The Extremogram for Time Series: Theory and Examples

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Extremes and Time Series Modeling

Two strategies for thinking about modeling extremes in time series:

1. Fit a model to the entire data set (e.g., GARCH and SV for financial time series) and study the extreme value behavior associated with the fitted model as truth.
2. Construct and fit models only to the ***extremes*** (e.g., observations exceeding a large threshold).

Do fitted models actually capture the desired (*extremal)* characteristics of the data?

- How do we assess "fitted" (expected) with "observed"?
- Need a mechanism for measuring extremal dependence.

Goal of this talk: Describe the extremogram which may be useful as a tool for addressing this question. That is, how well does the "sample" extremogram match up with the "population" extremogram?

Characteristics of financial time series

Define $X_t = \ln(P_t) - \ln(P_{t-1})$ (log returns)

- heavy tailed

  $P(|X_1| > x) \sim RV(-\alpha), \quad 0 < \alpha < 4.$

- uncorrelated

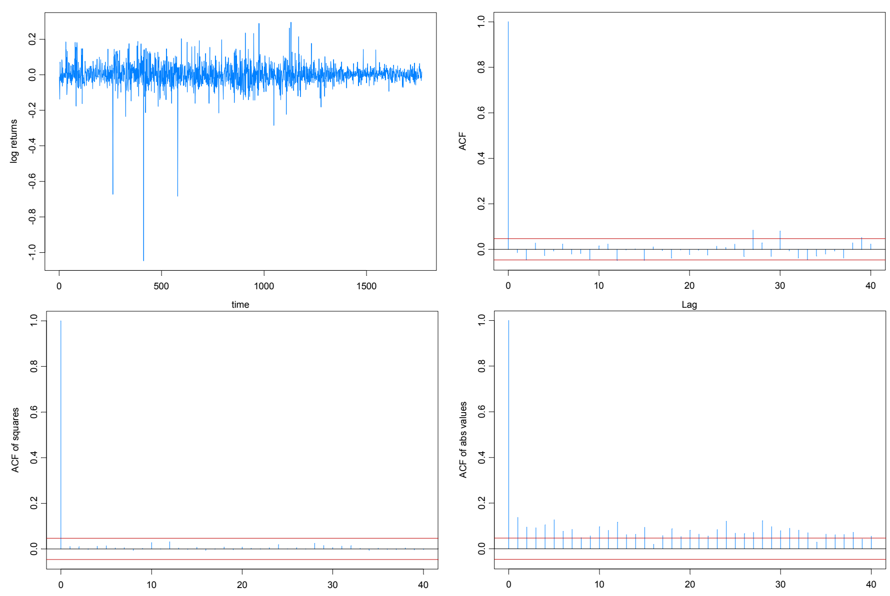
  $\hat{\rho}_X(h)$ near 0 for all lags h > 0

- $|X_t|$ and X_t^2 have slowly decaying autocorrelations

  $\hat{\rho}_{|X|}(h)$ and $\hat{\rho}_{X^2}(h)$ converge to 0 *slowly* as h increases.

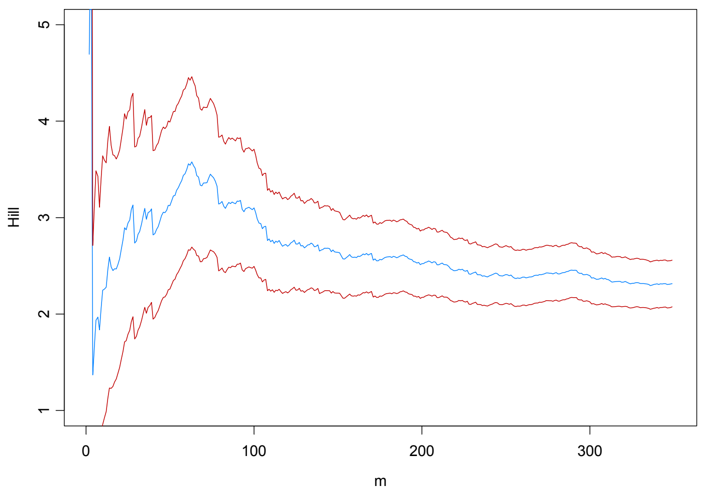
- process exhibits 'volatility clustering'.

Example: Amazon-returns (May 16, 1997 – June 16, 2004)



Example: Amazon-returns

Hill's estimate of alpha (Hill Horror plots-Resnick)



Starting point: GARCH vs SV

$X_t = \sigma_t Z_t$ (observation eqn in state-space formulation)

(i) GARCH(1,1)

$$X_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \ \{Z_t\} \sim \text{IID}(0,1)$$

(ii) Stochastic Volatility

$$X_t = \sigma_t Z_t, \quad \log \sigma_t^2 = \phi_0 + \phi_1 \log \sigma_{t-1}^2 + \varepsilon_t, \ \{\varepsilon_t\} \sim \text{IID N}(0,\sigma^2)$$

Key question:

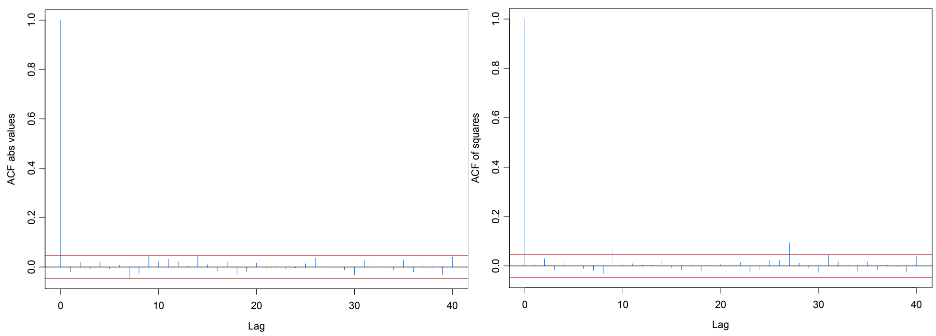
What intrinsic (extremal?) features in the data (*if any*) can be used to discriminate between these two models?

Amazon returns (GARCH model)

GARCH(1,1) model fit to Amazon returns:

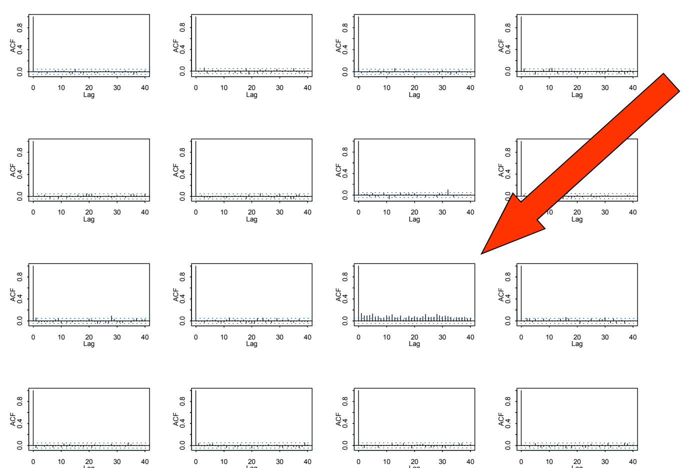
α_0= .00002493, α_1= .0385, β_1 = .957, $X_t=(\alpha_0+\alpha_1 X^2_{t-1}+\beta_1 \sigma^2_{t-1})^{1/2}Z_t$, $\{Z_t\}$~IID t(3.672)

Simulation from fitted GARCH(1,1) model



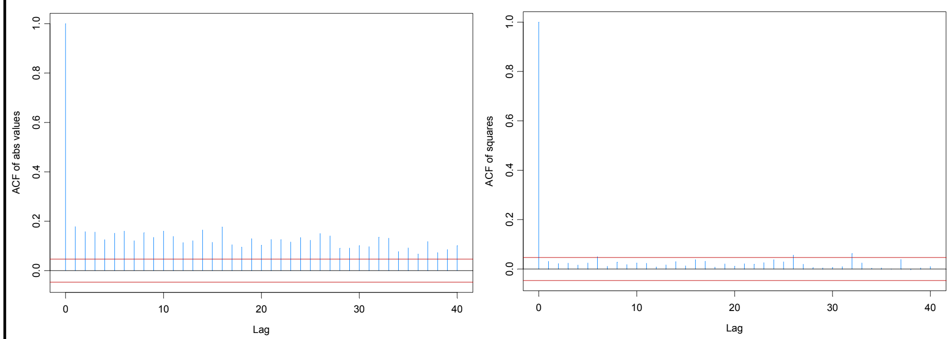
ACF Plots for Amazon

ACF of the absolute values from 15 simulated realizations from the GARCH model on previous slide.



Amazon returns (SV model)

Stochastic volatility model fit to Amazon returns: simulation based on fitted model.



Game Plan

- Extremes and time series modeling
  - A motivating example
  - Starting point: GARCH vs SV
- The Extremogram
  - Examples
  - Sufficient conditions for existence: regular variation
  - Empirical extremogram
  - Illustrations (permutation procedures)
  - Cross-extremogram (devolatilizing/deGARCHing)
- Connections with Return Times of Rare Events
- Bootstrapping the Extremogram
  - Theory & examples

The Extremogram

The extremogram of a stationary time series $\{X_t\}$ can be viewed as the analogue of the correlogram in time series for measuring dependence in extremes (see Davis and Mikosch (2009)).

Definition: For two sets A & B ***bounded away from 0***, the ***extremogram*** is defined as

$$\rho_{A,B}(h) = \lim_{x\to\infty} P(\mathbf{X}_h \in xB \mid \mathbf{X}_0 \in xA)$$

$$= \lim_{x\to\infty} P(\mathbf{X}_0 \in xA, \mathbf{X}_h \in xB)/P(\mathbf{X}_0 \in xA),$$

for h = 0, 1, …, provided the limit exists, where $\mathbf{X}_h=(X_h,X_{h+1},\ldots,X_{h+k})$.

Remark: This definition requires that the limit exists.

a) exists for heavy-tailed time series (see forthcoming slide)
b) exists for some light-tailed time series w/ special choices of A and B.
c) extremal dependence ***depends*** on the choice of sets A & B.

The Extremogram (cont)

If one takes $A=B=(1,\infty)$ and k = 0, then

$$\rho_{A,B}(h) = \lim_{x\to\infty} P(X_h > x \mid X_0 > x) = \lambda(X_0,X_h)$$

often called the ***extremal dependence coefficient*** ($\lambda = 0$ means independence or asymptotic independence).

Other choices of A and B can lead to interesting extremograms:

- $P(X_1 < -x \mid X_0 < -x)$ (negative return followed by a neg return)
- $P(X_1 > x \mid X_0 < -x)$ (neg return followed by a pos return)
- $P(X_1 + \cdots + X_4 > 2x \mid X_0 < -x)$ (neg return followed by a big pos return aggregated over next 4 days)
- $P(X_1 > x, \ldots, X_4 > x \mid X_0 > x)$ (pos return followed by a pos return in next 4 days)
- $P(\min\{X_2, X_3, X_4\} > x \mid X_0 > x, X_1 > x)$ (2 pos returns⇒pos return)

The Extremogram: examples

Let A = B = $(1,\infty)$, then

$$\rho_{A,B}(h) = \lim_{x\to\infty} P(X_0 > x, X_h > x)/P(X_0 > x)$$

Gaussian Processes: In this case,

$$\rho_{A,B}(h) = 0 \text{ for all } h > 0 \quad \text{(asymptotic independence).}$$

GARCH: In this case

$$\rho_{A,B}(h) > 0 \text{ for all } h > 0,$$

but decays to 0 geometrically fast.

SV process: $X_t = \sigma_t Z_t, \quad \log\sigma_t^2 = \mu + \sum_{j=0}^{\infty}\psi_j \varepsilon_{t-j}, \ \{\varepsilon_t\} \sim \mathrm{IIDN}(0,\sigma^2)$

In this case,

$$\rho_{A,B}(h) = 0 \text{ for all } h > 0.$$

The Extremogram: examples

Let A = B = $(1,\infty)$, then

$$\rho_{A,B}(h) = \lim_{x\to\infty} P(X_0 > x, X_h > x)/P(X_0 > x)$$

AR(1): $X_t = \phi X_{t-1} + Z_t$, $\{Z_t\}$~IID Cauchy. Then

$$\rho_{A,B}(h) = \max(0, \phi^h).$$

Note if $\phi < 0$, then extremogram alternates between positive #'s and 0

MaxMA(2): Let (Z_t) be iid with Pareto distribution, i.e., $P(Z_1 > x) = x^{-\alpha}$ for $x \geq 1$, and set $X_t = \max(Z_t, Z_{t-1}, Z_{t-2})$. Then

$$\rho_{A,B}(h) = 1 \quad \text{for } h = 0.$$
$$= 2/3 \quad \text{for } h = 1$$
$$= 1/3 \quad \text{for } h = 2$$
$$= 0, \quad \text{for } h > 2.$$

Regular Variation and the Extremogram

Facts

1. The extremogram of a RV stationary time series $\{X_t\}$ exists for all choices of sets A & B bounded away from the origin.
2. Many heavy-tailed time series (GARCH and SV) are regularly varying.

The Empirical Extremogram

A natural estimator of the extremogram,

$$\rho_{A,B}(h) = \lim_{x\to\infty} P(X_h \in xB \mid X_0 \in xA)$$

based on observations, $X_1, \ldots, X_n$, is the empirical extremogram defined by

$$\hat{\rho}_{A,B}(h) = \frac{\frac{m}{n}\sum_{t=1}^{n-h} I_{\{a_m^{-1}X_t \in A, a_m^{-1}X_{t+h} \in B\}}}{\frac{m}{n}\sum_{t=1}^{n} I_{\{a_m^{-1}X_t \in A\}}},$$

where a_m is the $1 - m/n$ quantile of $|X_t|$. For the theory to work, need

$$m_n \to \infty \text{ with } m/n \to 0.$$

Under suitable mixing conditions, a CLT for the empirical estimate is established in D&M (2009).

The Empirical Extremogram — central limit theorem

$$\hat{\rho}_{A,B}(h) = \frac{\frac{m}{n}\sum_{t=1}^{n-h} I_{\{a_m^{-1}X_t \in A, a_m^{-1}X_{t+h} \in B\}}}{\frac{m}{n}\sum_{t=1}^{n} I_{\{a_m^{-1}X_t \in A\}}}$$

After first establishing a joint CLT for the numerator and denominator, we obtain the limit result

$$(n/m)^{1/2}(\hat{\rho}_{A,B}(h) - \rho_m(h)) \rightarrow_d N(0, \sigma^2(A,B)),$$

where $\rho_m(h)$ is the ratio of expectations (***pre-asymptotic extremogram***),

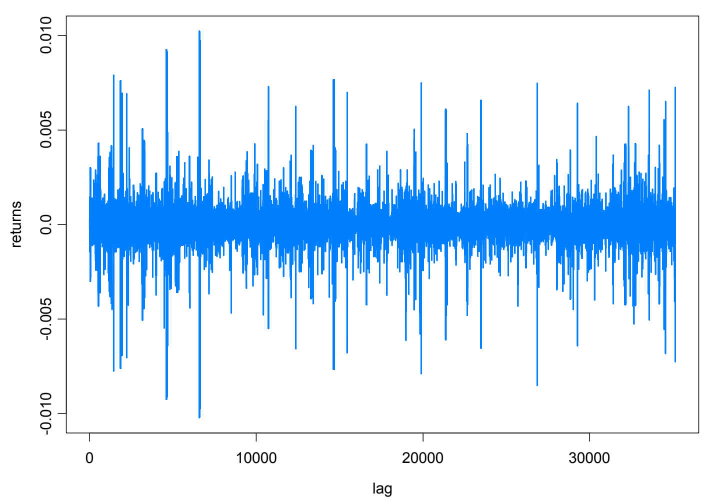
$$P(a_m^{-1}X_0 \in A,\ a_m^{-1}X_h \in B) / P(a_m^{-1}X_0 \in A).$$

Now provided a bias condition, such as

$$(n/m)^{1/2}\left(mP(a_m^{-1}X_0 \in A,\ a_m^{-1}X_h \in B) - \mu_h(A \times B)\right) \rightarrow 0,$$

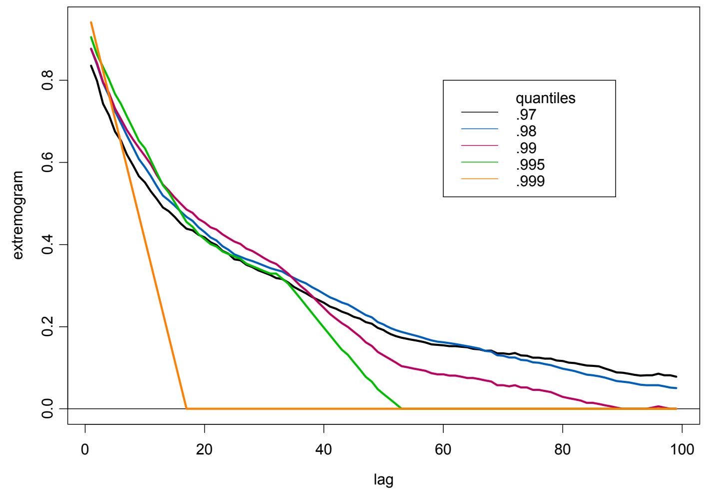
holds, then $\rho_m(h)$ can be replaced with $\rho_{A,B}(h)$. This condition can often be difficult to check.

Application to Five-Minute Return Data (US/DM) exchange



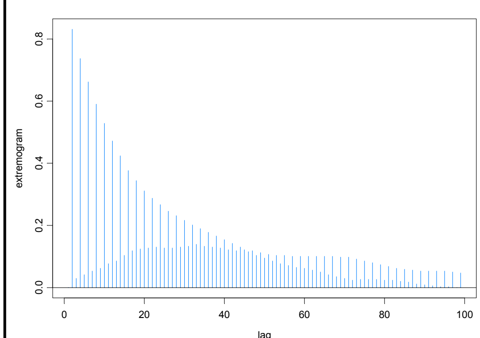
Application to Five-Minute Return Data (US/DM) exchange

Extremogram absolute values: choice of threshold a_m



Application to Five-Minute Return Data (US/DM) exchange

Extremogram A=B=$(1,\infty)$



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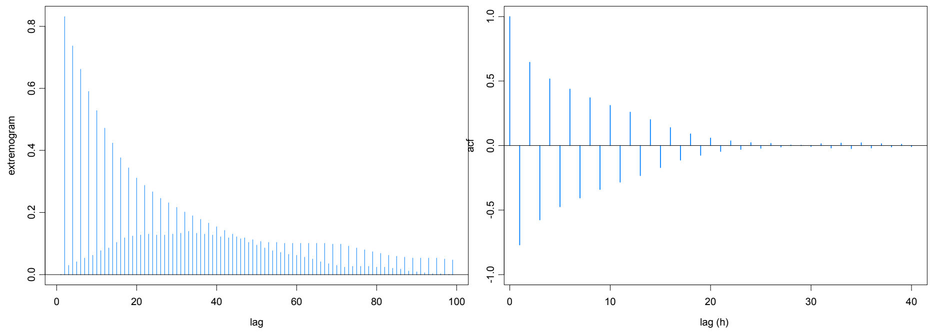
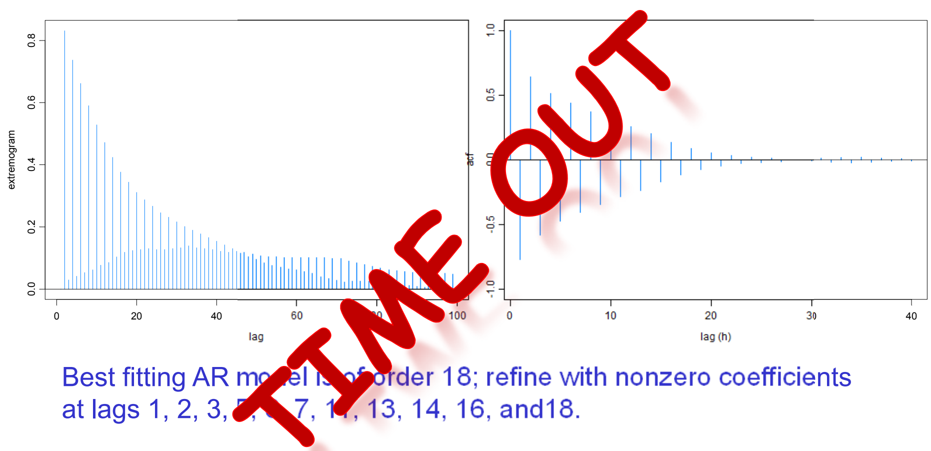


Best fitting AR model is of order 18; refine with nonzero coefficients at lags 1, 2, 3, 5, 6, 7, 11, 13, 14, 16, and18.

Application to Five-Minute Return Data (US/DM) exchange

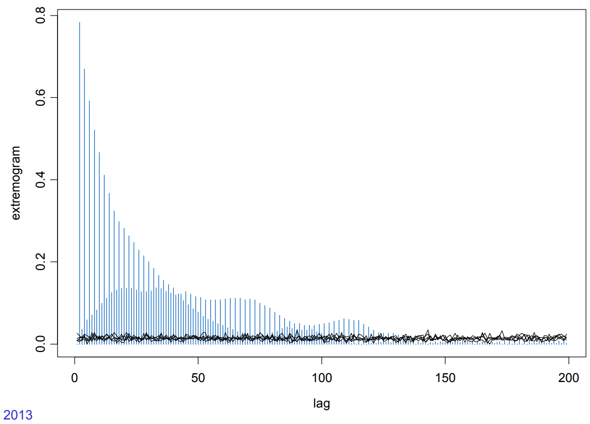
Extremogram A=B=(1,∞)



Best fitting AR model is of order 18; refine with nonzero coefficients at lags 1, 2, 3, 5, 6, 7, 11, 13, 14, 16, and18.

Time out: Resampling and Testing for Serial Dependence

A natural way (*not often used in time series*) for testing serial correlation is to compute the ACF for random permutations of the data. If the sample ACF appears more ***extreme*** than the ACFs based on random permutations, then there is evidence of serial correlation. We apply the same idea to the extremogram.



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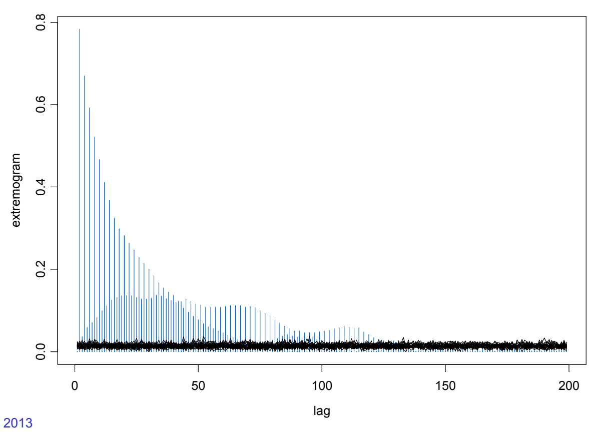

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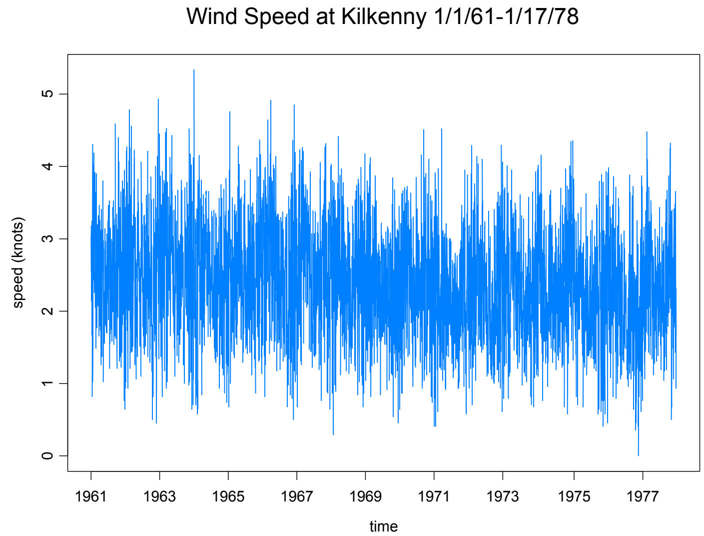
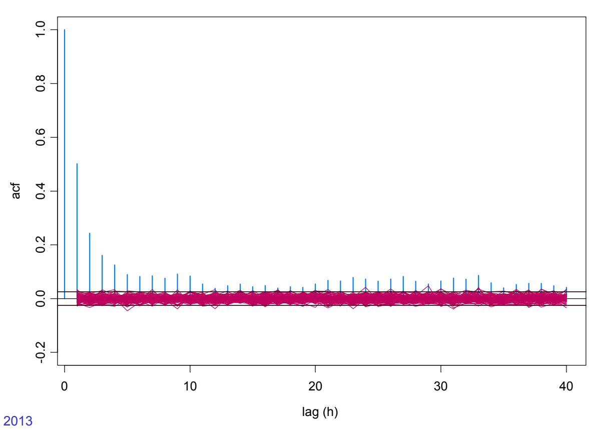
Time out: Illustration with ACF (Windspeed at Kilkenny)



Time out: Illustration with ACF

In plotting the sample ACF, one normally includes the ±1.96/sqrt(n) bounds (95% CI under the assumption of iid noise). One could use the permutation idea here as well.



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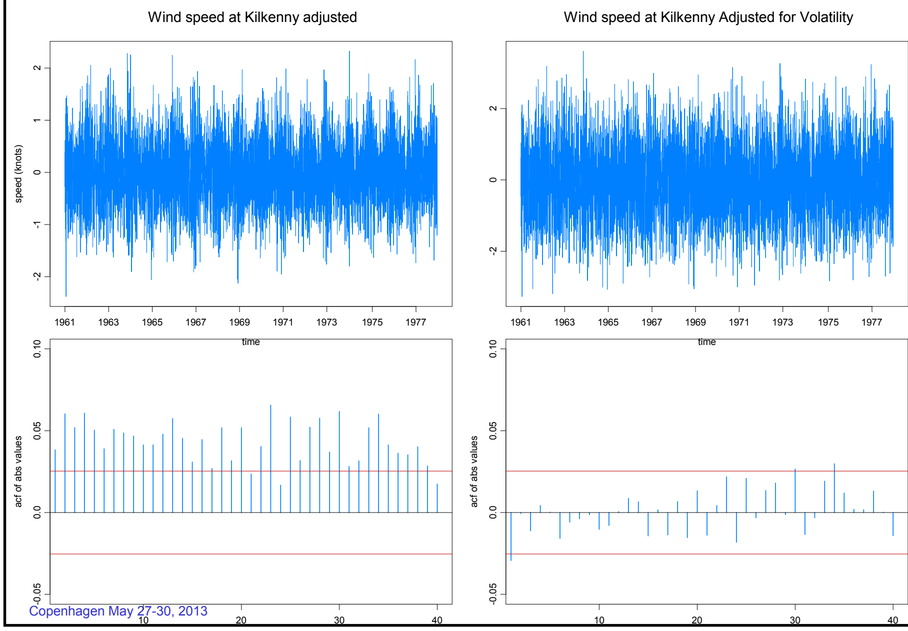

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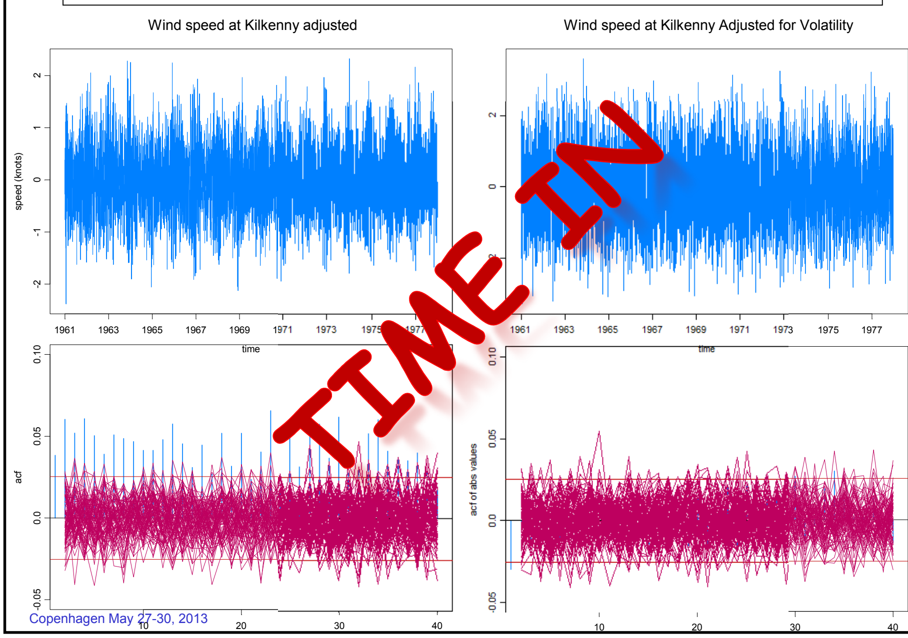
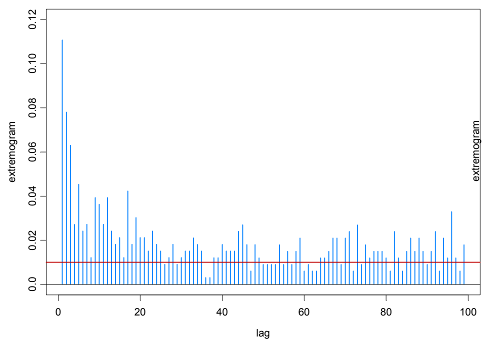


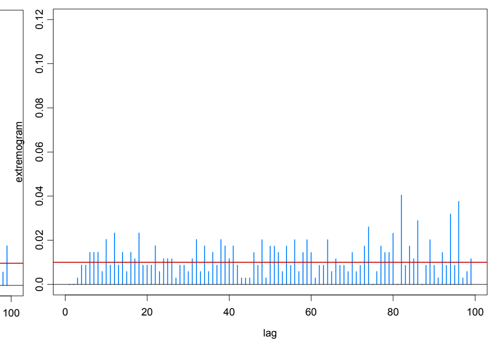
Application to Five-Minute Return Data (US/DM) exchange

Extremogram for residuals from subset AR(18) and from GARCH
A=B=$(1,\infty)$

Residuals from AR



Residuals from GARCH



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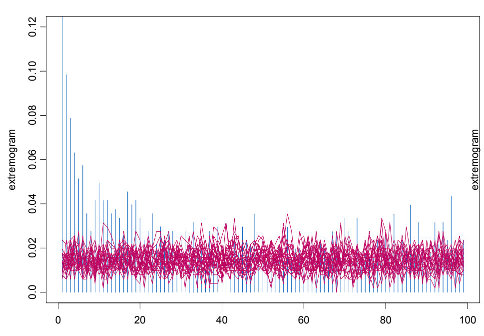
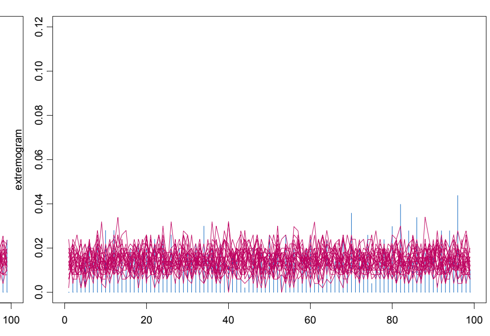


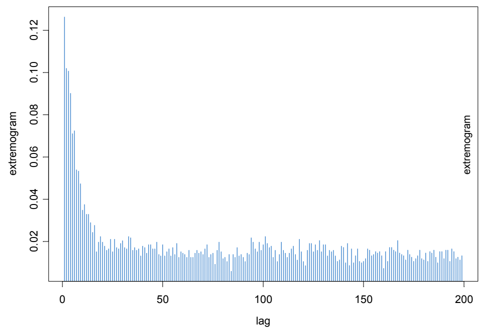
Residuals from GARCH



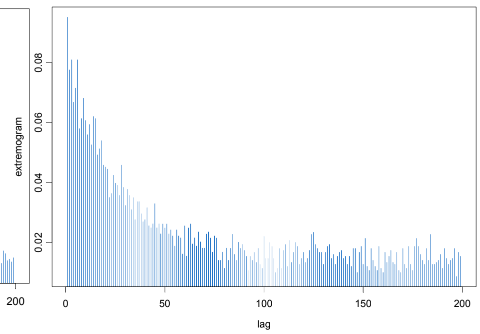
Extremogram of a SV Process

SV Process: $X_t = \sigma_t Z_t$, $\{Z_t\} \sim$ IID t_4 ; $\log \sigma_t = .9 \log \sigma_{t-1} + \varepsilon_t$

GARCH(1,1): $X_t = (.1 + .14 X^2_{t-1} + .83\sigma^2_{t-1})^{1/2} Z_t$, $\{Z_t\} \sim$ IID N(0,1),



SV



GARCH

Threshold = .97 quantile

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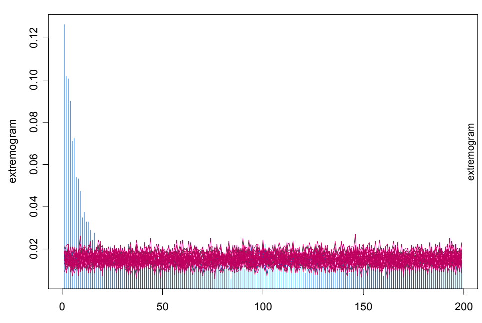
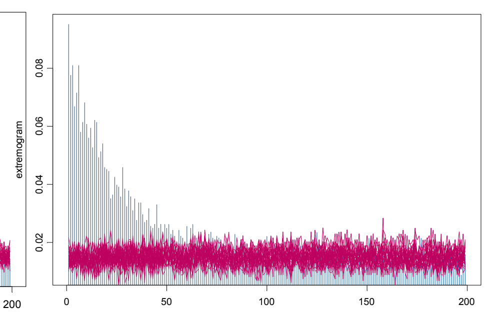


SV



GARCH

Threshold = .97 quantile

Extremogram of a Max-MA(2)

Example: Let (Z_t) be iid with Pareto distribution, i.e., $P(Z_1 > x) = x^{-\alpha}$ for $x \geq 1$, and set $X_t = \max(Z_t, Z_{t-1}, Z_{t-2})$. Then

$$nP(X_1 > xn^{1/\alpha}) \to 3x^{-\alpha} \text{ and } F^n(xn^{1/\alpha}) \to \exp(-3x^{-\alpha}).$$

On the other hand,

$$P(n^{-1/\alpha} M_n \leq x) = P(n^{-1/\alpha} \max(Z_{-1}, \ldots, Z_n) \leq x) \to \exp(-x^{-\alpha}) = \exp(-1/3\; 3x^{-\alpha}),$$

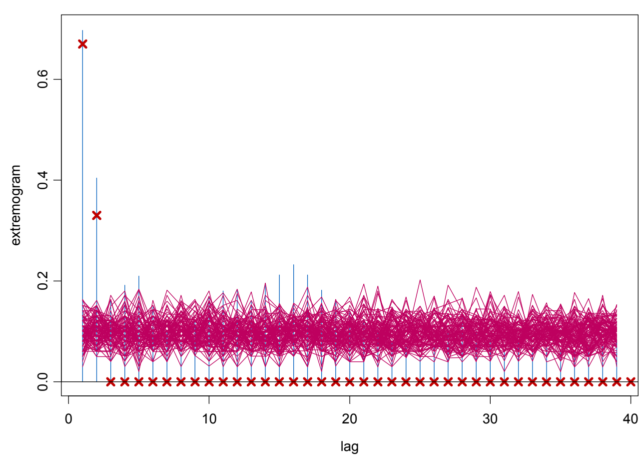
which implies that the extremal index is $\theta = 1/3$.

The extremogram with $A = B = (1, \infty)$ is

$$\lim_n P(X_h > n^{1/\alpha} \mid X_0 > n^{1/\alpha}) \begin{array}{ll} = 1 & \text{for } h = 0. \\ = 2/3 & \text{for } h = 1 \\ = 1/3 & \text{for } h = 2 \\ = 0, & \text{for } h > 2. \end{array}$$

Extremogram of a Max-MA(2)

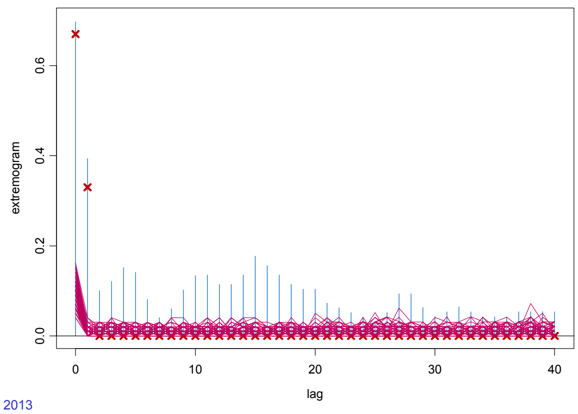
Extremogram: $\lim_n P(X_h > n^{1/\alpha} \mid X_0 > n^{1/\alpha}) = 2/3, 1/3, 0$ for $h = 1$, $h=2$, and for $h > 3$, respectively. Blue = sample



Extremogram of a Max-MA(2)

Extremogram: $\lim_n P(\min(X_h,X_{h+1}) > n^{1/\alpha} \mid X_0 > n^{1/\alpha}) = 2/3, 1/3, 0$ for $h = 0$, $h=1$, and for $h > 2$, respectively.

Note: Confidence intervals are narrow—how come?



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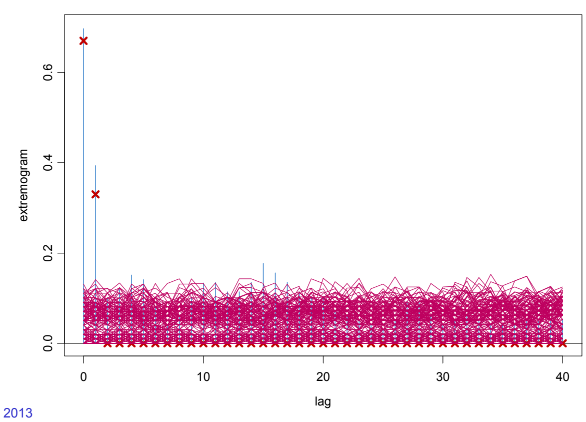

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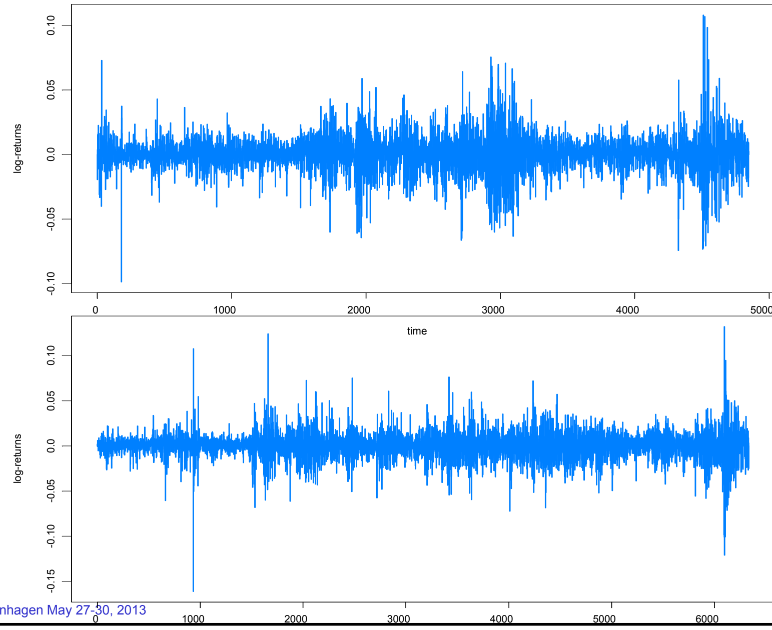
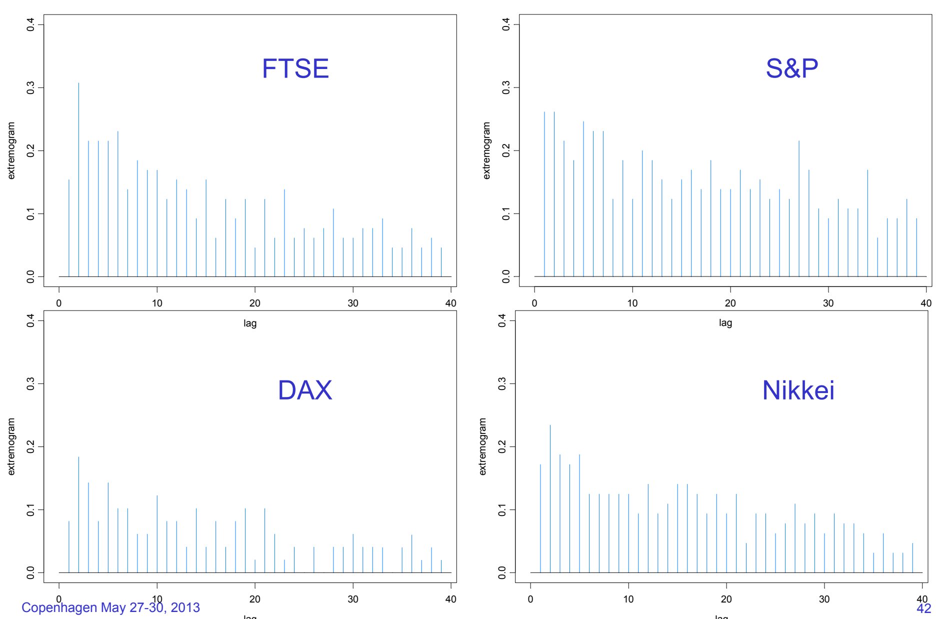
Log-returns for DAX and Nikkei (Apr 4, `84-Oct 2, `09



Extremogram for FTSE, S&P, DAX, Nikkei



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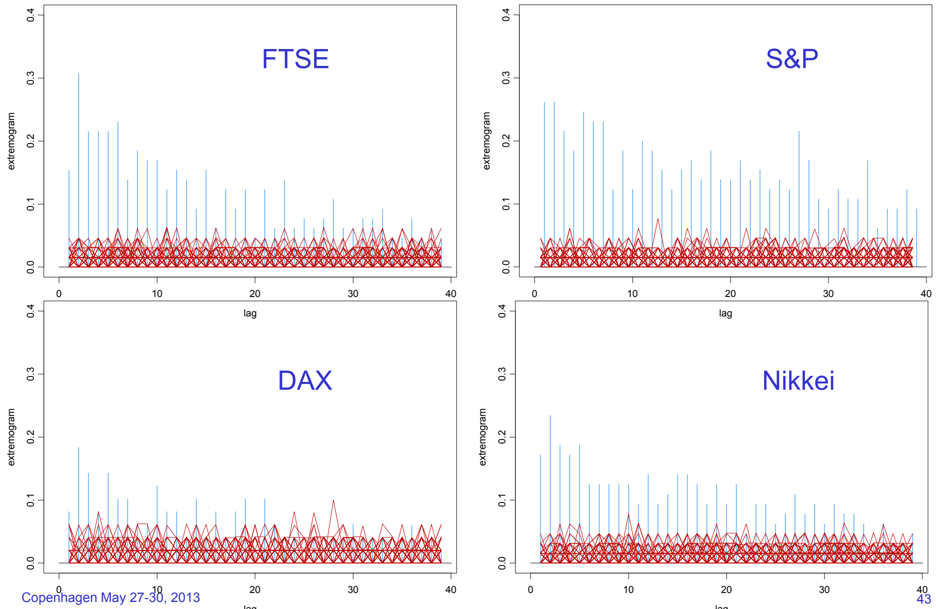


Cross-Extremogram

The cross-extremogram measures extremal dependence between two or more series. Suppose we have two time series $\{X_t\}$ and $\{Y_t\}$

Definition: For two sets A & B ***bounded away from 0***, the ***cross-extremogram*** is defined as

$$\rho_{A,B}(h) = \lim_{x\to\infty} P(Y_h \epsilon\, xB \mid X_0 \epsilon\, xA)$$

For example, if X_t and Y_t represent log-returns of two stocks, then one might be interested in extremal dependence of negative returns. It may seem natural to take $A = B = (-\infty, -1]$, so that

$$\rho_{A,B}(h) = \lim_{x\to\infty} P(Y_h < -x \mid X_0 < -x).$$

Cross-Extremogram

As before, we estimate

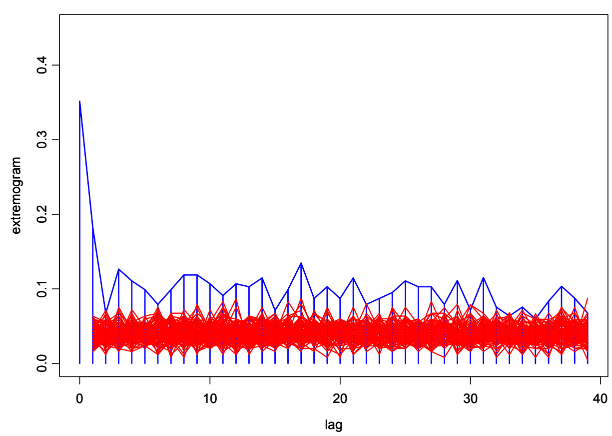
$$\rho_{A,B}(h) = \lim_{x\to\infty} P(Y_h \epsilon\, xB \mid X_0 \epsilon\, xA)$$

by

$$\hat{\rho}_{A,B}(h) = \frac{\frac{m}{n}\sum_{t=1}^{n-h} I_{\{a_{m,1}^{-1}X_t \in A,\, a_{m,2}^{-1}Y_{t+h} \in B\}}}{\frac{m}{n}\sum_{t=1}^{n} I_{\{a_{m,1}^{-1}X_t \in A\}}}$$

Problem: For log-returns, heteroskedasticity can produce ***spurious*** extremograms. That is, volatility in both series (which tend to happen in unison) produce large extremograms.

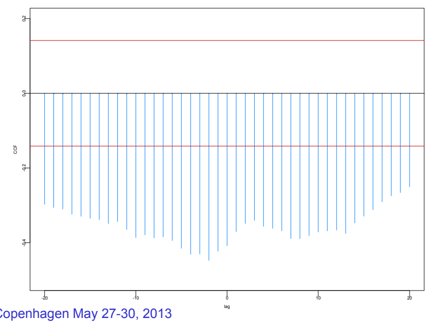
Cross-Extremogram FTSE and SP



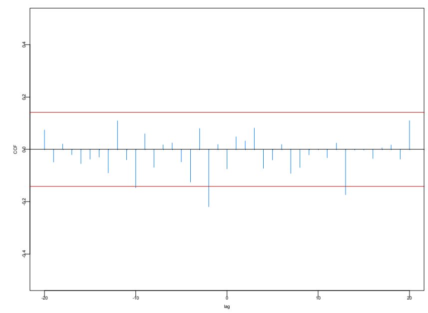
Cross-Extremogram

Strategy: Devolatilize the component series before computing the extremogram. This is ***analogous*** to the issue of spurious cross-correlations in a time series without whitening the series first.

Cross-correlation between two "independent" AR(1)'s



Cross-correlation between the ***whitened*** series'

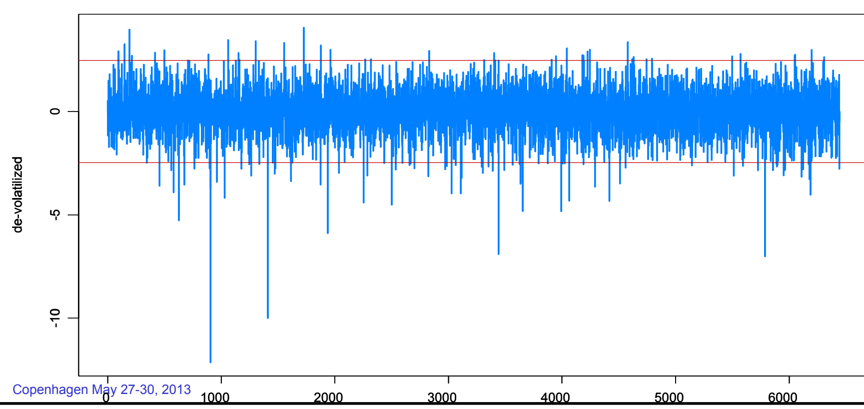


Devolatilizing (deGARCHing) S&P

Extremogram for S&P: significant for large number of lags ~40+

Devolatilize S&P by fitting GARCH(1,1):

$$X_t = (6.28e-7 + .057X_{t-1}^2 + .939\sigma_{t-1}^2)^{1/2}Z_t, \quad \{Z_t\} \sim IID\ t(6.14),$$



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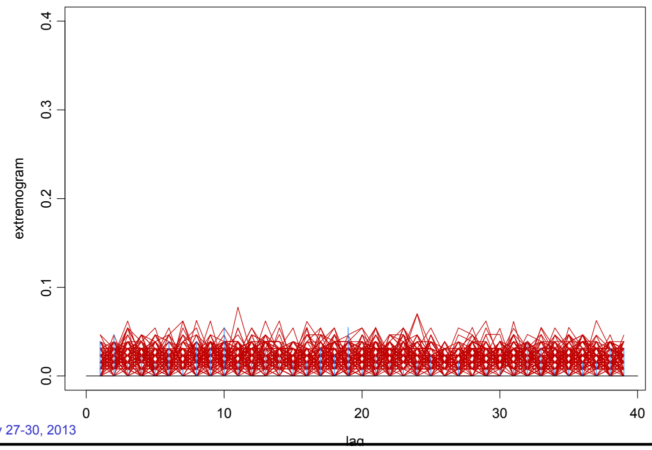
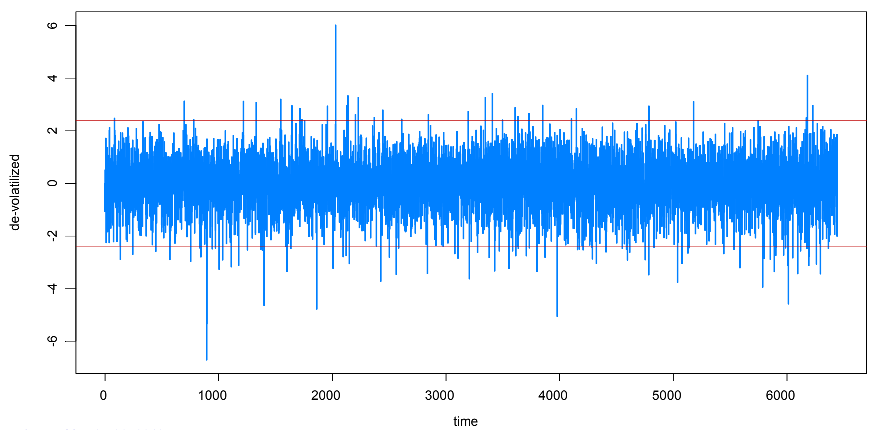


Devolatilizing (deGARCHing) FTSE

Extremogram for FTSE: significant for large number of lags ~40+

Devolatilize FTSE by fitting GARCH(1,1):

$$X_t=(1.32e-6+.084\, X_{t-1}^2 + .904\sigma_{t-1}^2)^{1/2} Z_t, \quad \{Z_t\} \sim IID\ t(13),$$



Devolatilizing FTSE

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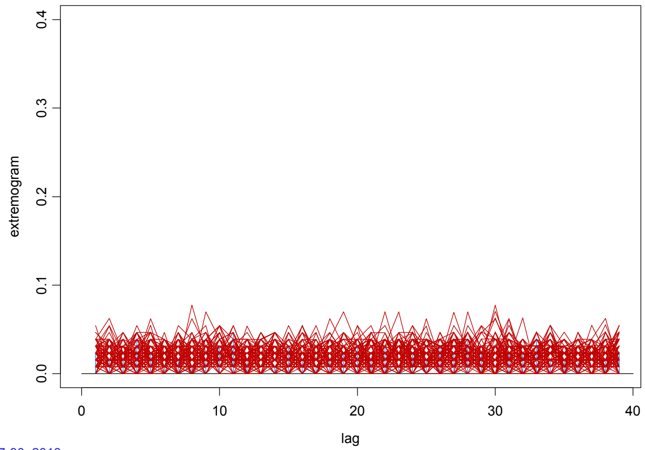
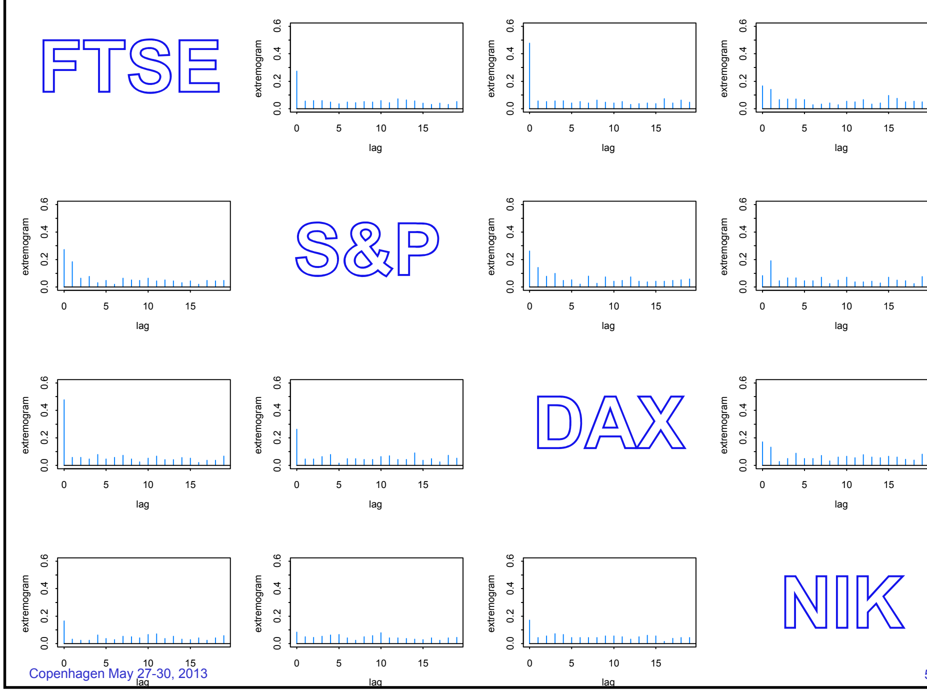
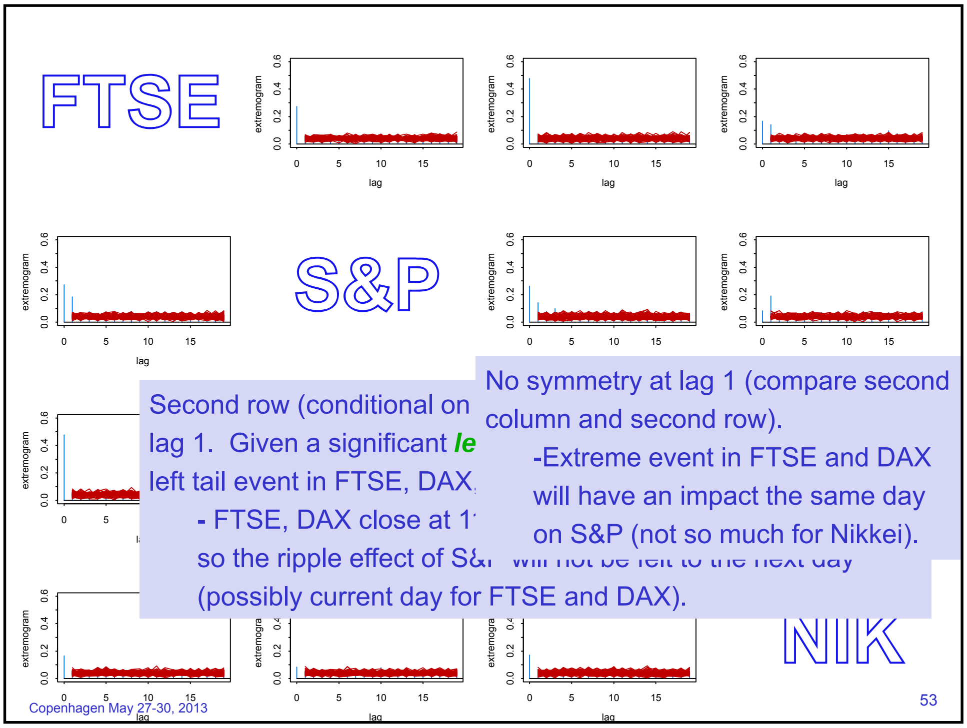






Cross-Extremogram for 3 Time Series

We extend the cross-extremogram to 3 time series.

Definition: For three sets A, B & C ***bounded away from 0***, the ***cross-extremogram*** is defined as

$$\rho_{A,B,C}(h) = \lim_{x\to\infty} P(Z_h \in xC, Y_h \in xB \mid X_0 \in xA)$$

We estimate $\rho_{A,B,C}(h)$ as before the empirical cross-extremogram.

To illustrate, we will look at 5 min log-returns , Dec 1, `04-July 26, `06.

X_t = 5 minute log-returns Bank of America

Y_t = 5 minute log-returns Citibank

Z_t = 5 minute log-returns Microsoft

Two cases:

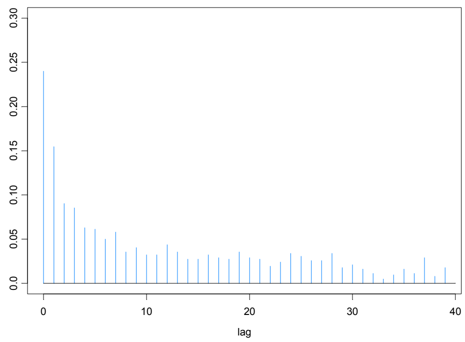
(i) $\rho_{A,B,C}(h) = \lim_{x\to\infty} P(|Z_h| > x \text{ or } |Y_h| > x \mid |X_0| > x)$

(ii) $\rho_{A,B,C}(h) = \lim_{x\to\infty} P(|Z_h| > x \mid |Y_0| > x \text{ or } |X_0| > x)$

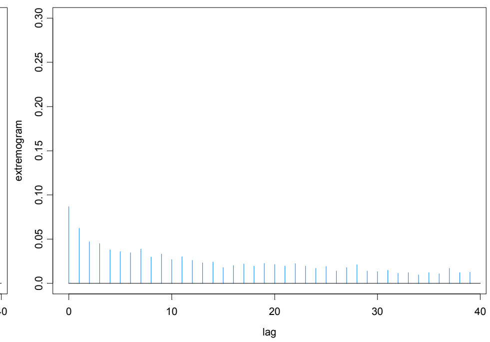
Cross-Extremogram for 3 Time Series

X_t = BAC, Y_t = Citibank, Z_t = MSFT

(i) $P(|Z_h| > x \; or \; |Y_h| > x \mid |X_0| > x)$



(ii) $P(|Z_h| > x \mid |Y_0| > or \; |X_0|| > x)$



Cross-Extremogram for 3 Time Series

X_t = BAC, Y_t = Citibank, Z_t = MSFT

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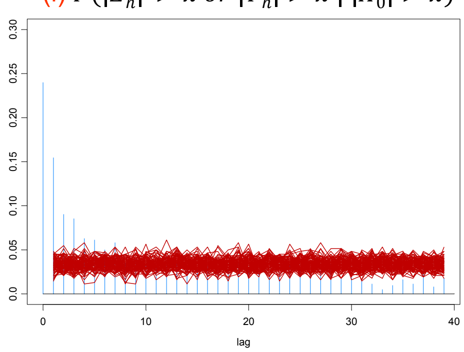
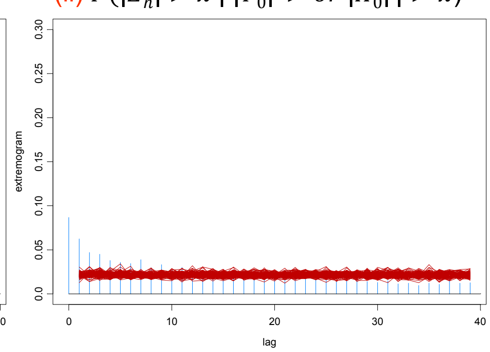


(ii) $P(|Z_h| > x \mid |Y_0| > or \; |X_0|| > x)$



(i) Given BAC is large ⇒ CB or MSFT is large in same time period or 5 minutes hence.

(ii) Given BAC or CB large ⇒ MSFT is large in one time period.

Connections with Return Times (of rare events)

This is an idea due to ***Geman and Chang (2009)***:

Setup:

- $\{X_t\}$ time series—think log-returns, for example.
- ξ_v, ξ_{1-v} are the vth and (1-v)th quantile of the of the marginal distribution.

Define the exceedance (or stopping times) times τ_j by

$$\tau_1 = \min\{ t \geq 1: X_t < \xi_v \text{ or } X_t < \xi_{1-v}\}$$

$$\tau_{j+1} = \min\{ t \geq \tau_j: X_t < \xi_v \text{ or } X_t < \xi_{1-v}\}, \; j \geq 0.$$

The inter-arrival (or return times) are

$$T_j = \tau_j - \tau_{j-1} \quad j \geq 1.$$

These are the times between occurrences of rare events (number of tosses of a coin until next head).

Connections with Return Times (of rare events)

For ***nice*** time series, like iid observations, the T_j's are iid with a geometric distribution,

$$P(T_j = k) = (1-p)^{k-1}p, \; k=1,2, \ldots,$$

$$p = P(X_t < \xi_v \text{ or } X_t > \xi_{1-v}) = 2v.$$

Recall for a geometric rv,

$$E(T_1) = 1/p.$$

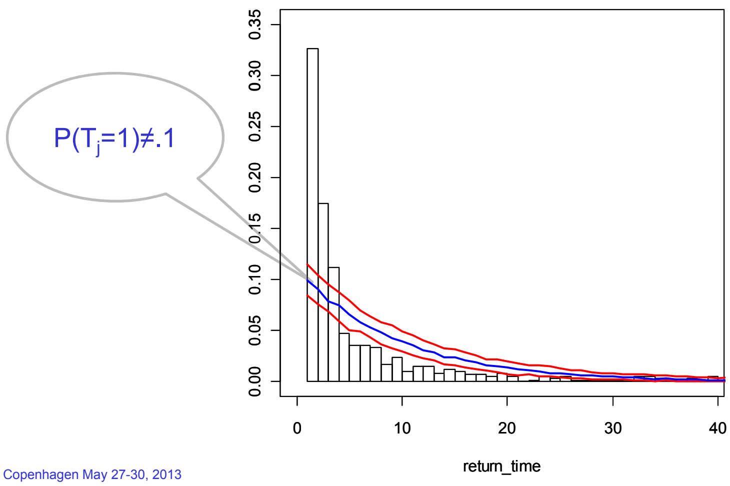
Note: This is the ***backstory*** behind the term 100 year flood, or 100 year *blank*, which corresponds to the threshold x such that the expected time until x is exceeded is 100. (In this case, p = .01, x= $\xi_{.99}$.)

Idea: For v fixed (can do one-sided tail), look at the histogram of return times and compare against a geometric distribution.

Connections with Return Times (of rare events)

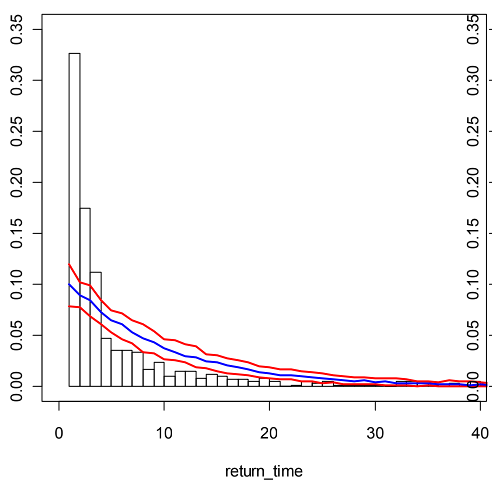
Idea: For v fixed (can do one sided tail), look at the histogram of return times and compare against a geometric distribution.

Example with BAC, v=.05 ⇒ geometric(p=.1)

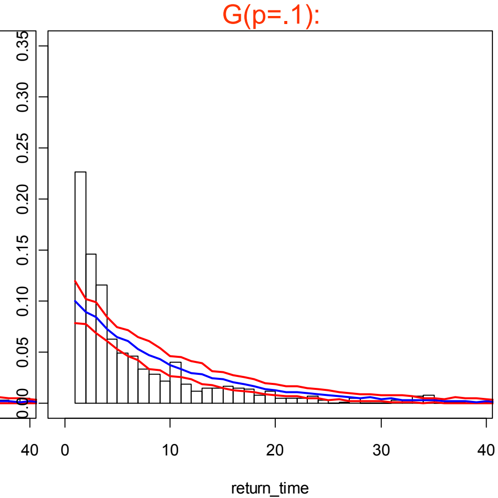


Connections with Return Times (Daily Returns for BAC)

BAC, 2 tail, v=.05 ⇒ G(p=.1)



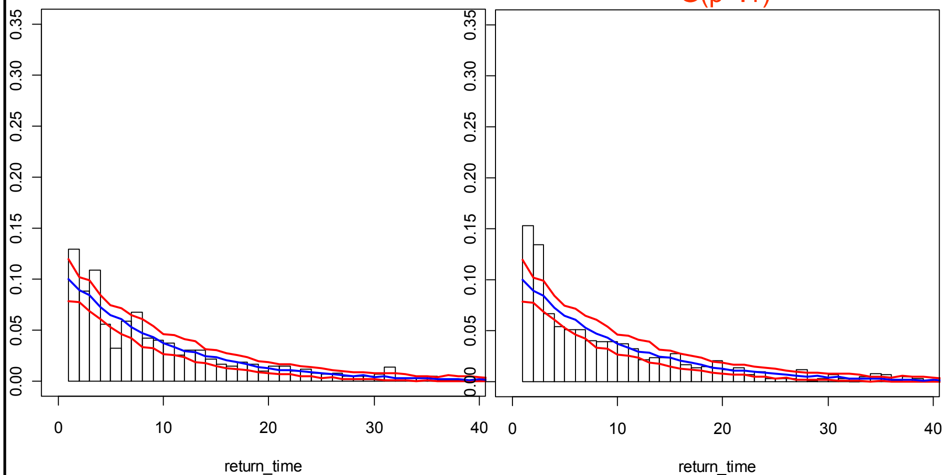
BAC, lower only v=.01 ⇒ G(p=.1):



Connections with Return Times

BAC devolatilized
v=.05 ⇒ G(p=.1)

BAC devolatilized
lower tail only v=.01 ⇒
G(p=.1)



Connections with Return Times (of rare events)

Question: What is the connection with the extremogram?

Answer: The estimated distribution for the return times is exactly the extremogram for specially chosen sets A & B. For example, in the upper tail case, $P(T_1 = 1)$ is estimated by

$$\hat{P}(T=1) = \frac{\sum_{t=1}^{n-1} I_{\{X_t \ge a_m, X_{t+1} \ge a_m\}}}{\sum_{t=1}^{n} I_{\{X_t \ge a_m\}}} = \frac{\#\,\text{consecutive pairs} > \mathrm{a_m}}{\#\,\text{observations} > \mathrm{a_m}}$$

$$\hat{\rho}_{A,B}(1) = \frac{\frac{m}{n}\sum_{t=1}^{n-1} I_{\{X_t \ge a_m, X_{t+1} \ge a_m\}}}{\frac{m}{n}\sum_{t=1}^{n} I_{\{X_t \ge a_m\}}}$$

Remark: So theory and methodology (permutation/bootstrapping) developed for the extremogram applies to the histogram

Bootstrapping the Extremogram

The stationary bootstrap, introduced by Politis and Romano (1994) seems well suited for the extremogram.

Stationary Bootstrap Setup: Have data $X_1, \ldots, X_n$ and construct BS sample as follows:

- $K_1, K_2, \ldots,$ be iid uniform on $\{1,\ldots, n\}$
- $L_1, L_2, \ldots,$ be iid geometric(p_n)

The BS sample $X_1^*,\ldots,X_n^*$ is given by the first n observations in the sequence.

$$X_{K_1},\ldots,X_{K_1+L_1-1},X_{K_2},\ldots,X_{K_2+L_2-1},\ldots,X_{K_N},\ldots,X_{K_N+L_N-1}$$

where

$$N=\inf\{i\geq 1: L_1+\cdots+L_i\geq n\}.$$

Bootstrapping the Extremogram

$$X_{K_1},\ldots,X_{K_1+L_1-1},X_{K_2},\ldots,X_{K_2+L_2-1},\ldots,X_{K_N},\ldots,X_{K_N+L_N-1}$$

- $K_1, K_2, \ldots,$ be iid uniform on $\{1,\ldots, n\}$
- $L_1, L_2, \ldots,$ be iid geometric(p_n)

Remarks

- Procedure is similar to the block bootstrap method
- Each block has a random length given by independent geometrics, $L_1, L_2, \ldots$.
- Mean block size is $1/p_n$
- Mean number of blocks is np_n
- By the previous two bullet points, we require

$$p_n \to 0,\ np_n \to \infty.$$

Bootstrapping the Extremogram (cont)

The extremogram, computed from either the sample or BS sample, are ratios of partial sums of the form,

$$\hat{P}_n(C) = \frac{m_n}{n}\sum_{t=1}^{n} I_{\{a_m^{-1}X_t \in C\}} \quad \text{and} \quad \hat{P}_n^*(C) = \frac{m_n}{n}\sum_{t=1}^{n} I_{\{a_m^{-1}X_t^* \in C\}}.$$

Theorem . Assuming our general setup (mixing conditions + regular variation, etc), and the growth conditions,

$np_n \to \infty, \quad np^2/m_n \to \infty,$

we have $E^*\hat{P}_n^*(C) \xrightarrow{P} \mu(C)$ and $ms_n^2 = \mathrm{Var}^*((n/m)^{1/2}\hat{P}_n^*(C)) \xrightarrow{P} \sigma^2(C).$

Moreover,

$$\sup_x |P((n/m)^{1/2}(ms_n^2)^{-1/2}(\hat{P}_n^*(C) - \hat{P}_n(C)) \le x \mid X_1,\dots,X_n) - \Phi(x)| \xrightarrow{P} 0$$

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Bootstrapping the Extremogram (cont)

The sample extremogram and its BS counterpart are:

$$\hat{\rho}_{A,B}(h) = \frac{\frac{m}{n}\sum_{t=1}^{n-h} I_{\{a_m^{-1}X_t \in A, a_m^{-1}X_{t+h} \in B\}}}{\frac{m}{n}\sum_{t=1}^{n} I_{\{a_m^{-1}X_t \in A\}}} \qquad \hat{\rho}^*_{A,B}(h) = \frac{\frac{m}{n}\sum_{t=1}^{n-h} I_{\{a_m^{-1}X_t^* \in A, a_m^{-1}X_{t+h}^* \in B\}}}{\frac{m}{n}\sum_{t=1}^{n} I_{\{a_m^{-1}X_t^* \in A\}}}$$

Theorem . Assuming our general setup (mixing conditions + regular variation, etc), and the growth conditions,

$np_n \to \infty, \quad np^2/m_n \to \infty,$

we have

$$\sup_x |P((n/m)^{1/2}(\hat{\rho}^*_{A,B}(h) - \hat{\rho}_{A,B}(h)) \le x \mid X_1,\dots,X_n) -$$
$$P((n/m)^{1/2}(\hat{\rho}_{A,B}(h) - \rho_m(h)) \le x)| \xrightarrow{P} 0$$

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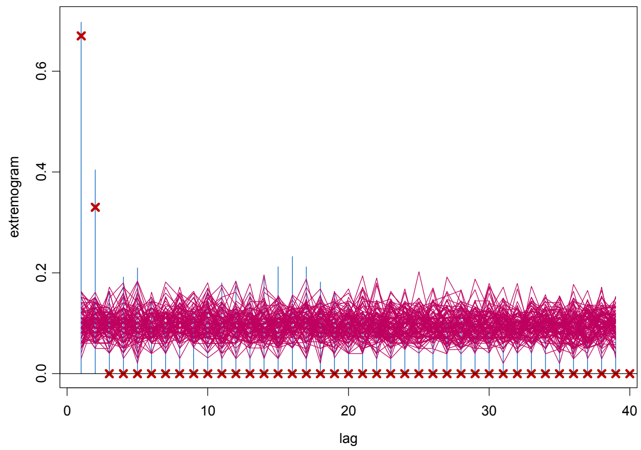
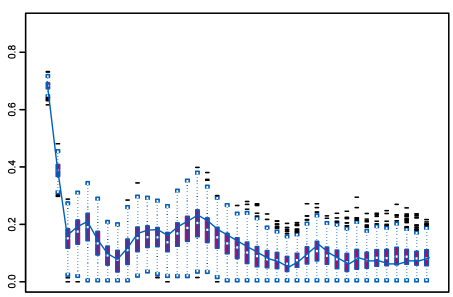
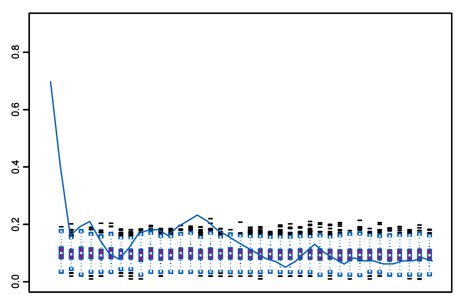
Bootstrap of the Extremogram of the Max-MA(2)



Bootstrap of the Extremogram of the Max-MA(2)





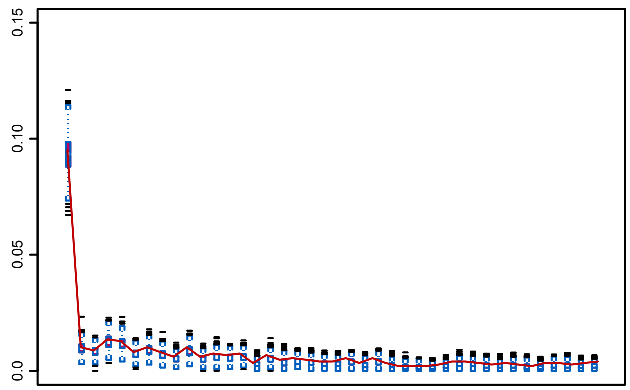
p_n =.02 (mean block size is 50) p_n = 1 (mean block size is 1)

BS reps = 1000

Extremogram of a GARCH(1,1)

GARCH(1,1): $X_t=(.1+.14\,X^2_{t-1}+.83\sigma^2_{t-1})^{1/2}Z_t$, $\{Z_t\} \sim$ IID N(0,1), $n=10^6$

3-dim extremogram ($\lim_n P(\min(X_h,X_{h+1}) > n^{1/\alpha} \mid X_0 > n^{1/\alpha})$)



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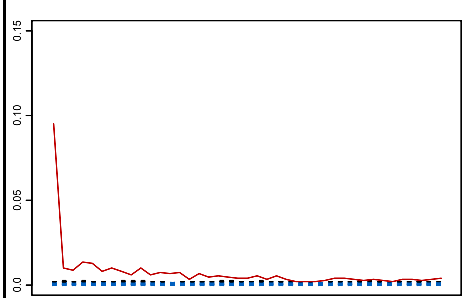
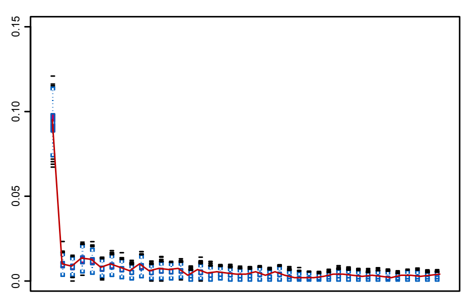




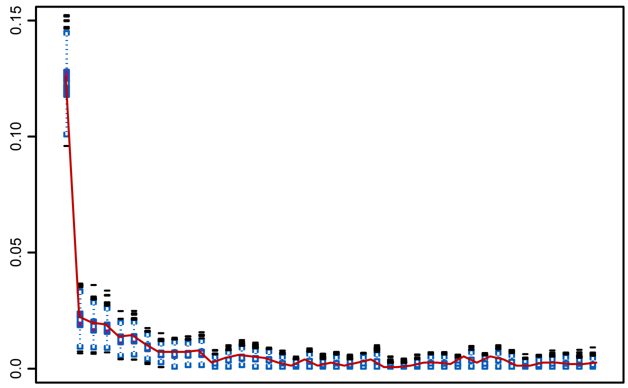
p_n =1 (mean block size is 1)

p_n = .02 (mean block size is 50)

BS reps = 1000

Extremogram of a SV

SV Process: $X_t=\sigma_t Z_t$, $\{Z_t\}$ ~ IID t_4 ; $\log \sigma_t = .9 \log \sigma_{t-1} + \varepsilon_t$, $n=10^6$
3-dim extremogram ($\lim_n P(\min(X_h,X_{h+1}) > n^{1/\alpha} \mid X_0 > n^{1/\alpha})$)



$p_n = .02$ (mean block size is 50) BS reps = 1000

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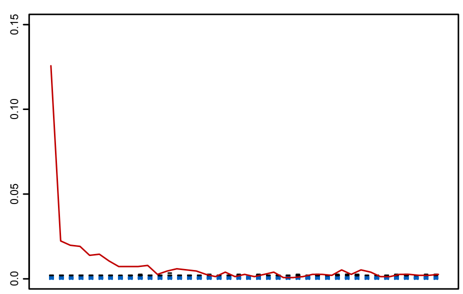

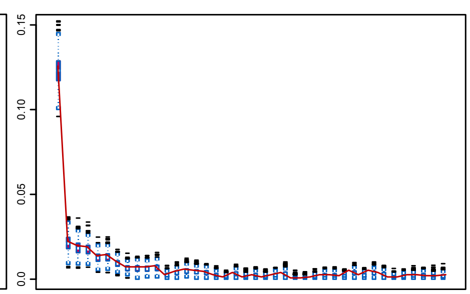

$p_n = 1$ (mean block size is 1) $p_n = .02$ (mean block size is 50)

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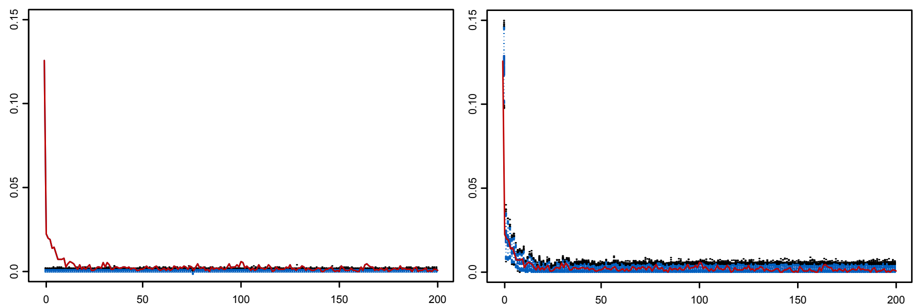
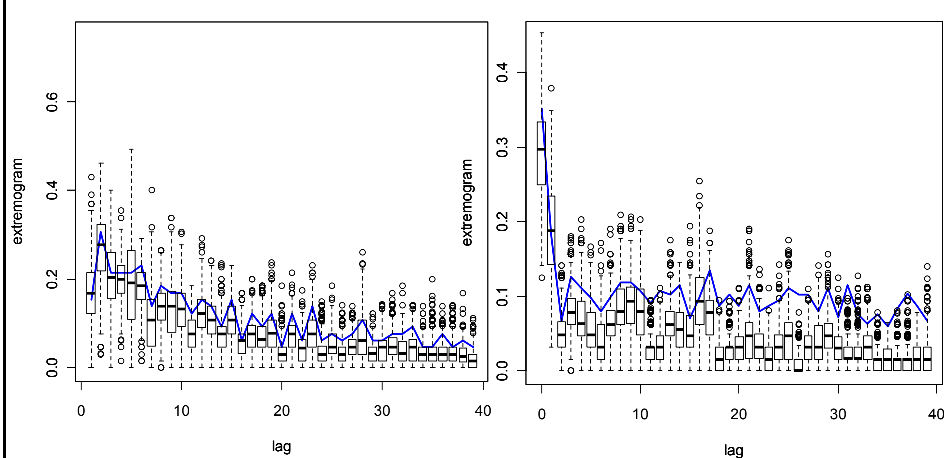


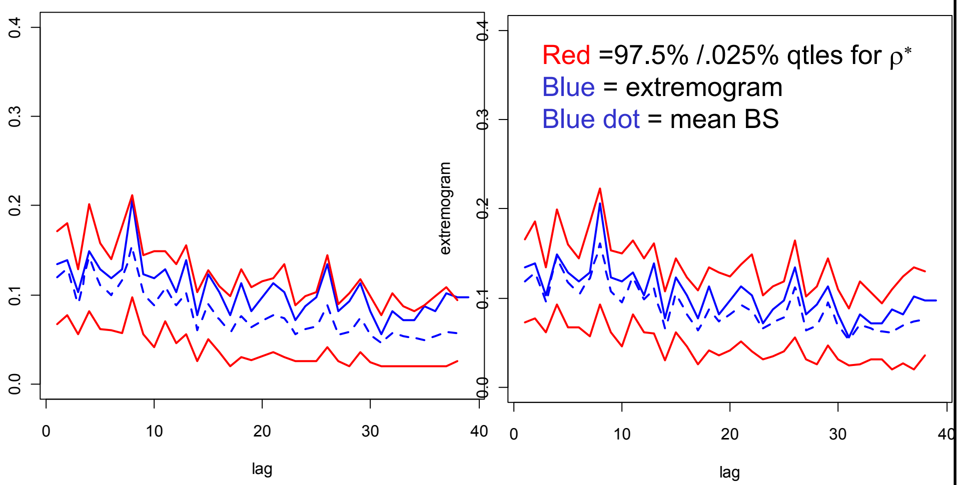
p_n =1 (mean block size is 1) p_n = .02 (mean block size is 50)

BS reps = 1000

Application to FTSE (left) and cross S&P | FTSE (right)

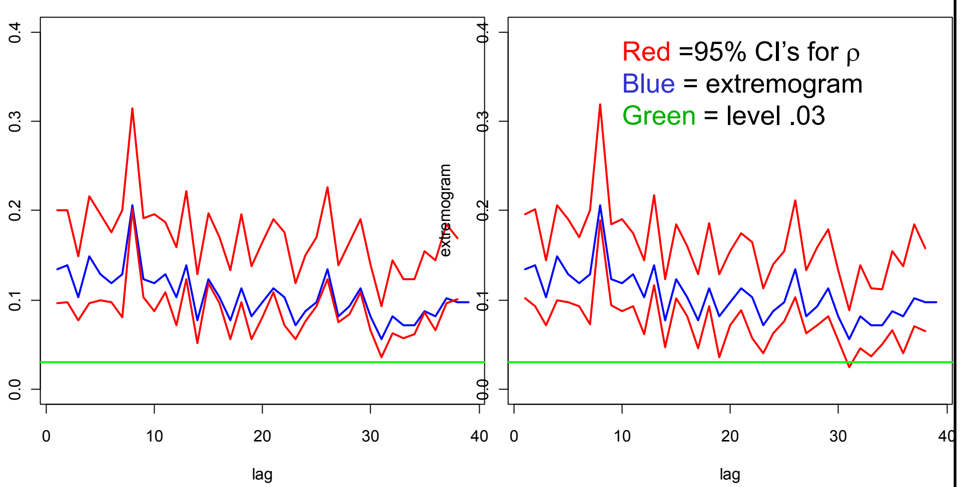


Application to FTSE (lower tail)



p_n =1/50 p_n =1/200

Application to FTSE (lower tail)



p_n =1/50 p_n =1/200

Wrap-up

- *Extremogram* is another potential tool for estimating extremal dependence that may be helpful for discriminating between models on the basis of extreme value behavior.
- Permutation procedures are a *quick* and *clean* way to test for significant values in the extremogram and other statistics.
- *Bootstrapping* may prove useful for constructing CI's for the extremogram and also for assessing extremal dependence.
- The *Extremogram* can provide insight on extremal dependence between components in a multivariate time series.
- Interesting connection between *return times* and the *extremogram.*