(K) = 2$. Since the three subcomplexes are strong collapsible by *Proposition 2.1.8* we have that $gcat(K) = 2$. Moreover, by *Corollary 3.1.11* we know that the simplicial category, in this case, doesn't change under subdivision, $cat(\mathbb{R}P^2) = scat(sd^n(K)) = scat(K) = 2$ for all $n > 0$.

3.4 Computation of simplicial L-S category: the Torus T^2

We compute the value of the simplicial category of the triangulation of the Torus T^2 by using the L-S category of the geometric realisation as a lower bound. The results in this section are due to the author.

Consider the following triangulation T^2 of the 2-dimensional Torus.

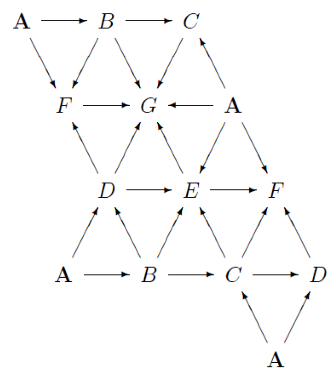


Figure 3.4.1: A triangulation for the 2-dimensional Torus T^2.

We can find a cover of T^2 of strongly collapsible subcomplexes given by $U_0 = st(B)$, $U_1 = st(F)$ and $U_2 = st(G)$. Note that if K is a simplicial complex and v one of its vertices then the star of the vertex v, $st(v)$ is a strongly collapsible subcomplex. In fact, $st(v) = \{\sigma \in K : \sigma \cup v \in K\}$ therefore all maximal simplex in $St(v)$ contains v therefore if v' is any other vertex in $st(v)$ then every maximal simplex in $st(v)$ that contain v' contain also v. That means that all vertex v' is $st(v)$ is dominated by v, that implies $st(v) \searrow\searrow v$. By *Theorem 2.4.4* we have that $scat(T^2) \leq gscat(T^2) \leq 2$. Moreover we know from *Example A.2.3* in *Appendix A* that $cat(|T^2|) = 2$. We can conclude by *Theorem 3.1.8* that

$$2 = cat(|T^2|) \leq scat(T^2) \leq gscat(T^2) \leq 2$$

Therefore $scat(T^2) = 2$. Moreover, by *Corollary 3.1.11* we have that $scat(sd^n(T^2)) = 2$ for all $n \in \mathbb{N}$.

Appendix A

An upper bound and a lower bound for the L-S category

In this Appendix we want to present two useful results of the general theory of LS-category for topological spaces. We will show that there are lower and upper bounds for estimating the value of the L-S category. In fact, given a topological space X the cup-length of X gives a lower bound for the value of L-S category and the dimension of the space provides the upper bound. We will first recall the definition of cup product and cup-length in co-homology.

A.1 The cup length as lower bound for the L-S category

The definitions about co-homology and cup-product refer to [12], [6],[10] and [8], while *Theorem A.1.4* and *Definition A.1.3* can be found in [4]. We first remind the construction of the co-homology groups.
A *simplicial object* on a category C is a contravariant functor between the category Δ of ordered set $[n]_+ = [0, ..., n]$ of ordered abstract simplices Δ^n and order preserving map, and the category C. A *singular set* of a topological space X is a *simplicial set* (simplicial object in the category SET) $S(X) : \Delta \to SET$ such that for all $[n]_+ \in \Delta$

$$S(X)([n]_+) = Hom_{TOP}(|\Delta^n|, X)$$

$S(X)$ can be considered as a sequence of sets

$$S(X)_0 = Hom_{TOP}(|\Delta^0|, X), ..., S(X)_n = Hom_{TOP}(|\Delta^n|, X)$$

with specific face maps $d_i : S(X)_n \to S(X)_{n-1}$ and degeneracy maps $s_i : S(X)_n \to S(X)_{n+1}$. Singular sets and morphisms between them can be considered part of the category of Δ-sets, therefore given a topological space X we can associate trough a functor the singular set $S(X)$ and continuous function $f : X \to Y$ between topological spaces are associated to induced functions $f_* : S(X) \to S(Y)$ defined as the composition

$$f_*(\sigma : |\Delta^n| \to X) = f\sigma : |\Delta^n| \to Y$$

Then we can associate to the singular set $S(X)$ its simplicial chain complex $C_*(X)$ composed by the n-chains $C_n = \mathbb{Z}[(S)(X)_n]$ that are the formal finite sums $\sum_i n_i\sigma_i$, $n_i \in \mathbb{Z}$, $\sigma_i \in S(X)_i$ and boundary maps $\delta_n : C_n(X) \to C_{n-1}(X)$. Morphisms between singular sets $f_* : S(X) \to S(Y)$ in the category of Δ-sets correspond in the category of chain complexes C_* to chain maps

$$f_\circ : C_*(X) \to C_*(Y)$$

Finally we can consider the co-chain complex $C^*(X)$ composed by the dual

$$C^n(X) = Hom_{\mathbb{Z}}(C_n(X), \mathbb{Z})$$

and the boundary maps

$$\delta^n : C^n(X) \to C^{n+1}(X)$$

Now we can define the n *singular co-homology groups with coefficients in* $\mathbb{Z}$ by

$$H^n(X) = \frac{ker(\delta^{n+1})}{Im(\delta^n)}$$

Now we can define the notions of cup product and cup length. We will denote the value of a co-chain $\phi \in C^n(X)$ on a chain $a \in C_n(X)$ by $< \phi, a > \in \mathbb{Z}$.

Definition A.1.1. Let X be a topological space. Given two co-chain $\phi \in C^n(X)$ and $\psi \in C^m(X)$ the *cup product* of ϕ and ψ, $\phi \cup \psi$ is the co-chain $\phi \cup \psi \in C^{n+m}(X)$ such that for all simplices $\sigma : \Delta^{n+m} \to X$

$$< \phi \cup \psi, \sigma >= (-1)^{n+m} < \phi, \sigma|[v_0, ..., v_n] >< \psi, \sigma|[v_{n+1}, ..., v_{n+m}] >$$

where $\sigma|[v_0, ..., v_n] = [v_0, ..., v_n, 0..., 0]$ and $\sigma|[v_{n+1}, ..., v_{n+m}] = [0, ..., 0, v_{n+1}, ..., v_{n+m}]$

Therefore there is a product operation in the co-homologies classes $H^n(X) \otimes H^m(X) \to H^{n+m}(X)$ that gives to $H^*(X) = (H^0(X), H^1(X)...)$ the structure of a graded commutative ring.

Remark A.1.2. *It is possible to define the cup product from the cross product in co-homology. Given $a \in H^p(X)$ and $b \in H^q(X)$,*

$$a \cup b = \Delta^*(a \times b) \in H^{p+q}(X)$$

where Δ^* is induced by $\Delta : X \to X \times X$ is the diagonal map. The cross product of $a \otimes b \in H^p(X) \otimes H^q(Y)$ is defined as the image of $a \otimes b$ trough the composition

$$H^p(X) \otimes H^q(Y) \to H^{p+q}(S_*(X) \otimes S_*(Y)) \to H^{p+q}(X \times Y)$$

The first map is the map $\times^{alg} : H^p(C) \otimes H^q(D) \to H^{p+q}((C_* \otimes D_*)^*)$, where C_* and D_* are two chain complexes and it is defined by $\times^{alg}([a] \otimes [b]) = [\sum z_i \otimes w_i \to \sum < a(z_i) > \cdot < b(w_i) >]$ and $\cdot$ is the multiplication in the ring R. The second map is induced by the Eilenberg-Zilber map $A : S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$. The dual of this map is a chain homotopy equivalence $A^* : (S_*(X) \otimes S_*(Y))^* \to S^*(X \times Y)$ that induces an isomorphism in cohomology $A^* : H^*(S_*(X) \otimes S_*(Y)) \to H^*(X \times Y)$.
Therefore we can define the cup product for the homology of the pair

$$H^*(X, A; R) \otimes H^*(X, B; R) \to H^*(X, A \cup B; R)$$

$a \cup b = \Delta^*(a \times b)$ where Δ^* is induced by the diagonal map

$$\Delta : (X, A \cup B) \to (X, A) \times (X, B) = (X \times X, A \times X \cup X \times B)$$

A result, that we will need in the next proof, that follows from the naturality of A and the definition of cross product is the following. If $f : X' \to X$ and $g : Y' \to Y$ are continuous maps between topological spaces and $a \in H^*(X)$, $b \in H^*(Y)$ then

$$(f \times g)^*(a \times b) = f^*(a) \times g^*(b)$$

More details about this construction can be fond in [5].

Definition A.1.3. Let R be a commutative ring and X a topological space. The *cup length of X with coefficient in R* is the largest integer k (or ∞) such that the product in the co-homology ring $H^*(X;R)$ is $a_1 \cup ... \cup a_k \neq 0$ where the terms $a_j \in H^i(X)$ have degrees $i \geq 1$, we denote this integer by $cup_R(X)$.

Theorem A.1.4. *(Proposition 1.5 [4]) Let X be a topological space. Then the R-cup length of X is less or equal to the L-S category of X for all coefficients R, $cup_R(X) \leq cat(X)$.*

Proof. Suppose that $cat(X) = n$ and $\{U_1, ..., U_{n+1}\}$ a categorical cover of X. Let $x_1 \cup ... \cup x_{n+1}$ be a cup product. Consider the long exact sequence in co-homology for the pair (X, U_i), where $j_i : U_i \hookrightarrow X$ and $q : X \to (X, U_i)$

$$... \to H^m((X,U_i);R) \xrightarrow{q^*} H^m(X;R) \xrightarrow{j^*} H^m(U_i;R) \to ...$$

U_i is contractible in X, therefore $H^m(U_i;R) = 0$ and $j^* = 0$. By exactness

$$...0 \to H^m((X,U_i);R) \simeq H^m(X,R) \xrightarrow{j^*} 0 \to ...$$

For each U_i, x_i has a preimage x_i' such that $q^*(x_i') = x_i$.
Now, from the general description of the cup product, we have a map $H^*(X,A;R) \otimes H^*(X,B;R) \to H^*(X, A \cup B; R)$ defined by $a \cup b = \Delta^*(a \times b)$ and induced by

$$\Delta : (X, A \cup B) \to (X, A) \times (X, B) = (X \times X, A \times X \cup X \times B)$$

is the diagonal map (*Remark A.1.2*). By the commutativity of the diagram

$$\begin{array}{ccc} X & \xrightarrow{q} & (X, A \cup B) \\ \downarrow\Delta & & \downarrow\Delta \\ X \times X & \xrightarrow{q_1 \times q_2} & (X,A) \times (X,B) \end{array}$$

we have that $q^*\Delta^* = \Delta^*(q_1 \times q_2)^* = \Delta^*(q_1^* \times q_2^*)$ where the last inequality is provided by *Remark A.1.2*, therefore $q^*(a \cup b) = q_1^*(a) \cup q_2^*(b)$. Consider now the product $x_1' \cup ... \cup x_{n+1}'$ defined in $H^*(X, \cup U_i; R)$, $i = 1, ..., n+1$. The map $q : X \to (X, \cup U_i)$ induces an homomorphism $H^*(X, \cup U_i; R) \to H^*(X;R)$ that implies

$$q^*(x_1' \cup ... \cup x_{n+1}') = q_1^*(x_1') \cup ... \cup q_{n+1}^*(x_{n+1}') = x_1 \cup ... \cup x_{n+1}$$

Moreover $H^*(X, \cup U_i; R) = 0$ since $\cup U_i = X$, $i = 1, ..., n+1$, this implies that $x_1 \cup ... \cup x_{n+1} = q^*(0) = 0$ therefore $cup_R(X) \leq n = cat(X)$. □

Example A.1.5.
- *If X is a contractible topological space it has $cat(X) = 0$ and also $H^*(X; R) = 0$ for all R.*
- *The sphere S^n has cohomology groups $H^0(S^n) = H^n(S^n) = \mathbb{Z}$ and 0 the others, therefore $1 = cup_{\mathbb{Z}}(S^n)$. Moreover S^n can be covered by two contractible hemisphere therefore $1 = cup_{\mathbb{Z}}(S^n) \leq cat(S^n) \leq 1$, therefore $cat(S^n) = 1$.*
- *The real projective plane $\mathbb{R}P^2$ has cohomology groups with coefficients in $\mathbb{Z}/2\mathbb{Z}$ given by $H^0(\mathbb{R}P^2) = H^1(\mathbb{R}P^2) = H^2(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$ and 0 otherwise, therefore $cup_{\mathbb{Z}}(\mathbb{R}P^2) = 2$.*
- *The n-dimensional Torus T^n has cup length over $\mathbb{Q}$ $cup_{\mathbb{Q}}(T^n) = n$, therefore $cat(T^n) \geq 2$.*

A.2 The dimension as upper bound for the L-S category

We will now show that the dimension of a topological space provides an upper bound for the value of L-S category. We define as dimension of a paracompact space X, $dim(X)$ the *covering dimension* that is the least integer k so that any open cover has a refinement of order k. The *order* of an open cover V is the least integer k so that there exist $k+1$ members of V with non-trivial intersection but not $k+2$. In the case of C-W complex the covering dimension correspond with the dimension as a C-W complex. A *refinement* of a cover $\mathcal{U}$ of X is a cover $\mathcal{U}'$ of X such that every set in $\mathcal{U}'$ is contained in some set in $\mathcal{U}$. A space is said to be *paracompact* if every open cover has a locally finite open refinement.
The following results can be found in *Chapter 1* in [4]. In order to prove the main theorem we will state the following lemma due to Milnor, it is possible to find the proof in *Appendix A* in [4].

Lemma A.2.1. *(Lemma A.4 [4]) Let $\mathcal{U} = \{U_i\}$ be an open covering of X of order n with a partition of unity subordinate to the cover. Then there is an open covering of X refining $\mathcal{U}$, $\mathcal{G} = \{G_{i\beta}\}$ $i = 1, ..., n+1$ such that $G_{i\beta} \cap G_{i\beta'} = \emptyset$ for $\beta \neq \beta'$.*

In particular there is such a refinement if X is a pracompact space with covering dimension n and $\mathcal{U} = \{U_\alpha\}$ is any open covering of X.

Theorem A.2.2. *(Theorem 1.7 [4]) Let X be a path-connected locally contractible paracompact space, then $cat(X) \leq dim(X)$.*

Proof. Suppose that $dim(X) = k$ and that $\mathcal{U} = \{U_1, ..., U_{n+1}\}$ is a categorical cover of X. By the previous lemma there is an open covering $\{G_i\}$, $i = 1, ..., k+1$ of X refining $\mathcal{U}$ with the property that each G_i is the union of disjoint open sets each of which lies in some U_j. Since U_j is contractible in X and the sets forming G_i are disjoint then G_i are also contractible in X, because each component lies in a U_j, therefore each component is contractible and X is path-connected. $\{G_i\}$, $i = 1, ..., k+1$ is a categorical cover of X, therefore $cat(X) \leq k = dim(X)$. $\square$

Example A.2.3.

- *The projective space $\mathbb{R}P^n$ has dimension $dim(\mathbb{R}P^n) = n$. Combining this with the result in Exemple A.1.5 we have that $2 = cup_{\mathbb{Z}}(\mathbb{R}P^2) \leq cat(\mathbb{R}P^2) \leq dim(\mathbb{R}P^2) = 2$. So we obtain that $cat(\mathbb{R}P^2) = 2$.*

- *The n-dimensional Torus T^n has L-S category $cat(T^n) \leq n$ and by Example A.1.5 $n \leq cat(T^n)$. This implies that $cat(T^n) = n$.*

Appendix B

The closed category

In this appendix we want to discuss the L-S category given by the minimal categorical cover of closed subsets, that we denote by cat^{cl}. In general the value of cat^{cl} is different from the value of the L-S category defined with the open cover, but we will show that in the specific case of the geometric realisation of a finite simplicial complex the two values coincide. Therefore, we have that for every finite simplicial complex K, $cat(|K|) = cat^{cl}(|K|)$. With this aim, we will first discuss the topology of the geometric realisation and in *Section B.2* we will give the definition of *closed category* and some results. Finally we will prove the main theorem about the closed category of the realisation of a finite simplicial complex.

B.1 The topology of the geometric realisation

In this section we discuss the topology of the geometric realisation. We define two different topologies on it and then prove that they coincide in the case of the realisation of finite simplicial complex (even locally finite). The results in this section refer to *Chapter 2* [11], *Appendix* [2] and *Chapter 5* in [15]. As we described in *Section 3.1*, let K be a simplicial complex and K^0 its set of vertices, we can define the realisation of K as the subset

$$|K| = \{f : K^0 \to \mathbb{R} : f(K^0) \subseteq [0,1], supp(f) \in K, \sum_{v \in K^0} f(v)\}$$

of the vector space $\mathbb{R}[K^0] = \{f : K^0 \to \mathbb{R} : supp(f) \text{ finite}\}$ with basis the functions $v : K^0 \to \mathbb{R}$ defined for all $v' \in K^0$ by $v(v') = 1$ if $v = v'$ and $v(v') = 0$ otherwise. We can write an element α of the realisation as

$$\alpha(v') = \sum_{v' \in K^0} \alpha_v v(v')$$

for all $v' \in K^0$. Therefore the realisation of a simplex σ in K, $\sigma = \{v_0, ..., v_n\}$ where $v_i \in K^0$, is given by

$$|\sigma| = \{\sum_{v\in\sigma} \alpha_v v | \alpha_v \geq 0, \sum_{v\in\sigma} \alpha_v = 1\}$$

where v denote a vertex in K and an element of base for the vector space $\mathbb{R}[K^0]$, $\alpha_v \in \mathbb{R}$. The realisation of K, $|K|$ can be regarded as the union of the realisations of the simplices σ in K

$$|K| = \cup|\sigma|$$

We can define on $|K|$ a topology given by the metric defined on $\mathbb{R}[K^0]$ for all α, $\beta \in |K|$ by

$$d(\alpha, \beta) = (\sum_{v\in K^0} (\alpha_v - \beta_v)^2)^{\frac{1}{2}}$$

we will denote $|K|_d$ the realisation of K equipped with the *metric topology.* For all simplex σ in K the realisation $|\sigma|$ is a compact subset of $\mathbb{R}[K^0]$, moreover $|\sigma|$ is a compact and therefore closed subset of the Hausdorff space $|K|_d$ (all metric spaces are Hausdorff).
In the text [15] we will refer to, the metric topology is called *barycentric topology* and it is defined as the initial topology given by the barycentric coordinates. We can define a map $m : K^0 \to [0, 1]^{|K|}$ such that for all vertex v in K $m(v) = b_v$ where b_v is the barycentric coordinate, that is a map from $|K|$ to $[0, 1]$ defined for all $\phi \in |K|$ by $b_v(\phi) = \phi(v)$. The initial topology on $|K|$ given by the family of functions $\{b_v\}_{v\in K^0}$ is the smallest topology τ for which each $b_v : (|K|, \tau) \to [0, 1]$ is continuous, ($[0, 1]$ is equipped by the Euclidean topology given by the Euclidean metric on $\mathbb{R}$). The barycentric topology is metrizable with the following metric

$$d(\alpha, \beta) = \sum_{v\in K^0} |b_v(\alpha) - b_v(\beta)|$$

that is equivalent to the one previously defined, therefore the two metrics induce the same topology on $|K|$, namely $|K|_d$.
We can also define a second topology on $|K|$, the so called *coherent topology.* Consider $|K| = \cup|\sigma|$, then the topology coherent with the closed cover
$\{|\sigma| : \sigma \in K\}$ of $|K|$ is defined by

$$L \subseteq |K| \text{ is open (closed) in } |K| \iff L\cap|\sigma| \text{ is open (closed) in the metric space } |\sigma|, \forall\sigma \in K$$

$|\sigma| \subseteq \mathbb{R}[K^0]$. We denote by $|K|$ the realisation of K equipped by the coherent topology. The sets L that are open (closed) in the metric space $|K|_d$ are open (closed) in $|K|$ and therefore $|K|$ is normal Hausdorff.

Proposition B.1.1. *Let L be a subcomplex of K, then the realisation $|L|$ is a closed subset of $|K|$.*

Proof. $|L| = \cup_{\tau \in L} \tau$ is a closed subset of $|K|$ if $|L| \cap |\sigma|$ is closed in $|\sigma|$ for all $\sigma \in K$. Now, $|\sigma| \cap |\tau| = |\sigma \cap \tau|$ (eventually empty if the intersection is empty) therefore $|L| \cap |\sigma| = \cup_{\tau \in L, \tau \subseteq \sigma} |\tau|$ that is closed in $|\sigma|$ because it is the finite union of realisations of the faces of σ that are in L. □

Moreover, since $|K|$ has the coherent topology with respect to $\{|\sigma| : \sigma \in K\}$, it has the final topology with respect to the inclusion maps $i_{|\sigma|} : |\sigma| \to |K|$ for all $\sigma \in K$ (that is the finest topology that make the inclusion maps continuous). Therefore a map $f : |K| \to Y$ is continuous if and only if $f i_{|\sigma|} : |\sigma| \to Y$ is continuous for all $\sigma \in K$. We can now define the *open cells* in the realisation of a simplicial complex.

Definition B.1.2. The *open cell* $< \sigma >$ of the simplex σ in K is the subset of the cell $|\sigma|$ defined as

$$< \sigma >= \{t \in |K| : \forall v \in K^0, v \in \sigma \Leftrightarrow t(v) > 0\}$$

In particular we have that $|K| = \cup < \sigma >$, where the union is disjoint. Note that the open cell $< \sigma >$ is open in $|\sigma|$ but in general is not open in $|K|$.

Proposition B.1.3. *Consider the topologies τ, τ_d. Then $\tau_d \subseteq \tau$.*

Proof. If a set is open in $|K|_d$ then it is also open is $|K|$, then the coherent topology is finer then the metric topology. □

Proposition B.1.4. *(Theorem 2.5 [11]) Let K be a locally finite simplicial complex, then $|K| = |K|_d$. In particular the statement holds for finite simplicial complexes.*

Proof. If K is locally finite then $\{|\sigma| : \sigma \in K\}$ is a locally finite closed cover of $|K|_d$. Then by the *Glueing lemma* we have that for any map $f : |K|_d \to Y$, $f : |\sigma| \to Y$ is continuous for all $\sigma \in K$ implies that $f : |K|_d \to Y$ is continuous. The inclusion maps $i_{|\sigma|} : |\sigma| \to |K|$ are continuous for all simplices in K, therefore the identity map $id : |K|_d \to |K|$ is continuous, that is $\tau \subseteq \tau_d$. By *Proposition B.1.3* follows that $\tau = \tau_d$ and so $|K|_d = |K|$. In particular, a finite simplicial complex is locally finite, therefore if K is a finite simplicial complex then $|K|_d = |K|$. □

B.2 The closed category

The analogous concept of L-S category that uses closed covers has been used in mathematics, especially in analysis. In general the closed category does not coincide with the open L-S category. In this section we want to define the closed category and show that in the particular case of the realisation of finite simplicial complexes the closed and open category have the same value. In this section we denote cat^{op} the L-S category defined with open sets and cat^{cl} the close category. The results refer to *Chapter 1* in [4] and *Chapter 5,6* in [15]. The main theorem in this section, *Theorem B.2.6*, is due to the author.

Definition B.2.1. Let X be a topological space, the *closed category* of X, $cat^{cl}(X)$ is the least integer k such that there is a cover of X, $U_0, ..., U_k$ of $k+1$ closed subsets contractible to a point in X.

We recall the definition of ANR space and normal space.

Definition B.2.2 (Appendix A, [4])**.** A topological space X is *normal* if for each closed set $A \subset X$ and open neighbourhood U of A, there exists an open set V with $A \subset V \subset \overline{V} \subset U$.

Definition B.2.3 (Appendix A [4])**.** A metrizable space is an *absolute neighbourhood retract*, ANR if for any metrizable space X and closed subset A any continuous map $f : A \to Y$ has an extension $f' : U \to Y$ for some neighbourhood U of $A \subseteq X$.

Proposition B.2.4 (Proposition 1.10, [4])**.** *If X is a normal ANR, then $cat^{op}(X) = cat^{cl}(X)$.*

In general a topological space X has the homotopy type of a finite CW-complex if and only if it has the homotopy type of a compact ANR. A proof and a detailed description of this result can be found in [15] in particular from the proof in 6-20 in [15] we can deduce that if K is a finite simplicial complex then $|K|$ is a normal ANR, in order to prove this result we will state a lemma from [15].

Lemma B.2.5 (6-18 [15])**.** *Let K be a simplicial complex then $|K|_d$ is an ANR.*

Theorem B.2.6. *Let K be a finite simplicial complex, then $cat^{op}(|K|) = cat^{cl}(|K|)$.*

Proof. Let K be a finite simplicial complex. We showed that $|K|$ is normal and we know by *Lemma B.2.5* that $|K|_d$ ia an ANR. Moreover, by *Proposition B.1.3* $|K| = |K|_d$. We can conclude that $|K|$ is a normal ANR and by *Proposition B.2.4* we have that $cat^{op}(|K|) = cat^{cl}(|K|)$. □

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