Large deviations for (pseudo-)regenerative Markov chains

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Motivation: characterization of the limit of partial sums

Let $(X_t)_{t \geqslant 1}$ be a process with dependent extreme values.

Motivation

Characterization of the limit of $S_n = \sum_{t=1}^{n} X_t$ under tractable hypothesis?

Example (Errors of empirical statistics)

1. Empirical mean $\overline{X}_n = \frac{1}{n} \sum_{t=1}^{n} X_t$ when $\mathbb{E}|X| < \infty$ but $\mathbb{E}X^2 = \infty$, limit distribution of the error $(\overline{X}_n - \mathbb{E}(X))$ correctly normalized?
2. Empirical autocovariances: for any lag $h \geqslant 1$ we have

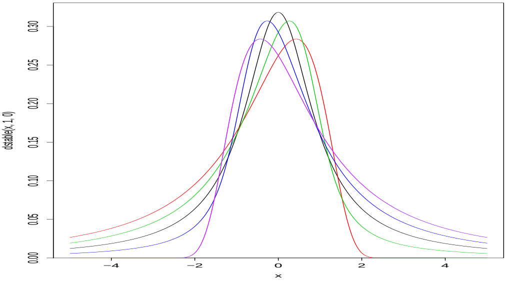
$$\hat{\gamma}_n(h) = \frac{1}{n-h} \sum_{j=1}^{n-h} (X_j - \overline{X}_n)(X_{j+h} - \overline{X}_n).$$

Strictly stable r.v.

Definition

A r.v. Y is strictly α-stable distributed iff $\exists\ a > 0$, Y_1 and Y_2 independent, distributed as Y such that $Y_1 + Y_2 = aY$ in distribution.
Then Y is strictly α-stable with $0 < \alpha \leqslant 2$ and c.f. $\exp(-|x|^{\alpha}\chi_{\alpha}(x, b_+, b_-))$,

$$\chi_{\alpha}(x, b_+, b_-) = \frac{\Gamma(2-\alpha)}{1-\alpha}((b_+ + b_-)\cos(\pi\alpha/2) - i \pm_x (b_+ - b_-)\sin(\pi\,\alpha/2)).$$



Strictly stable central limit theorem

Theorem (Feller, 1977)

If $\exists (a_n)$, $a_n > 0$ *and* Y *strict. stable such that*

$$a_n^{-1} S_n \to Y \tag{SSL}$$

then X_t *are iid* $RV(\alpha)$ *centered r.v. if* $\alpha > 1$.
For $\alpha < 2$ *and* $a_n = L(n) n^{1/\alpha}$ *s.t.* $\lim_n n \mathbb{P}(|X| > a_n) = 1$ *then* $b_+ + b_- = 1$.

Remark that if $0 < \alpha < 1$ then $\mathbb{E}|X| = \infty$.

Regularly varying sequences

Stationary RV(α) processes, Basrak & Segers (2009)

(X_t) is RV(α) iff $\exists$ its spectral tail process (Θ_t) defined for $k \geqslant 0$, $u \geqslant 1$ when $x \to \infty$

$$\mathbb{P}(X_0 > ux, |X_0|^{-1}(X_0, \dots, X_k) \in \cdot \mid |X_0| > x) \xrightarrow{w} u^{-\alpha}\mathbb{P}((\Theta_0, \dots, \Theta_k) \in \cdot).$$

Example

If (X_t) is iid, $\Theta_t = 0$ for $t \geqslant 1$ and $b_\pm = \mathbb{E}[{\Theta_0}_\pm^\alpha]$ for $\alpha \in (1, 2)$.

Remark that $b_+ + b_- = \mathbb{E}[{\Theta_0}_+^\alpha] + \mathbb{E}[{\Theta_0}_-^\alpha] = \mathbb{E}|\Theta_0|^\alpha = 1$ because $|\Theta_0| = 1$.

A necessary condition

Theorem (Jakubowski, 1993)

If (SSL) *with* $a_n = L(n)n^{1/\alpha}$ *then it exists a sequence* $k_n, n/k_n \to \infty$ *such that*

$$|\mathbb{E}(e^{ixa_n^{-1}S_n}) - \mathbb{E}(e^{ixa_n^{-1}S_{n/k_n}})^{k_n}| \to 0. \qquad \text{(MX)}$$

Example

(MX) is satisfied for

1. (X_t) iid,
2. $X_t = Y$ strictly stable for all $t \geqslant 1$!!!

Toward coupling conditions

Remark that $X_t = Y \in RV(\alpha)$ is a stationary sequence satisfying

1. RV(α),
2. (MX).

However, (SSL) holds iff Y is strictly α-stable.

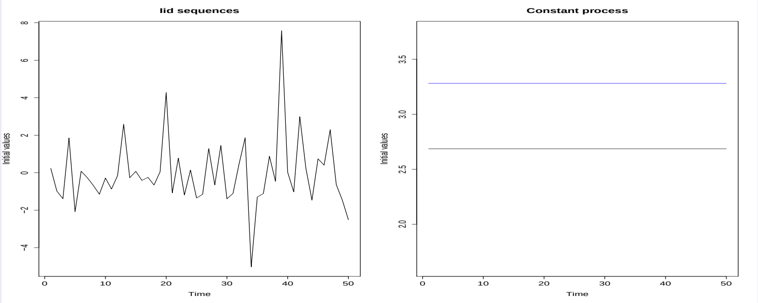
Mixing type conditions sufficient for (MX) excluding the case $X_t = Y$.

Coupling conditions

Assume that $X_t = f(\Phi_t)$ where (Φ_t) is a Markov chain:
$\Phi_t = F(\Phi_{t-1}, \xi_t)$, where (ξ_t) is iid.

Definition (Coupling scheme, Thorisson (2000))

Consider $X_t^* = f(\Phi_t^*)$ with $\Phi_t^* = F(\Phi_{t-1}^*, \xi_t)$ for $t \geqslant 1$ and (Φ_0^*, Φ_0) iid:

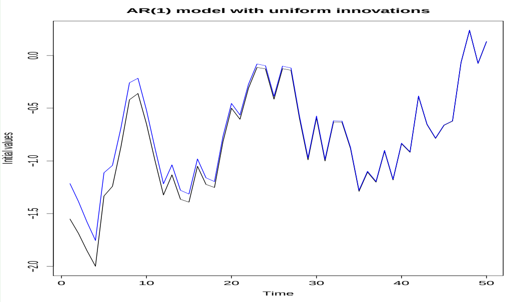


Coupling conditions

Proposition

If $\sum_t \mathbb{E}|X_t - X_t^*| < \infty$ then (MX) is satisfied

Example (AR(1): $X_t = \rho^t X_0 + \sum_{j=1}^t \rho^{t-j} \xi_j$)



When small jumps matter.

The point process approach deals with $\sum_{t=1}^{n} \delta_{X_t/a_n}$ on some set vanishing around 0.

Example (Coupled regularly varying Markov chain)

For (T_t) iid positive RV(α'), (B_t) iid Rademacher, (ξ_t) iid centered RV(α) with $\alpha > \alpha' > 1$ consider $X_t = B_{N_T(t)} + \xi_t$, $N_T(t) = \inf\{k \geqslant 1,\ T_1 + \cdots + T_k \geqslant t\}$.

Then $\begin{cases} \sum_{t=1}^{n} \delta_{X_t/a_n} \sim \sum_{t=1}^{n} \delta_{\xi_t/a_n} & \Rightarrow\ \alpha\text{-stable limit,} \\ S_n \sim \sum_{j=1}^{N_T(n)} \pm T_j,\ N_T(n)\mathbb{E}(T) \sim n & \Rightarrow\ L(n)n^{-\alpha'} S_n\ \alpha'\text{-stable limit.} \end{cases}$

Remark

- $\mathbb{E}|X_t - X_t^*| = \mathbb{E}|B_{N_T(t)} - B^*_{N_{T^*}(t)}| \leqslant 2\mathbb{P}(T_1 \geqslant t) = 2L(t)t^{-\alpha'}$.
- Does not work for $0 < \alpha' < 1$.

Vanishing small values condition

Additional hypothesis

Davis and Hsing (1995)

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \mathbb{P}\Big(\Big| \sum_{t=1}^{n} X_t I_{\{|X_t| \leqslant \epsilon a_n\}} - \mathbb{E}(X_t I_{\{|X_t| \leqslant \epsilon a_n\}}) \Big| > x\, a_n \Big) = 0, \quad x > 0. \tag{VSV}$$

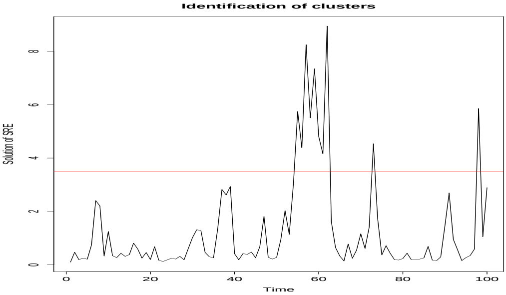
Example

Iid (X_t) satisfies (VSV).

Condition (VSV) has to be verified for dependent (X_t).

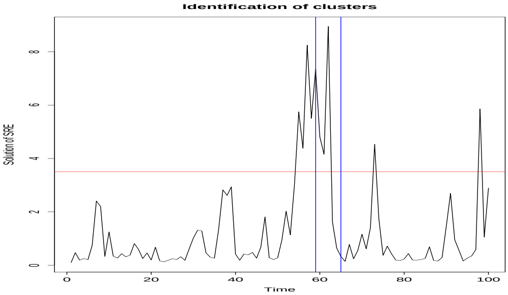
Identification of the clusters

SRE: $X_t = A_t X_{t-1} + B_t$, $t \geqslant 1$ with (A_t, B_t) iid, $A_t > 0$, $\mathbb{E}A_0^\alpha = 1$ and $\mathbb{E}|B_0|^{\alpha+\varepsilon} < \infty$, $\varepsilon > 0$. The unique stationary solution (X_t) is RV(α).



How to identify the clusters?

Approximation by local dependance (Rootzen, 1978)



When is it a good approximation when $m \to \infty$?

Davis & Hsing (1995), Basrak & Segers (2009)

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\Big(\max_{m \leqslant |i| \leqslant n/k_n} |X_i| > x\, a_n \mid |X_0| > x\, a_n \Big) = 0\,, \quad x > 0. \qquad \text{(ALD)}$$

Under (ALD) $\theta > 0$, i.e. average size of clusters are finite.

Drift condition (DCp)

Two issues

1. Condition (MX) or sufficient coupling is not sufficient for (VSV),
2. Condition (ALD) is not very tractable.

One solution

Let $X_t = f(\Phi_t)$ where Φ_t is a nice Markov chain. It satisfies Condition (DCp) for $p > 0$ if there exist $\beta \in (0,1)$, $b > 0$ such that for any y,

$$\mathbb{E}(|f(\Phi_1)|^p \mid \Phi_0 = y) \leqslant \beta |f(y)|^p + b. \tag{DCp}$$

Remark that (DCp) implies (DCp') for $p > p'$ (Jensen's inequality).

Examples for (DCp)

Examples

1. (X_t) iid RV(α) then $\mathbb{E}(|X_1|^p \mid X_0 = y) = \mathbb{E}|X_1|^p =: b, \qquad 0 < p < \alpha,$
2. AR(1): $X_t = \rho X_{t+1} + \xi_t$ with (ξ_t) iid RV(α) then

$$\mathbb{E}(|\rho y + \xi_1|^p \mid X_0 = y) \leqslant (|\rho| y + (\mathbb{E}|\xi_1|^p)^{1/p})^p \leqslant \beta y^p + b$$

   for $|\rho|^p < \beta < 1$ and all $1 \leqslant p < \alpha$,
3. $X_t = Y$ then $\mathbb{E}(|X_1|^p \mid X_0 = y) = |y|^p$ does not satisfied (DCp).

Examples for (DCp)

Example

SRE: $X_t = A_t X_{t-1} + B_t$ with $\mathbb{E}A_0^\alpha = 1$ and $\mathbb{E}B_0^{\alpha+\varepsilon} < \infty$ then

$$\mathbb{E}(|A_1 y + B_1|^p \mid X_0 = y) \leqslant ((\mathbb{E}A_0^p)^{1/p} y + (\mathbb{E}|\xi_1|^p)^{1/p})^p \leqslant \beta y^p + b$$

for $\mathbb{E}A_0^p < \beta < 1$ as $(\mathbb{E}A_0^p)^{1/p} < (\mathbb{E}A_0^\alpha)^{1/\alpha} = 1$ for $1 \leqslant p < \alpha$,

Conjecture

If the Markov chain $(\Phi_t) \in \mathrm{RV}(\alpha)$ then it satisfies (DCp).

Regeneration of Markov chains with an accessible atom (Doeblin, 1939)

Definition

(Φ_t) is a Markov chain of kernel P on $\mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$.

- A is an atom if $\exists$ a measure ν on $\mathcal{B}(\mathbb{R}^d)$ st $P(x, B) = \nu(B)$ for all $x \in A$.
- A is accessible, i.e. $\sum_k P^k(x, A) > 0$ for all $x \in \mathbb{R}^d$.

Let $(\tau_A(j))_{j \geqslant 1}$ visiting times to the set A, i.e.
$\tau_A(1) = \tau_A = \min\{k > 0 : X_k \in A\}$ and $\tau_A(j+1) = \min\{k > \tau_A(j) : X_k \in A\}$.

Regeneration cycles

1. $N_A(t) = \#\{j \geqslant 1 : \tau_A(j) \leqslant t\}$, $t \geqslant 0$, is a renewal process,
2. The cycles $(\Phi_{\tau_A(t)+1}, \ldots, \Phi_{\tau_A(t+1)})$ are iid.

Irreducible Markov chain and Nummelin scheme

Definition (Minorization condition, Meyn and Tweedie, 1993)

$\exists\ \delta > 0$, a small set $C \in \mathcal{B}(\mathbb{R}^d)$ and a distribution ν on C such that

$$P^k(x, B) \geqslant \delta\nu(B), \qquad x \in C, \quad B \in \mathcal{B}(\mathbb{R}^d). \tag{MCk}$$

(MC1) is called the strongly aperiodic case.

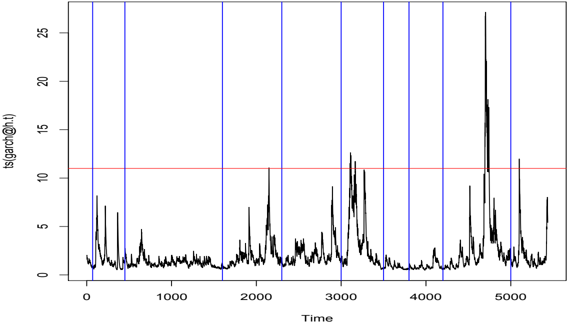
Any irreducible aperiodic Markov chain (Φ_t) satisfies (MCk) for some $k \geqslant 1$.

Nummelin splitting scheme for pseudo-regenerative Markov chain

Under (MC1) an enlargement of (Φ_t) on $\mathbb{R}^d \times \{0, 1\} \subset \mathbb{R}^{d+1}$ possesses an accessible atom $A = C \times \{1\} \Longrightarrow$ the enlarged Markov chain regenerates.

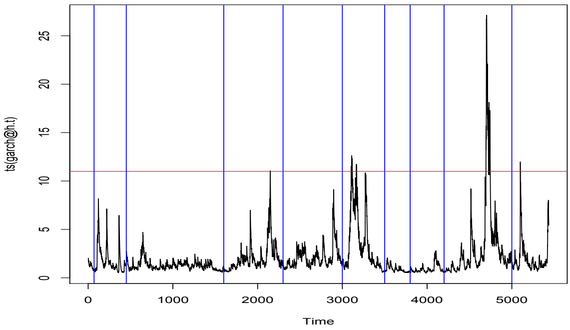
Inference on real data, Bertail and Clemencon (2009)

Squared of log-ratios $X_t = \log(P_t/P_{t-1})^2$ where (P_t) are CAC 40 prices.



Small sets $C = \{X_t^2 \leqslant a_n\}$ for any $a_n > 0$ (T-chains).

Coupling under (DCp)



Under (DCp) then $\mathbb{E}e^{c\tau_A(1)} < \infty$, $\mathbb{P}(\tau_A(1) \geqslant t) \leqslant \mathbb{E}e^{c\tau_A(1)}e^{-ct}$ and

$$\mathbb{E}|X_t - X_t^*| \leqslant 2\mathbb{E}|X_t|\mathbb{P}(\tau_A(1) \geqslant t) \leqslant 2\mathbb{E}|X_t|Ce^{-ct}.$$

(SSL) for sums of m-dependent r.v.

Assume $(X_t,\ t \leqslant 0)$ is independent of $(X_t,\ t \geqslant m)$ then $\Theta_t = 0$ for $|t| \geqslant m$.

Theorem

If (X_t) is centered RV(α) with $\alpha > 1$ then it satisfies (SSL) $a_n^{-1} S_n \to Y$ where Y has c.f. $\exp(-|x|^\alpha \chi_\alpha(x, b_+, b_-))$ with cluster indices

$$b_\pm = \mathbb{E}\Big[\Big(\sum_{t=0}^{m-1} \Theta_t\Big)_\pm^\alpha - \Big(\sum_{t=1}^{m-1} \Theta_t\Big)_\pm^\alpha\Big].$$

Large deviations for function of Markov chains

Assume $(X_t = f(\Phi_t))$ where (Φ_t) (possibly enlarged) possesses an accessible atom A and an invariant measure π s.t. $\Phi_0 \sim \pi$.

Theorem

If (X_t) *is centered RV(*α*) with* $\alpha > 1$ *and satisfies* (DCp) *for* $p < \alpha$ *then it satisfies* (SSL) *with cluster indices*

$$b_\pm = \mathbb{E}\Big[\Big(\sum_{t=0}^{\infty} \Theta_t\Big)_\pm^\alpha - \Big(\sum_{t=1}^{\infty} \Theta_t\Big)_\pm^\alpha\Big].$$

Sketch of the proof

Under (DCp) we have $\mathbb{E}|\Theta_k|^p \leqslant C\rho^k$ for some $C > 0$, $0 < \rho < 1$.
In particular (Θ_t) is a convergent series in $\mathbb{L}^{\alpha-1}$.
By the mean value theorem we have there exists $C > 0$

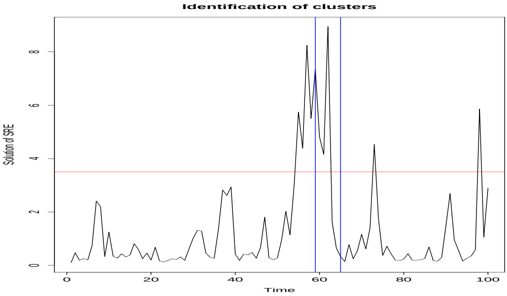
$$\mathbb{E}\Big[\Big(\sum_{t=0}^{m-1}\Theta_t\Big)_{\pm}^{\alpha} - \Big(\sum_{t=1}^{m-1}\Theta_t\Big)_{\pm}^{\alpha}\Big] \leqslant C\mathbb{E}\Big|\sum_{t=0}^{m-1}\Theta_t\Big|^{\alpha-1}.$$

By the dominated convergence theorem the cluster index exists.

Approximation by local dependence

SRE: $X_t = A_t X_{t-1} + B_t$, then $\Theta_t = \prod_{j=1}^{t} A_j \Theta_0$ satisfies

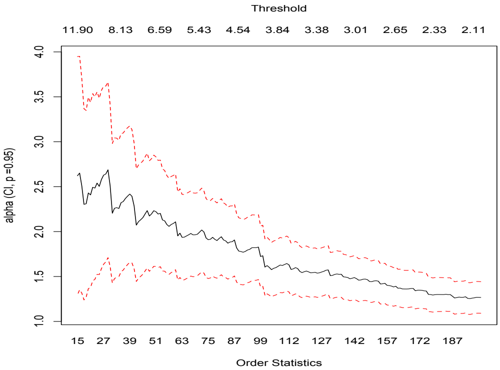
$$\mathbb{E}|\Theta_t|^\alpha = 1 \Longrightarrow \mathbb{E}\Big(\sum_{t=1}^{\infty} |\Theta_t|^\alpha\Big) = \infty.$$



Under (DCp), good approximation in $\mathbb{L}^p$, $p < \alpha$ when $m \to \infty$.

Application to autocorrelograms of squared log-ratios

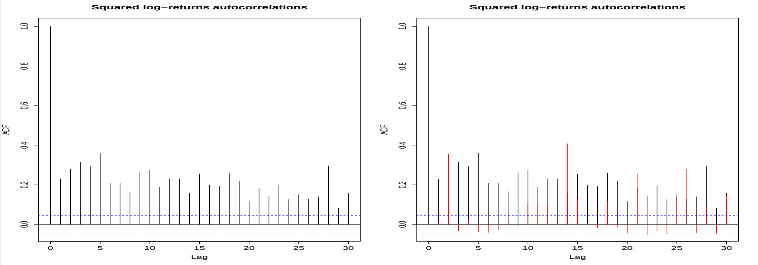
Assume that $X_t = \log(P_t/P_{t-1})^2$ is RV(α) satisfying (DCp).



Hill's estimator: $\hat{\alpha} \approx 2$.

Autocorrelogram in presence of extremes

$\hat{\gamma}_n(h) \approx \gamma(h) + Y_1(h)$ asymptotically $\alpha \approx$ 1-stable asymmetric distributed.



Analysis on basis of autocorrelogram are not adapted to heavy tailed cases.

Regular variation of cycles

Denoting the independent cycles $S_A(t) = \sum_{i=1}^{\tau_A(t+1)} f(\Phi_{\tau_A(t)+i})$,

$$S_n = \sum_{1}^{\tau_A} X_i + \sum_{t=1}^{N_A(n)-1} S_A(t) + \sum_{\tau_A(N_A(n))+1}^{n} X_i.$$

Theorem

If (X_t) *RV(*$_\alpha$*) with* $\alpha > 0$, $\alpha \notin \mathbb{N}$ *and* (DCp) *with* $p < \alpha$ *and* $b\pm \neq 0$ *then*

$$\mathbb{P}_A\Big(S_A(1) > x\Big) \sim_{x\to\infty} b_\pm \, \mathbb{E}_A(\tau_A)\, \mathbb{P}(|X| > x).$$

Remarks

1. The full cycles $S_A(t) = \sum_{i=1}^{\tau_A(t+1)} f(\Phi_{\tau_A(t)+i})$ are regularly varying with the same index $\alpha > 0$ than X_t,
2. If τ_A is independent of (X_t) then $\mathbb{P}_A(S_A(1) > x) \sim_{x\to\infty} \mathbb{E}_A(\tau_A)\, \mathbb{P}(X > x)$,
3. Under (DCp) and $\mathbb{E}|X|^p$ then $\mathbb{E}_A|S_A(1)|^p < \infty$.

Precise large deviations for sums

Corollary (Under the hypothesis of the Theorem)

If $0 < \alpha < 1$ *then* $\lim_{n\to\infty} \sup_{x\geqslant b_n} \left| \frac{\mathbb{P}(\pm S_n > x)}{n\,\mathbb{P}(|X| > x)} - b_\pm \right| = 0,$ *where* $b_n = n^{1/\alpha\wedge 1/2+\varepsilon}$ *else, if* $\mathbb{P}(\tau_A > n) = o(n\mathbb{P}(|X| > c_n))$*,*

$$\lim_{n\to\infty} \sup_{b_n\leqslant x\leqslant c_n} \left| \frac{\mathbb{P}(\pm S_n > x)}{n\,\mathbb{P}(|X| > x)} - b_\pm \right| = 0.$$

Determination of the constant in LD of Davis and Hsing (1995) valid for $\alpha < 2$.

Sketch of the proof:
Under $\mathbb{P}(\tau_A > n) = o(n\mathbb{P}(|X| > c_n))$,

$$S_n \approx \sum_{t=1}^{N_A(n)-1} S_A(t).$$

Use Nagaev's precise LD result on the iid regularly varying cycles $S_A(t)$.

Link between extremal and cluster index, $\Theta_0 = 1$

Under RV(α) and (DCp), extremal index $\theta_+ = \mathbb{E}[(\sup_{t\geqslant 0}\Theta_t)_+^\alpha - (\sup_{t\geqslant 1}\Theta_t)_+^\alpha]$.

Example (Asymptotic independence)

$\Theta_t = 0$ for all $t > 0$ then $b_+ = \theta_+ = 1$.

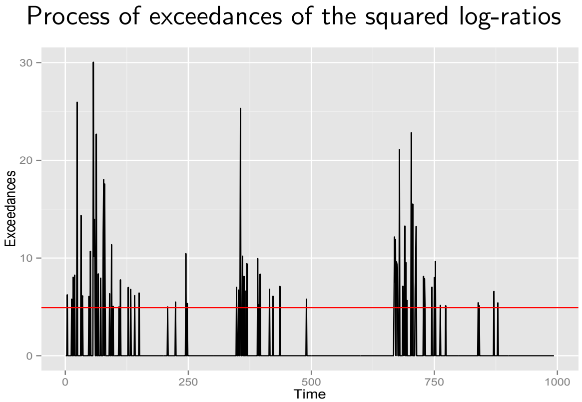
Example (AR(1): $X_t = \rho X_{t-1} + \xi_t$, $\forall t \in \mathbb{Z}$ with $\rho > 0$)

$\Theta_t = \rho^t$ for all $t \geqslant 0$ then $\theta_+ = 1 - \rho^\alpha$ and $b_+ = \theta_+/(1-\rho)^\alpha$.

Example (GARCH(1,1)^2: $X_t^2 = \sigma_t^2 Z_t^2$, $\sigma_t^2 = \alpha_0^* + \alpha_1^* X_{t-1}^2 + \beta_1^* \sigma_{t-1}^2$)

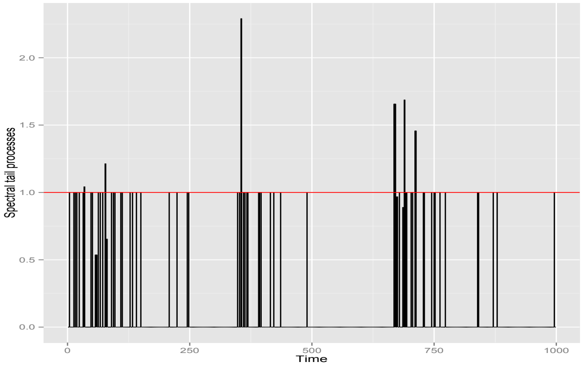
$\Theta_t = (Z_t/Z_0)^2 \prod_{i=1}^t (\alpha_1^* Z_{i-1}^2 + \beta_1^*)$ for all $t \geqslant 0$ then b_+ and θ_+ are explicit.

Peaks over thresholds

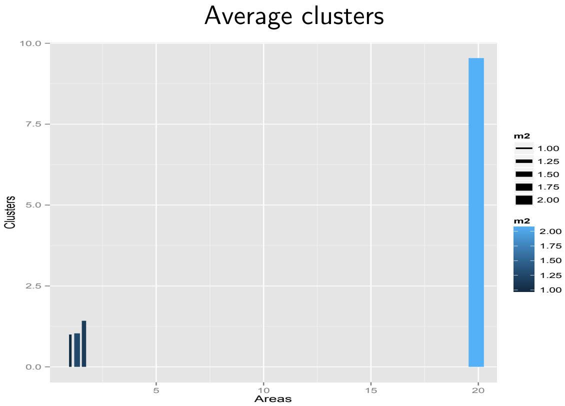


Description of the clusters

Renormalization by the first exceedance in the cluster



Representation of the average clusters



As. ind., observations, AR(1) GARCH(1,1)^2

Conclusions and perspectives on the extremes

- Conclusions
  1. Cluster indices $b_\pm$ determine the asymptotic distribution of the sums of dependent and regularly varying variables,
  2. The extremal and cluster indices describe the clusters of extreme values.

- Perspectives
  1. We use Markovian processes and their regenerative structures
     $\Longrightarrow$ use also regenerative structures to identify the clusters.
  2. Model the extremal dependence in view of the observed clusters
     $\Longrightarrow$ introduce new models with extremal behaviors similar than the observed ones.