Local robust and asymptotically unbiased estimation of conditional Pareto-type tails

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Topics

1. Introduction
   - Density power divergence
   - Pareto-type distributions
2. Estimation procedure
3. Asymptotic properties
4. Simulation results

1. Introduction: density power divergence

- **Basu, Harris, Hjort and Jones (1998):** density power divergence between density functions f and g

$$\Delta_\alpha(f,g) := \begin{cases} \int_{\mathbb{R}} \left[g^{1+\alpha}(y) - \left(1+\frac{1}{\alpha}\right) g^\alpha(y) f(y) + \frac{1}{\alpha} f^{1+\alpha}(y)\right] dy, & \alpha > 0, \\ \int_{\mathbb{R}} \log \frac{f(y)}{g(y)} f(y) dy, & \alpha = 0. \end{cases}$$

- Assume that the density function g depends on a parameter vector θ

  Let $Y_1, \ldots, Y_n$ be a independent and identically distributed (i.i.d.) random variables according to density function f.

  The **minimum density power divergence (MDPD) estimator** is the value of θ minimizing the empirical density power divergence. For $\alpha > 0$:

$$\widehat{\Delta}_\alpha(\theta) := \int_{\mathbb{R}} g^{1+\alpha}(y) dy - \left(1 + \frac{1}{\alpha}\right) \frac{1}{n} \sum_{i=1}^{n} g^\alpha(Y_i),$$

and for $\alpha = 0$:

$$\widehat{\Delta}_0(\theta) := -\frac{1}{n}\sum_{i=1}^{n} \log g(Y_i),$$

Note: for $\alpha = 0$ the method corresponds to fitting the density function g with the maximum likelihood method.

- The parameter α controls the **trade-off between efficiency and robustness** of the MDPD estimator:

  the estimator becomes more efficient but less robust against outliers as α gets closer to zero,

  whereas for increasing α the robustness increases and the efficiency decreases.

- We want to use the MDPD method to obtain a **robust nonparametric and asymptotically unbiased estimation method for Pareto-type distributions when there are random covariates.**

1. Introduction: Pareto-type distribution

- A distribution function F is said to be of *Pareto-type* if for some $\gamma > 0$

$$1 - F(y) = y^{-\frac{1}{\gamma}} \ell(y), \qquad y > 0, \tag{1}$$

with ℓ a *slowly varying function at infinity* :

$$\frac{\ell(\lambda y)}{\ell(y)} \to 1 \text{ as } y \to \infty, \qquad \forall \lambda > 0.$$

- γ: extreme-value index **First order tail parameter**

- Example: strict Pareto, F, Burr, $|t|$, log-gamma, . . .

- Estimation of γ has received a lot of attention. **Classical estimators are non-robust and typically show an asymptotic bias.** See Beirlant *et al.* (2004) and de Haan and Ferreira (2006)

- Robust estimation:

  Juárez and Schucany (2004), Kim and Lee (2008), Vandewalle and Beirlant (2007), Pend and Welsh (2001), Hubert *et al.* (2012), . . .

  Dierckx *et al.* (2013): fit the extended Pareto distribution with the MDPD technique

  – Robust and asymptotically unbiased
  – Asymptotic properties are available

- Extension to **regression case:** assume that together with Y we observe a **random covariate** X

  $F(y;x)$: conditional distribution function of the response variable Y given $X = x$,

  $b(x)$: density function of $X \in \mathbb{R}^p$.

  $F(y;x)$ is assumed to be of Pareto-type, i.e. there exists a positive function $\gamma(x)$ such that $\bar{F}(y;x) := 1 - F(y;x)$ is of the form

$$\bar{F}(y;x) = y^{-1/\gamma(x)} \ell(y;x), \qquad y > 0, \tag{2}$$

  $\gamma(x)$ describes the tail heaviness of $F(y;x)$ and has to be adequately estimated from the data.

  We use here a **nonparametric approach based on local estimation.**

  Local estimation: Daouia *et al.* (2011)

  Local asymptotically unbiased estimation: Goegebeur *et al.* (2013)

  But: these procedures are not robust!

2. Estimation procedure

The theoretical study of estimators for $\gamma(x)$ generally requires a **second order condition.**

Condition $(\mathcal{R})$. *Let* $\gamma(x) > 0$ *and* $\rho(x) < 0$ *be constants. The conditional distribution function* $F(y;x)$ *is such that* $y^{1/\gamma(x)}\bar{F}(y;x) \to C(x) \in (0,\infty)$ *as* $y \to \infty$ *and the function* $\delta(.;x)$ *defined via*

$$\bar{F}(y;x) = C(x)y^{-1/\gamma(x)}(1+\gamma(x)^{-1}\delta(y;x)),$$

is ultimately nonzero, of constant sign and $|\delta| \in RV_{\rho(x)/\gamma(x)}$, *i.e.*

$$\frac{\delta(ty;x)}{\delta(t;x)} \to y^{\rho(x)/\gamma(x)} \text{ as } t \to \infty,\ \forall y > 0.$$

Taking this second order structure into account during the estimation phase allows to obtain bias-corrected estimators.

Consider the **extended Pareto distribution** (Beirlant *et al., 2004*, Beirlant *et al.*, 2009), with distribution function given by

$$G(z;\gamma,\delta,\rho) = \begin{cases} 1 - [z(1+\delta-\delta z^{\rho/\gamma})]^{-1/\gamma}, & z > 1, \\ 0, & z \leq 1, \end{cases} \tag{3}$$

and density function

$$g(z;\gamma,\delta,\rho) = \begin{cases} \frac{1}{\gamma} z^{-1/\gamma-1}[1+\delta(1-z^{\rho/\gamma})]^{-1/\gamma-1}[1+\delta(1-(1+\rho/\gamma)z^{\rho/\gamma})], & z > 1, \\ 0, & z \leq 1, \end{cases}$$

where $\gamma > 0$, $\rho < 0$, and $\delta > \max\{-1, \gamma/\rho\}$.

For distribution functions satisfying $(\mathcal{R})$, one can **approximate** the conditional distribution function of $Z := Y/u$, given that $Y > u$, where u denotes a threshold value, by the extended Pareto distribution:

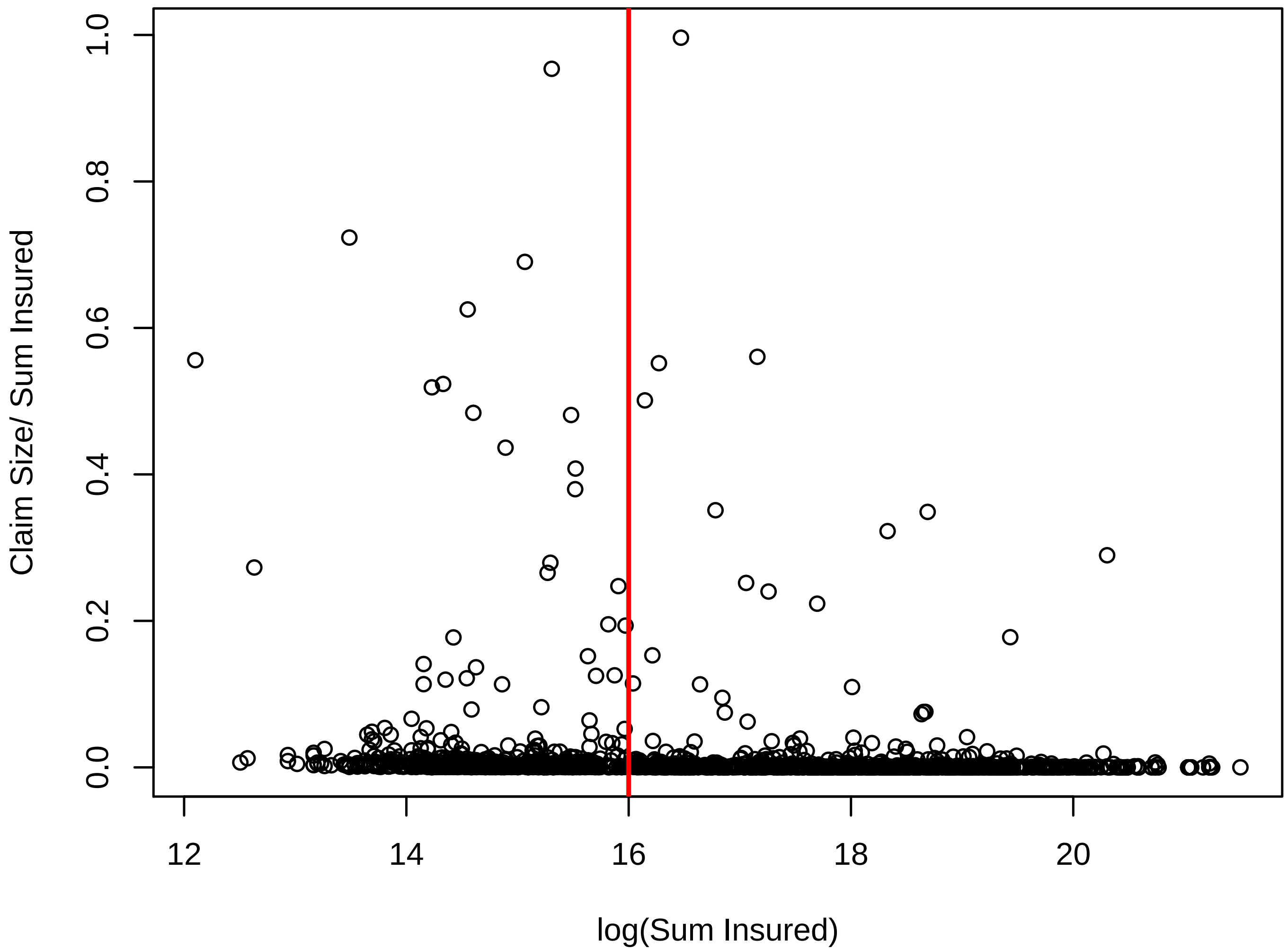
$$\frac{\bar{F}(uz;x)}{\bar{F}(u;x)} \approx \bar{G}(z;\gamma(x),\delta(u;x),\rho(x))$$

for large u.

Formally, as shown in Beirlant *et al.* (2009), one has that

$$\sup_{z\geq 1}\left|\frac{\bar{F}(uz;x)}{\bar{F}(u;x)} - \bar{G}(z;\gamma(x),\delta(u;x),\rho(x))\right| = o(\delta(u;x)), \qquad \text{if } u \to \infty.$$

Estimation of $\gamma(x)$: Let (X_i, Y_i), $i = 1, \ldots, n$, be independent realizations of the random vector $(X, Y) \in \mathbb{R}^p \times \mathbb{R}_{+,0}$, where X has a distribution with joint density function b, and $\bar{F}(y;x)$ satisfies $(\mathcal{R})$.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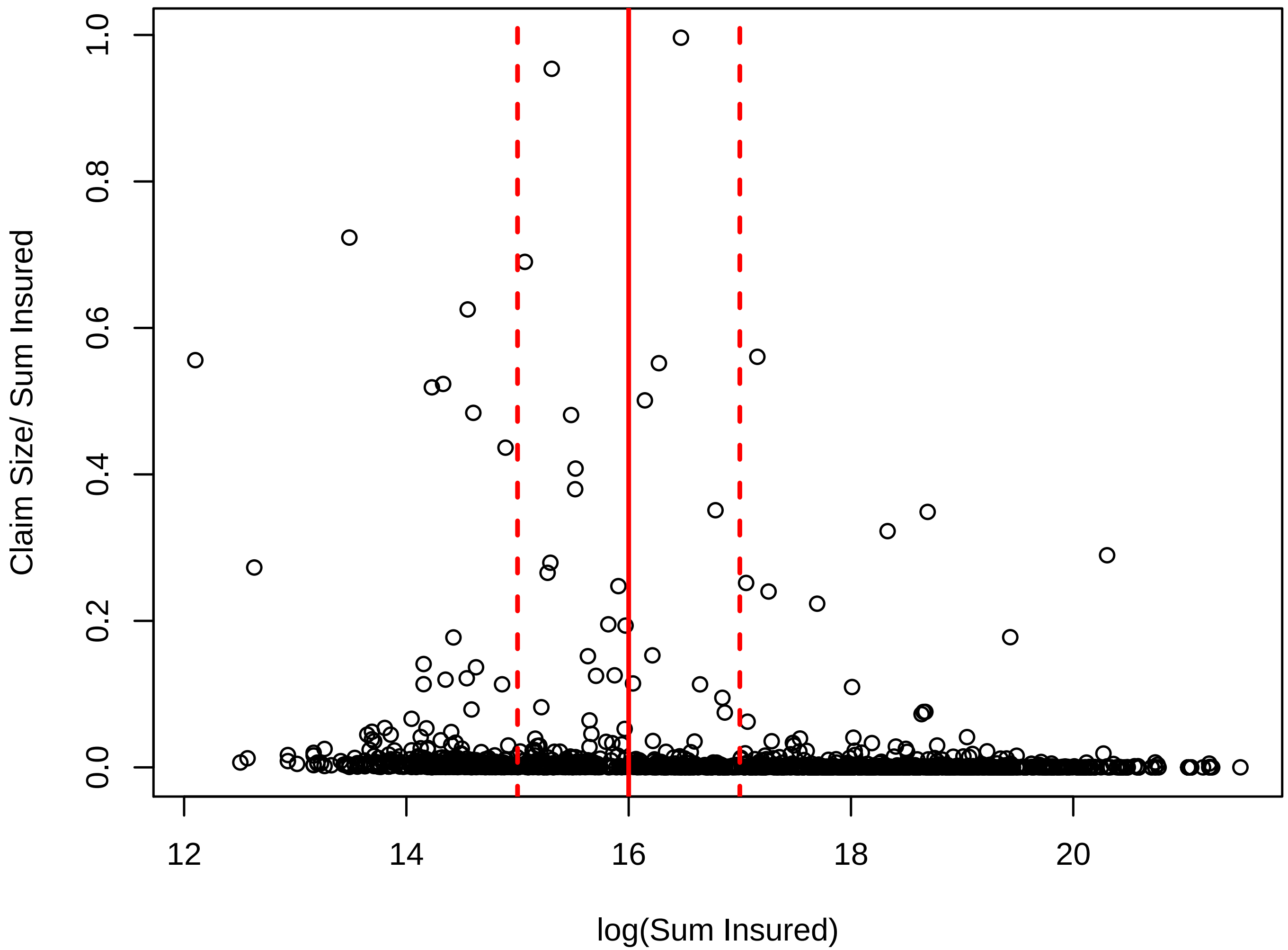

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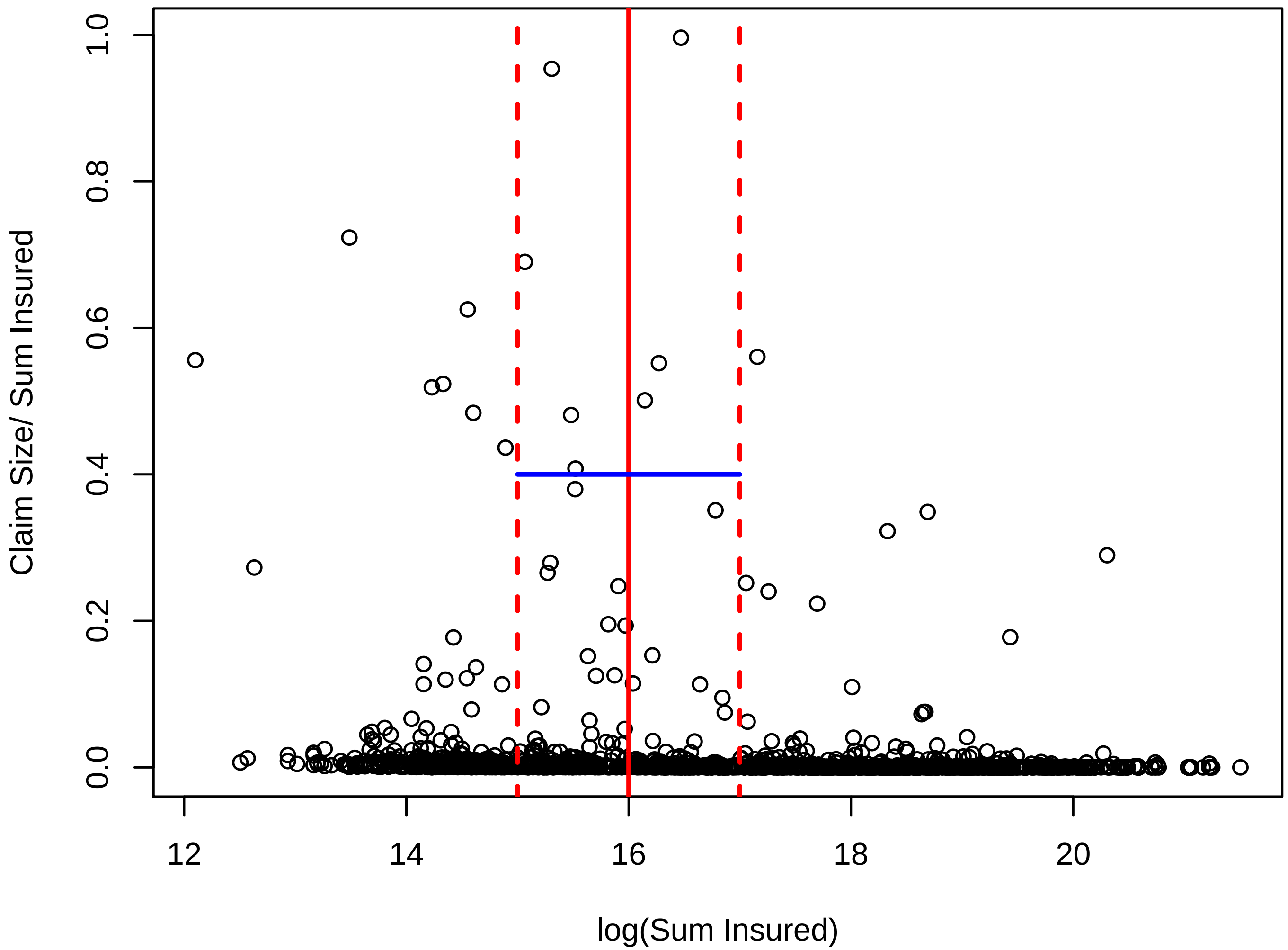

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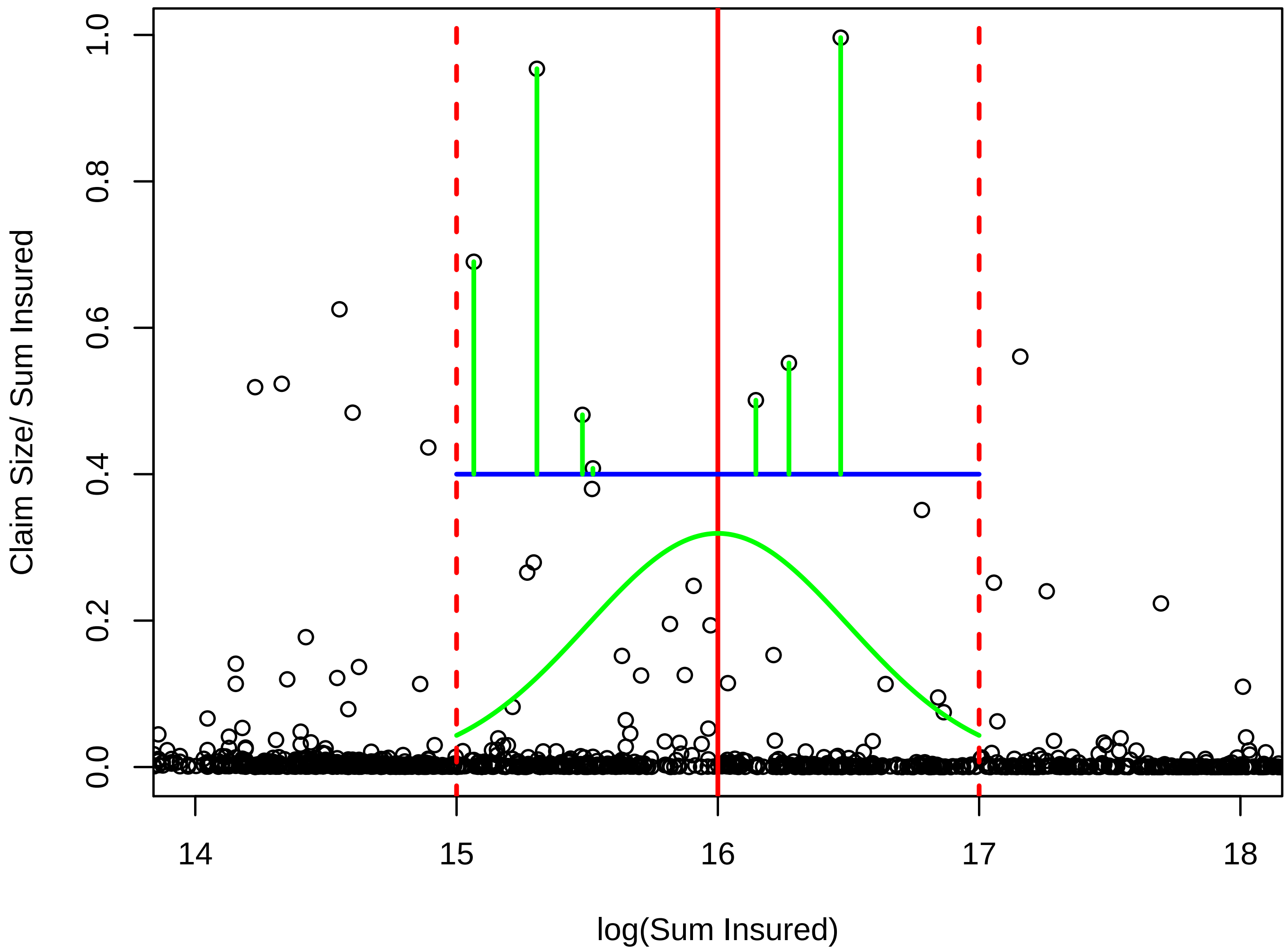

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Fit g locally to the relative excesses $Z_i := Y_i/u_n$, $i = 1, \ldots, n$, by **MDPD, adjusted to locally weighted estimation,** i.e. we minimize

$$\widehat{\Delta}_\alpha(\gamma, \delta; \rho) :=$$

$$\frac{1}{n}\sum_{i=1}^{n} K_{h_n}(x - X_i)\left\{\int_1^\infty g^{1+\alpha}(z;\gamma,\delta,\rho)dz - \left(1+\frac{1}{\alpha}\right) g^\alpha(Z_i;\gamma,\delta,\rho)\right\}\mathbf{1}\{Y_i > u_n\},$$

in case $\alpha > 0$ and

$$\widehat{\Delta}_0(\gamma, \delta; \rho) \quad := \quad -\frac{1}{n}\sum_{i=1}^{n} K_{h_n}(x - X_i)\ln g(Z_i;\gamma,\delta,\rho)\mathbf{1}\{Y_i > u_n\},$$

in case $\alpha = 0$,

where

$K_{h_n}(x) := K(x/h_n)/h_n^p$, K is a joint density function on $\mathbb{R}^p$,

h_n is a positive non-random sequence of bandwidths with $h_n \to 0$ if $n \to \infty$,

u_n is a local non-random threshold sequence satisfying $u_n \to \infty$ if $n \to \infty$.

The MDPD estimator for $(\gamma(x), \delta(u_n; x))$ satisfies the **estimating equations**

$$
\begin{aligned}
0 &= \frac{1}{n}\sum_{i=1}^{n} K_{h_n}(x - X_i)\mathbf{1}\{Y_i > u_n\}\int_1^\infty g^\alpha(z;\gamma,\delta,\rho)\frac{\partial g(z;\gamma,\delta,\rho)}{\partial\gamma}dz \\
&\quad -\frac{1}{n}\sum_{i=1}^{n} K_{h_n}(x - X_i)g^{\alpha-1}(Z_i;\gamma,\delta,\rho)\frac{\partial g(Z_i;\gamma,\delta,\rho)}{\partial\gamma}\mathbf{1}\{Y_i > u_n\}, \qquad (4) \\
0 &= \frac{1}{n}\sum_{i=1}^{n} K_{h_n}(x - X_i)\mathbf{1}\{Y_i > u_n\}\int_1^\infty g^\alpha(z;\gamma,\delta,\rho)\frac{\partial g(z;\gamma,\delta,\rho)}{\partial\delta}dz \\
&\quad -\frac{1}{n}\sum_{i=1}^{n} K_{h_n}(x - X_i)g^{\alpha-1}(Z_i;\gamma,\delta,\rho)\frac{\partial g(Z_i;\gamma,\delta,\rho)}{\partial\delta}\mathbf{1}\{Y_i > u_n\}. \qquad (5)
\end{aligned}
$$

Note:

Only $\gamma(x)$ and $\delta(u_n; x)$ are estimated by the MDPD method.

The rate parameter $\rho(x)$ **will either be fixed or estimated externally in a consistent way.**

3. Asymptotic properties

For all $x_1, x_2 \in \mathbb{R}^p$, the Euclidean distance between x_1 and x_2 is denoted by $d(x_1, x_2)$.

Assumption $(\mathcal{B})$ *There exists* $c_b > 0$ *such that* $|b(x_1) - b(x_2)| \leq c_b d(x_1, x_2)$ *for all* x_1, $x_2 \in \mathbb{R}^p$.

Assumption $(\mathcal{K})$ K *is a bounded density function on* $\mathbb{R}^p$, *with support* Ω *included in the unit hypersphere in* $\mathbb{R}^p$.

We also need to control the oscillation of $F(y;x)$ when considered as a function of its second argument.

Consider the conditional expectation

$$m(u_n, s, t; x) := \mathbb{E}\left[\left(\frac{Y}{u_n}\right)^s \left(\ln_+ \frac{Y}{u_n}\right)^t \mathbf{1}\{Y > u_n\} \middle| X = x\right],$$

with $s \leq 0$, $t \geq 0$.

Assumption $(\mathcal{M})$ *The function* $m(u_n, s, t; x)$ *satisfies that, for* $u_n \to \infty$, $h_n \to 0$, *and some* $S < 0$ *and* $T > 0$,

$$\Phi(u_n, h_n; x) := \sup_{(s,t)\in[S,0]\times[0,T]} \sup_{z\in\Omega} \left| \frac{m(u_n, s, t; x - h_n z)}{m(u_n, s, t; x)} - 1 \right| \to 0 \text{ if } n \to \infty.$$

- **Case 1: $\rho_0(x)$ known**

Theorem 1. (Existence and consistency) *Let* $(X_1, Y_1), \ldots, (X_n, Y_n)$ *be a sample of independent copies of the random vector* (X, Y) *where* $Y|X = x$ *satisfies* $(\mathcal{R})$, $X \sim b$, *and assume* $(\mathcal{B})$, $(\mathcal{K})$ *and* $(\mathcal{M})$ *hold. For all* $x \in \mathbb{R}^p$ *where* $b(x) > 0$, *we have that if* $h_n \to 0$, $u_n \to \infty$ *with* $nh_n^p \bar{F}(u_n; x) \to \infty$, *then with probability tending to 1 there exists sequences of solutions* $(\hat{\gamma}_n(x), \hat{\delta}_n(x))$ *of the estimating equations (4) and (5), with* ρ *fixed at* $\rho_0(x)$, *such that* $(\hat{\gamma}_n(x), \hat{\delta}_n(x)) \xrightarrow{\mathbb{P}} (\gamma_0(x), 0)$, *as* $n \to \infty$.

Theorem 2. (Asymptotic normality) *Let* $(X_1, Y_1), \ldots, (X_n, Y_n)$ *be a sample of independent copies of the random vector* (X, Y) *where* $Y|X = x$ *satisfies* $(\mathcal{R})$, $X \sim b$, *and assume* $(\mathcal{B})$, $(\mathcal{K})$ *and* $(\mathcal{M})$ *hold.*

Consider $(\hat{\gamma}_n(x), \hat{\delta}_n(x))$, *a consistent sequence of estimators for* $(\gamma_0(x), 0)$ *satisfying (4) and (5), with* ρ *fixed at* $\rho_0(x)$.

For all $x \in \mathbb{R}^p$ *where* $b(x) > 0$, *we have that if* $h_n \to 0$, $u_n \to \infty$ *with* $nh_n^p \bar{F}(u_n; x) \to \infty$, $\sqrt{nh_n^p \bar{F}(u_n; x)}\delta(u_n; x) \to \lambda \in \mathbb{R}$, $\sqrt{nh_n^p \bar{F}(u_n; x)}h_n \to 0$, *and* $\sqrt{nh_n^p \bar{F}(u_n; x)}\Phi(u_n, h_n; x) \to 0$, *then*

$$\sqrt{nh_n^p \bar{F}(u_n; x) b(x)} \begin{bmatrix} \hat{\gamma}_n(x) - \gamma_0(x) \\ \hat{\delta}_n(x) - \delta(u_n; x) \end{bmatrix}$$
$$\rightsquigarrow N_2(\mathbf{0}, \mathbb{C}^{-1}(\rho_0(x))\mathbb{B}(\rho_0(x))\boldsymbol{\Sigma}(\rho_0(x))\mathbb{B}'(\rho_0(x))\mathbb{C}^{-1}(\rho_0(x))).$$

$\rightarrow$ **Bias-corrected!**

- **Case 2: ρ fixed at some value $\tilde{\rho}(x) < 0$**

Proposition 1. *Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a sample of independent copies of the random vector (X, Y) where $Y|X = x$ satisfies $(\mathcal{R})$ and assume the parameter ρ is fixed at $\tilde{\rho}(x)$ in (4) and (5). Suppose also that $X \sim b$, and assume $(\mathcal{B})$, $(\mathcal{M})$ and $(\mathcal{K})$ hold. For all $x \in \mathbb{R}^p$ where $b(x) > 0$, we have that if $h_n \to 0$, $u_n \to \infty$ with $nh_n^p \bar{F}(u_n; x) \to \infty$, when $n \to \infty$, then with probability tending to 1 there exists sequences of solutions $(\hat{\gamma}_n(x), \hat{\delta}_n(x))$ of the estimating equations (4) and (5) such that $(\hat{\gamma}_n(x), \hat{\delta}_n(x)) \xrightarrow{\mathbb{P}} (\gamma_0(x), 0)$. If additionally $\sqrt{nh_n^p \bar{F}(u_n; x)}\delta(u_n; x) \to \lambda \in \mathbb{R}$, $\sqrt{nh_n^p \bar{F}(u_n; x)} h_n \to 0$, and $\sqrt{nh_n^p \bar{F}(u_n; x)}\Phi(u_n, h_n; x) \to 0$, then*

$$
r_n \left[\begin{array}{c} \hat{\gamma}_n(x) - \gamma_0(x) \\ \hat{\delta}_n(x) \end{array} \right] \rightsquigarrow N_2(-\lambda\sqrt{b(x)}\mathbb{C}^{-1}(\tilde{\rho}(x))\mathbb{B}(\tilde{\rho}(x))\tilde{\mathbb{D}},
$$

$$
\mathbb{C}^{-1}(\tilde{\rho}(x))\mathbb{B}(\tilde{\rho}(x))\boldsymbol{\Sigma}(\tilde{\rho}(x))\mathbb{B}'(\tilde{\rho}(x))\mathbb{C}^{-1}(\tilde{\rho}(x))),
$$

- **Case 3: $\rho_0(x)$ estimated externally in a consistent way.**

  Theorem 3. *The result of Theorem 1 and 2 continues to hold if ρ is replaced by an external consistent estimator $\hat{\rho}_n(x)$ in (4) and (5).*

  E.g. use the consistent estimator for $\rho(x)$ from Goegebeur *et al.* (2013).

4. Simulation results

Estimators

- Non-robust local estimator

$$\hat{\gamma}_n^{(2)}(x,t,K,K) = \frac{1}{t+1}\frac{\sum_{i=1}^n K_{h_n}(x-X_i)(\ln Y_i - \ln u_n)_+^{t+1}\mathbf{1}\{Y_i > u_n\}}{\sum_{i=1}^n K_{h_n}(x-X_i)(\ln Y_i - \ln u_n)_+^{t}\mathbf{1}\{Y_i > u_n\}}$$

with $t = 0$.

Bias-corrected version

$$\hat{\gamma}_n^{(2)}(x,\beta) = \beta\hat{\gamma}_n^{(2)}(x,0,K,K) + (1-\beta)\hat{\gamma}_n^{(2)}(x,1,K,K)$$

with $\beta = -1$ and $\beta = 1/\hat{\rho}(x)$. See Goegebeur *et al.* (2013) for details

- Robust local MDPD estimators

  Estimator with $\delta = 0$ in G (not bias-corrected)

  Bias-corrected: MDPD with γ and δ estimated jointly

    $\rho(x)$ fixed at -1 and $\rho(x)$ estimated

All kernels are taken to be the bi-quadratic kernel function

$$K(x) = \frac{15}{16}(1 - x^2)^2 \mathbf{1}\{x \in [-1, 1]\}, \qquad x \in \mathbb{R}.$$

For practical implementation we have **two tuning parameters** that have to be determined, namely

- the bandwidth parameter h_n
- the threshold u_n

Tuning parameter selection methods:

- Oracle strategy: min MSE
- Heuristic, data driven method

4. Simulation results: uncontaminated case

We simulate $N = 100$ samples of size $n = 1\,000$, with $X \sim U(0,1)$ and $Y|X = x$ is generated from the following Burr distribution

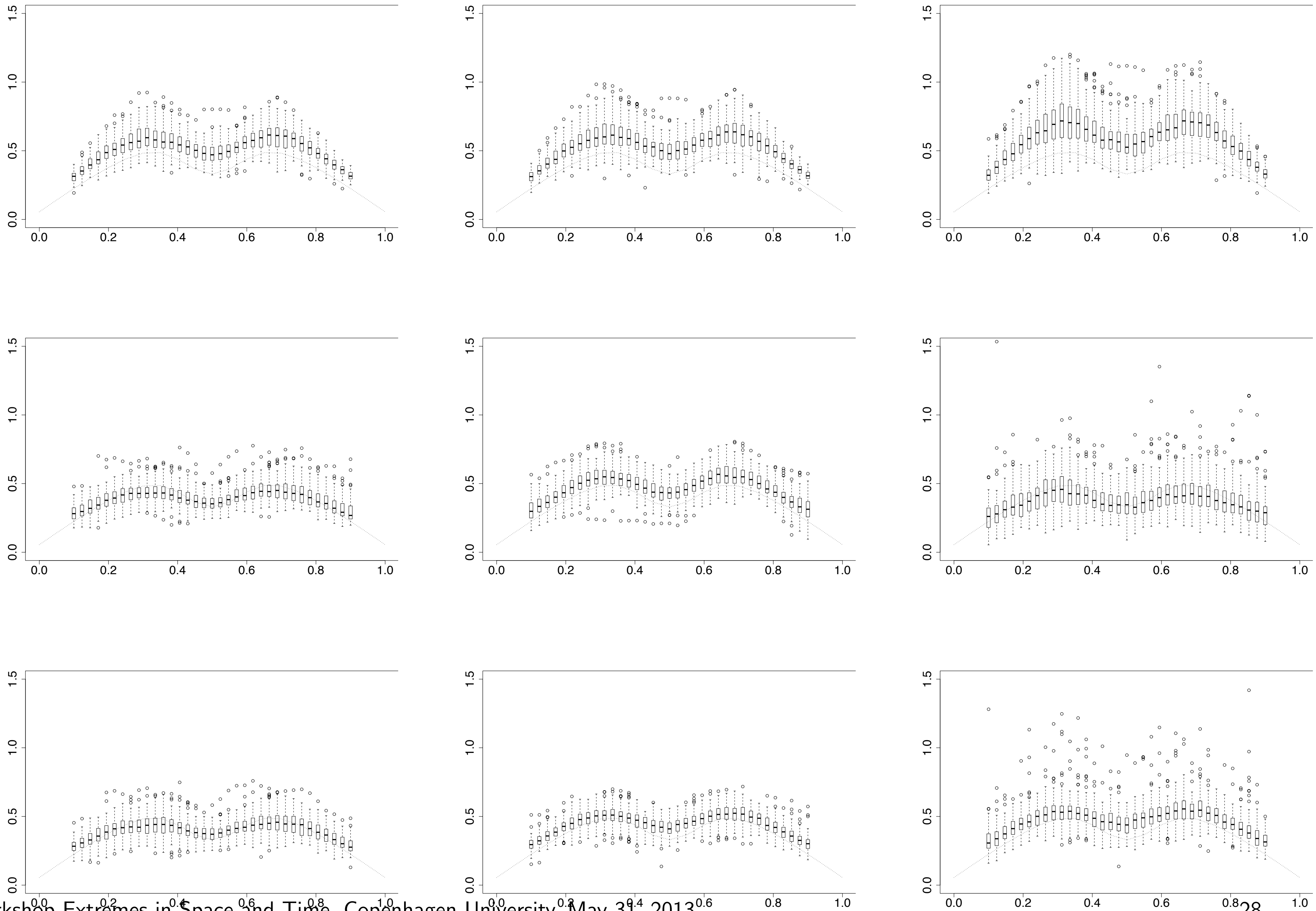
$$1 - F(y;x) = \left(1 + y^{-\rho(x)/\gamma(x)}\right)^{1/\rho(x)}, \qquad y > 0,$$

where

$$\gamma(x) = 0.5\,(0.1 + \sin(\pi x))\left(1.1 - 0.5\exp(-64(x-0.5)^2)\right) \text{ and } \rho(x) = -1.$$

MSE of the different estimators

Non Robust/Robust	Estimator	Oracle strategy	Data driven method
non robust	biased	0.006	0.019
non robust	bias-corrected $\rho(x) = -1$	0.003	0.006
non robust	bias-corrected $\rho(x) = \hat{\rho}(x)$	0.007	0.007
robust $\alpha = 0.1$	biased	0.006	0.025
robust $\alpha = 0.1$	bias-corrected $\rho(x) = -1$	0.007	0.011
robust $\alpha = 0.1$	bias-corrected $\rho(x) = \hat{\rho}(x)$	0.006	0.007
robust $\alpha = 0.5$	biased	0.008	0.055
robust $\alpha = 0.5$	bias-corrected $\rho(x) = -1$	0.007	0.017
robust $\alpha = 0.5$	bias-corrected $\rho(x) = \hat{\rho}(x)$	0.007	0.019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4. Simulation results: Contaminated case 1

Burr distribution

$$1 - F(y;x) = \left(1 + y^{-\rho(x)/\gamma(x)}\right)^{1/\rho(x)}, \qquad y > 0,$$

where

$$\gamma(x) = 0.5\,(0.1 + \sin(\pi x))\left(1.1 - 0.5\exp(-64(x-0.5)^2)\right) \text{ and } \rho(x) = -1.$$

Contaminated distribution

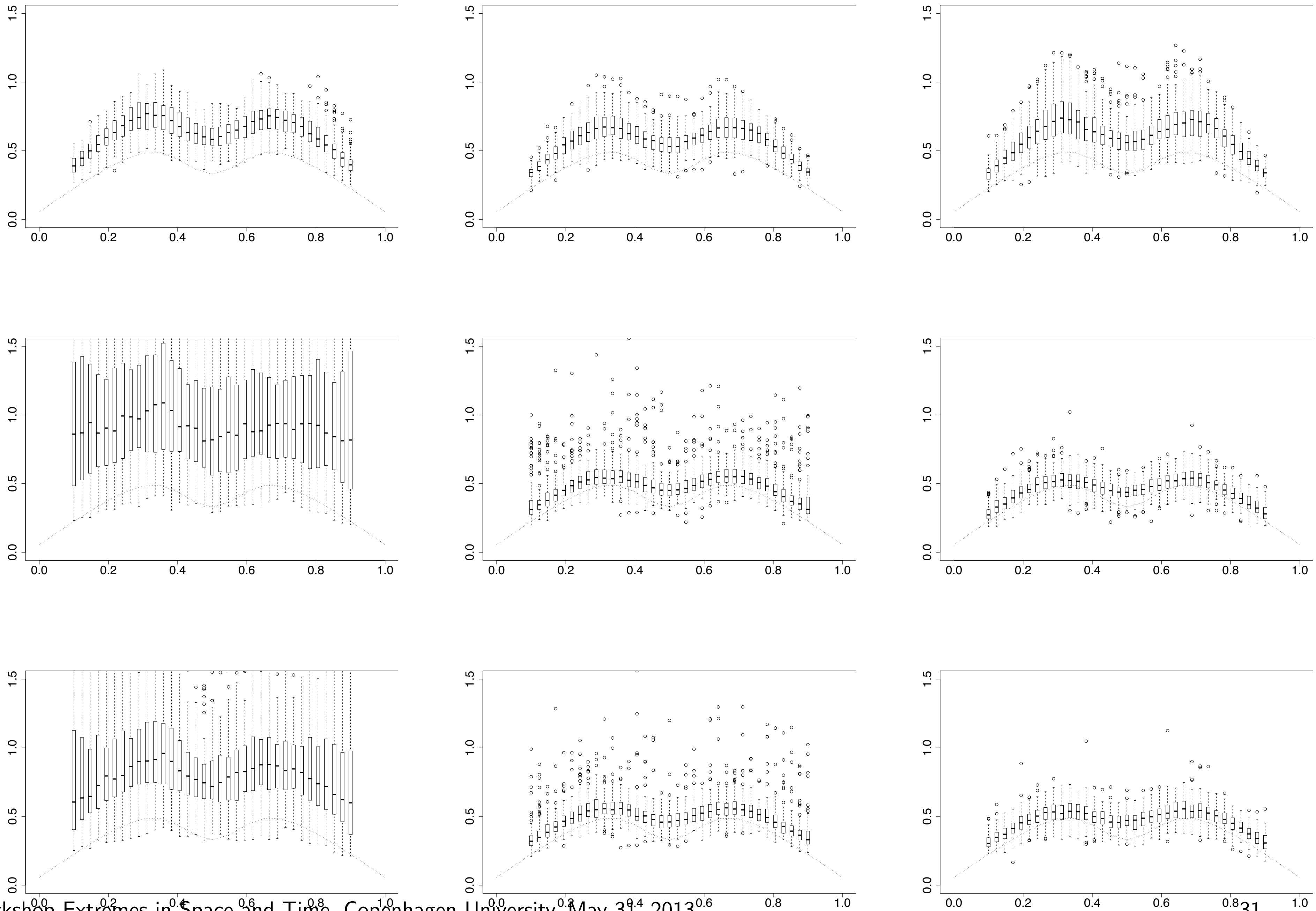
$$F_\epsilon(y;x) = (1-\epsilon)F(y;x) + \epsilon\tilde{F}(y;x)$$

where $\tilde{F}(y;x) = 1 - \left(\frac{y}{x_c}\right)^{-0.5}, y > x_c$,

We set $\epsilon = 0.01$, x_c= 1.2 times the 99.99% quantile of $F(y;x)$

MSE of the different estimators

Non Robust/Robust	Estimator	Oracle strategy	Data driven method
non robust	biased	0.053	0.069
non robust	bias-corrected $\rho(x) = -1$	0.291	0.977
non robust	bias-corrected $\rho(x) = \hat{\rho}(x)$	0.447	0.470
robust $\alpha = 0.1$	biased	0.020	0.039
robust $\alpha = 0.1$	bias-corrected $\rho(x) = -1$	0.011	0.025
robust $\alpha = 0.1$	bias-corrected $\rho(x) = \hat{\rho}(x)$	0.014	0.023
robust $\alpha = 0.5$	biased	0.012	0.060
robust $\alpha = 0.5$	bias-corrected $\rho(x) = -1$	0.007	0.009
robust $\alpha = 0.5$	bias-corrected $\rho(x) = \hat{\rho}(x)$	0.009	0.012



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