LECTURES ON HOMOTOPICAL GROUP ACTIONS (OUTLINE OF 3 TALKS AT UNIV. COPENHAGEN)

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ABSTRACT. 3 lectures, each 2 x 45 min.

Outline. .

Lecture 1 will explain what is introduce basic equivariant homotopy theory, and begin to study how equivariant homotopy theory relates to the homotopy theory of diagrams.

Lecture 2 will examine, for a $\mathbb{Z}/p$-space X the relationship between X and $X^{\mathbb{Z}/p}$, part of what is usually called Smith theory. It is "classical" Smith theory when X is assumed finite. When X is not assumed finite, Elmendorf's theorem shows that the homotopy type of X says basically nothing about that of $X^{\mathbb{Z}/p}$ However, surprisingly, the niceness of the finite case resumes, if one replaces "fixed-points" by "homotopy fixed-points". This "homotopy smith theory" uses the Steenrod algebra and the T functor of Lannes and it's interpretation by Dwyer-Wilkerson. The link to the classical case is precisely described via the Sullivan conjecture.

Lecture 3 ...

1. Lecture 1: Equivariant homotopy theory and the theory of diagrams

Ref: [12, 11, 9]

1.1. Assumptions and setup: .

G will be a finite group.

G-space X is a topological space with a G-action, which admits the structure of a G-CW-complex. (Since G is only amounts to assuming that X admits the structure of a CW-complex.)

G-map $f : X \to Y$ is a map which respect the G-action, i.e., $f(g \cdot x) = g \cdot f(x)$

Two maps $f, f' : X \to Y$ are called G-homotopic, if there exists a G-map $F : X \times I \to Y$ such that $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$.

An important point is that there will be two different notions of G-equivalence:

Definition 1.1. *A G-map $f : X \to Y$ is called a* G-equivalence *(or sometimes strong G-equivalence) if there exists a G-map $g : Y \to X$, such that $f \circ g$ and $g \circ f$ are G-homotopic.*

A map G-map $f : X \to Y$ is called an hG-equivalence *(or sometimes weak G-equivalence) if f is a homotopy equivalence as a non-equivariant map.*

equivequiv **Theorem 1.2** (Equivariant Whitehead theorem). *A G-map $f : X \to Y$ is a G-equivalence if the induced map $f^H : X^H \to Y^H$ is non-equivariant equivalence for all subgroups $H \leq G$.*

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Sketch of proof. That G-equivalences induces equivalences on fixed-points follows directly from the definition.

Conversely, if a G-map is an equivalence on all fixed-points, on can use induction of G-cells to construct an inverse. □

Remark 1.3. Note that, contrary to the above, a map being an hG-equivalence tells us nothing on what happens on the fixed-points for non-trivial subgroups H. E.g, for any G-space X, $X \times EG \to X$ is an hG-equivalence, but $(X \times EG)^H = \emptyset$ for all $1 \lneq H \leq G$. (Where EG as usual denotes a free contractible G-space.)

In the next subsection, we will introduce a sort of derived fixed-points, the homotopy fixed-points, which we can still work with in absence of meaningful actual fixed-points, and which by deep theorems agree with actual fixed-points in good cases (e.g. X finite, and "at a prime p").

1.2. Equivariant homotopy theory, and the diagram categories: Elmendorf's theorem. Ref: [6]

Proposition 1.2 shows that equivariant equivalences can be described as non-equivariant equivalences on fixed-points.

Note that the category of G-spaces and hG-equivalences can be described as functor category, with object-wise equivalences, namely the category of functors from G (viewed as a category with one object, and G as morphisms) to spaces.

In this subsection we will address two natural questions:

- Can the the category of G-spaces and G-equivalences be understood via non-equivariant homotopy theory as a certain functor category? (yes, Prop)
- Is there a relationship between the homotopy type of X and that of, say, $X^{\mathbb{Z}/p}$? (in general little by Theorem , under finiteness and "at p" assumptions a lot, cf later).

To warm up to Elmendorf's theorem, we do the following baby case first.

Definition 1.4 (The orbit category and the p-orbit category)**.** *Let $\mathbf{O}(G)$ denote the orbit category of G, i.e., the category with objects G/P, where P runs through the subgroups of G, and morphisms G-maps.*

The p-orbit category $\mathbf{O}_p(G)$ is the full subcategory of $\mathbf{O}(G)$, where P is assumed to be a p-subgroup.

Note that

$$\mathrm{Hom}_{\mathbf{O}(G)}(G/P, G/Q) = \{g \in G | g^{-1}Pg \leq Q\}/Q = (G/Q)^P$$

and in particular $\mathrm{Hom}_{\mathbf{O}(G)}(G/P, G/P) = N_G(P)/P$.

Will, in fact, most often be interested in the *opposite* orbit and p-orbit categories. Let us do a couple of examples:

Example 1.5.

- Suppose that $G = \mathbb{Z}/p$, then the p-orbit category has two objects G/G and G/e, with G as self-maps of G/e and only the identity self-map of G/G, and exactly one morphism between them. Diagrammatic the opposite p-orbit category looks like:

$$\begin{tikzcd} G/G \arrow[r] & G/e \arrow[loop above] \end{tikzcd}$$

- Suppose $G = \Sigma_3$. The opposite 2-orbit category looks like

$$\begin{tikzcd} G/P \arrow[r, Rightarrow] & G/e \arrow[loop above] \end{tikzcd}$$

that is only the identity self-map of G/P, G worth of self-map of G/e, and 3 maps between them, which naturally identify with G/P.

Theorem 1.6 (Elmendorf [6])**.** *The functor*

$$X \mapsto \{X^H\}_{G/H \in \mathbf{O}(G)^{\mathrm{op}}}$$

induces a $1-1$*-correspondence*

$$\{G\text{-}spaces\}/G\text{-}equivalence \longleftrightarrow \{Fun(\mathbf{O}(G)^{\mathrm{op}}, Spaces)\}/\,objectwise\ h.e.$$

In fact, with suitable definition of model category structures on the left- and right-hand side, this induces a Quillen equivalence of model categories. []

Before giving a sketch of proof of this theorem, let us do the special case $G = \mathbb{Z}/p$, which is already interesting.

Example 1.7 (Elmendorf's theorem for $G = \mathbb{Z}/p$)**.** Elmendorf's theorem for $\mathbb{Z}/p$ claims that giving a G-space X, up to G-equivalence is equivalent to giving space Y_0, a G-space Y_1 and a (non-equivariant) map $f : Y_0 \to Y_1^G$, up to equivalence of diagrams. We want to give an inverse functor.

We claim that we can take X to be the homotopy pushout of the diagram

$$\begin{tikzcd} EG \times Y_0 \arrow[r, "proj"] \arrow[d, "EG\times f"'] & Y_0 \\ EG \times Y_1 & \end{tikzcd}$$

First one observes that the G-homotopy homotopy type of X only depends, up to objectwise equivalence of diagrams.

Then one checks that these procedures are each other inverses:

Taking fixed-points on the homotopy pushout one easily sees that one recovers the diagram $Y_0 \to Y_1$, up to homotopy. Likewise, if one starts with a G-space, then homotopy colimit maps to X, and this map is a G-equivalence, since it induces an equivalence on all fixed-points.

Sketch of proof of Elmendorf's theorem; general case. The inverse $\Psi : Fun(\mathbf{O}(G)^{\mathrm{op}}, Spaces) \to G-spaces$ is given by the geometric realization of the simplicial space with n-simplices

$$(G/e \to G/P_0 \to G/P_1 \to \cdots \to G/P_n, x \in F(G/P_n))$$

and the obvious simplicial maps.

(This is by definition equival to the two-sided bar construction $B(E, \mathbf{O}(G)^{\mathrm{op}}, F)$, where $E : \mathbf{O}(G) \to G$-spaces is the identity functor which sends G/H to the G-space G/H, and likewise identifies with $\mathrm{hocolim}_{E\mathbf{O}(G)^{\mathrm{op}}} F$, where $E\mathbf{O}(G) := G/e \downarrow \mathbf{O}(G)$.)

We want to construct the natural equivalence $\Phi C \Rightarrow Id$.

For this, note that $E\mathbf{O}(G)^H$ has objects $G/e \xrightarrow{f} G/P$ for $H \leq G_f$, and hence has terminal object G/H, so we have a natural equivalence

$$(\mathrm{hocolim}_{(E\mathbf{O}(G))^{\mathrm{op}}} F)^H = \mathrm{hocolim}_{(E\mathbf{O}(G)^H)^{\mathrm{op}}} F \xrightarrow{\cong} F(G/H)$$

by the "cofinality" property of hocolim (G/H is a terminal object in $(E\mathbf{O}(G)^H)^{\mathrm{op}}$). $\square$

1.3. **Invariants of hG-equivalence: Homotopy fixed points and homotopy orbits (aka Borel construction). Ref:** [5]

Fixed-points X^H and orbit spaces X/H, for $H \leq G$ are of course invariants of G-equivalence. In this subsection we will introduce the homotopy orbit space and homotopy fixed-points, which *are* invariants of the space, up to hG-equivalence.

Denote by BG the classifying space of G, i.e., $BG \simeq K(G,1)$, and let EG denote the universal contractible e G-space on which G acts freely. We have $BG \simeq EG/G$.

Definition 1.8. *Define the Borel construction, or homotopy orbit space, of an action of G on X as as $X_{hG} = (X \times EG)/G$, where G acts diagonally on the product.*

Dually define the homotopy fixed-points as $X^{hG} = \mathrm{map}_G(EG, X)$, i.e., the space of equivariant maps, or equivalently the G-fixed-points on the mapping space $\mathrm{map}(EG, X)$ where g acts on $f : EG \to X$ by $(g \cdot f)(x) = gf(g^{-1}x)$.

Note that we have a map $X^G \to X^{hG}$ induced by $EG \to *$. The homotopy fixed-points should be thought of as derived fixed-points.

sectionslemma

Lemma 1.9. *$X^{hG} \simeq$ space of sections of fibration $X_{hG} \to BG$.*

Proof. We will describe the maps in the two directions:

Given a G-equivariant map $EG \to X$, we get the commutative diagram of G-maps

$$\begin{tikzcd} EG \arrow[r, "\mathrm{diag}"] \arrow[dr, equal] & X \times EG \arrow[d, "proj"] \\ & EG \end{tikzcd}$$

which upon quotienting out by G produces the section.

In the other direction, given a section of the fibration $X_{hG} \to BG$ we get a G-equivariant map $EG \to X$ by pulling $X \times EG)/G \to BG$ back along $EG \to BG$, and taking the induced map $EG \to X \times EG \xrightarrow{proj} X$ □

invarianceprop

Proposition 1.10. *If $f : X \to Y$ is an hG-equivalence, then it induces a homotopy equivalence $f_{hG} : X_{hG} \to Y_{hG}$ and $f^{hG} : X^{hG} \to Y^{hG}$.*

Proof. If $X \to Y$ is an hG-equivalence, then $X \times EG \to Y \times EG$ is a G-equivalence, by Lemma 1.2, and hence $X_{hG} \to Y_{hG}$ is an equivalence (over BG) by the invariance property of orbit spaces, showing the first claim.

To see the second claim, note that by Lemma 1.9 the map $X^{hG} \to Y^{hG}$ identifies with the from the space of sections of $X_{hG} \to BG$ to the space of sections of $Y_{hG} \to BG$.

By elementary homotopy theory, we can choose an inverse equivalence *over* BG to the map $X_{hG} \to Y_{hG}$, giving the wanted homotopy inverse on the level of spaces of sections.

□

Proposition 1.11. *Applying $EG \times -$ or $map(EG, -)$ to a map $f : X \to Y$ turns an hG-equivalence into a G-equivalence.*

In particular G-spaces X and Y are hG-equivalent if and only if $X \times EG$ and $Y \times EG$ are G-equivalent if and only if $\mathrm{map}(EG, X)$ and $\mathrm{map}(EG, Y)$ are G-equivalent.

Proof. For $f \times EG$ and $map(EG, f)$ will obviously be hG equivalences. Furthermore, if $H \leq G$ is a nontrivial subgroup then map $(f \times EG)^H$ will a map between empty spaces and hence trivially an equivalence, likewise $\mathrm{map}(EG, f)^H$ will be an equivalence by the

hH-invariance of homotopy fixed-points (Prop. 1.10). It follows by Lemma 1.2 that the maps are in fact G-equivalences.

If X and Y are hG-equivalent they are connected by a zig-zag of hG-equivalences.

$$X = X_0 \to X_1 \leftarrow X_2 \to \cdots \leftarrow X_n = Y$$

Applying $EG \times -$ and $\mathrm{map}(EG, -)$ to this shows the one direction of the 'in particular'. The other direction is immediate, noting that $EG \times X \xrightarrow{proj} X$ and $X \to \mathrm{map}(EG, X)$ are hG-equivalences. □

Remark 1.12. Examining this a bit more closely, one sees that the category of G-spaces with hG-equivalence equivalent to the category of spaces over BG, and non-equivariant equivalences.

One can construct two model structures on hG-spaces, where $EG \times -$ and $\mathrm{map}(EG, -)$ are cofibrant respectively fibrant replacement.

2. Lecture 2: Smith theory: The fixed-points $X^{\mathbb{Z}/p}$ from X

We have now seen that without special assumptions on the space, there is little relationship between the homotopy type or hG-homotopy type of a space, and that of its fixed-points. We have also introduced the Borel construction and homotopy fixed-points as (so far quite undescribed) invariants of the hG-homotopy type of X.

In this section we want to see the following:

- When X is a finite CW-complex and V is an elementary abelian group, cohomology of $H^*(X^V \mathbb{F}_p)$ is determinable from cohomological information which only depends on the hV-homotopy type of X.
- More generally, when X is a finite CW-complex and P is a p-group there is a close relationship between the homotopy type of X, as a hP-homotopy type, and that of X^P, "at a prime p".
- Replacing fixed-points by homotopy fixed-points, we find a natural home for these theorems, and we can significantly weaken the finiteness assumption on X.

This will be used in the next section to explore how if we view all the X^{hP} together, via the p-orbit category, we can get strong invariants of X, up to hG-equivalence (at a prime p).

2.1. The localization theorem.

Define the Borel equivariant cohomology of X as the ordinary cohomology of the Borel construction, i.e., $H^*_{hG}(X) = H^*(X_{hG})$. Borel cohomology is an equivariant cohomology theory, and obviously only depend on X up to hG-equivalence.

Theorem 2.1 (Localization theorem, Borel, Quillen,...). *Let $V = \mathbb{Z}/p$ and let S denote the non-trivial elements in $H^1(V; \mathbb{F}_p)$ if $p = 2$ and the non-nilpotent elements in $H^2(V; \mathbb{F}_p)$ if p is odd. Then*

$$S^{-1}H^*_{hV}(X; \mathbb{F}_p) \xrightarrow{\cong} S^{-1}H^*_{hV}(X^V; \mathbb{F}_p) = S^{-1}H^*(V; \mathbb{F}_p) \otimes H^*(X^V; \mathbb{F}_p)$$

There is also a relative version, as well as a version involving two elementary abelian p-groups $W \le V$.

Sketch of proof. By long exact sequence in Borel cohomology and induction on cells, enough to prove for relative cells $(G/G \times D^n, G/G \times S^{n-1})$ and $(G/e \times D^n, G/e \times S^{n-1})$.

However, in both cases the result is clear: For G/G there is nothing to prove. And for G/e it is also clear, since both sides are seen to be zero.

□

Corollary 2.2 (P.A.Smith)**.** *X finite V-CW complex. If X is $\mathbb{F}_p$-acyclic, then so is X^V. If X is an $\mathbb{F}_p$-homology sphere, so is X^V.*

2.2. **Smith theory, Dwyer-Wilkerson and Lannes style.** The formula for the homology of the fixed-points, due to Dwyer-Wilkerson, where we'll explain the notation after the theorem.

dwtheorem **Theorem 2.3** (Dwyer-Wilkerson [3, 4, 8])**.** *Let X be a finite V-CW complex. IF*

$$H^*_{hV}(X^V;\mathbb{F}_p) \xrightarrow{\cong} Un(S^{-1}H^*_{hV}(X^V)) \xleftarrow{\cong} Un(S^{-1}(H^*_{hV}(X)))$$

and in particular

$$H^*(X^V;\mathbb{F}_p) = \mathbb{F}_p \otimes_{H^*(V)} Un(S^{-1}H^*_{hV}(X))$$

Here Un is the largest unstable module, i.e.,

Definition 2.4. *For a module M over the Steenrod algebra $\mathcal{A}_2$ define*

$$Un(M) = \{x \in M | Sq^I(x) = 0 \text{ if } excess(I) > |x|\}$$

i.e., the largest submodule which satisfies the instability condition. Similarly for odd primes.

We also need to say how we get a Steenrod action after inverting S. This is described as follows.

Observation 2.5. *For R an R-algebra over the Steenrod algebra, M an unstable R-module over the Steenrod algebra (i.e., $R \otimes M \to M$ satisfies Cartan formula), and S a multiplicative subset of R, then $S^{-1}M$ has a Steenrod algebra action given by the following formula*

$$P_\xi(x/s) = P_\xi(x)(P_\xi(s))^{-1} = P_\xi(x)(s + Sq^1(s)\xi + Sq^2(s)\xi^2 + \cdots)^{-1}$$

Sketch of proof of Theorem 2.3. By the localization theorem we just need to see that

$$Un(S^{-1}H^*(V) \otimes H^*(X^V)) = H^*(V) \otimes H^*(X^V)$$

By a small calculation this in fact holds for any unstable module M

$$H^*(V) \otimes M \xrightarrow{\cong} Un(S^{-1}H^*(V) \otimes M)$$

This is a small calculation. The key reason is that, by the formula above, elements in the denominator have quite long Steenrod squares on them, and presense of these in hence not compatible with instability....

The short paper by Dwyer-Wilkerson well is worth reading! □

Remark 2.6. Note how Theorem 2.3 should seem surprising, in that, a priori, there would be no reason to believe that the homology of $H^*(X^V;\mathbb{F}_p)$ should be determinable from the hV-homotopy type of X. (Historically, the first display of this was through the Sullivan conjecture; see below.)

Exercise 2.7. Use the Dwyer-Wilkerson theorem to prove the following: X finite $\mathbb{Z}/2$-CW complex with $H^*(X;\mathbb{F}_2) \cong H^*(\mathbb{R}P^n;\mathbb{F}_2) = \mathbb{F}_2[x]/x^{n+1}$ then $H^*(X^G;\mathbb{F}_2) = H^*(\mathbb{R}P^{i-1}) \oplus H^*(\mathbb{R}P^{j-1})$, $i + j = n + 1$. (These are obviously realized by flipping i of the axes.)

2.3. **Smith theory, Lannes style.**

- The formular above also holds for homotopy fixed-points.
- In fact, for *any* X (subject to some mild technical conditions) there exists a functor Fix, derived from the so-called Lannes T-functor, such that
$$H^*(X^{hV};\mathbb{F}_p) = \mathbb{F}_p \otimes_{H^*(V)} Fix(H^*_{hV}(X))$$

See [7, §4.9]

2.4. **Fixed-points vs homotopy fixed-points: The Sullivan conjecture.**

Theorem 2.8 (Sullivan Conjecture, Miller, Carlsson, Lannes)**.** *Let X be a finite P-CW-complex, P a finite group, then*

$$X^P{}^{\hat{}}_p \xrightarrow{\cong} X^{\hat{}}_p{}^{hP}$$

In particular, under mild assumptions e.g., fixed-points simply connected,

$$H^*(X^P;\mathbb{F}_p) \cong H^*(X^{hP};\mathbb{F}_p)$$

Guide to various proofs: Before you go, yeah, well, sure, note that in the case the action is trivial, the statement reads that $\mathrm{map}(BP, X) \xrightarrow{\simeq} X$, in particular $\mathrm{map}_*(\mathbb{R}P^\infty, X) \cong *$, when X is a finite complex, which are not at all obvious—Sullivan considered it a test case for the conjecture. There are several proofs of the general statement, none of them easy!

There's a (technical) proof by Dwyer-Miller-Neisendorfer [2], building on the original work by Miller in the trivial action case [10]. There is a technical proof by Carlsson [1], deriving in from the Segal conjecture which he proved. And, there is a proof by Lannes (simplified by Farjoun-Smith) [7] which is long but "robust", but builds up a lot of machinery along the way: It establishes an algebraic formula for $\mathrm{map}(BV, X)$ when X is arbitrary Y, and hence in particular $Y = X_{hV}$, which enables one to get the statement for arbitrary actions from things about trivial actions. You can give a reasonable sketch of this proof in a couple of lectures, but we don't have time here.... (I can do it in the fall if enough people are interested.) □

Remark 2.9. Note how the Sullivan conjecture relates to the Dwyer-Wilkerson formula: The formula of Dwyer-Wilkerson tells us that that X^V *can* be extracted from the hV-homotopy type of X, and even gives a concrete formula for the cohomology. The Sullivan conjecture is then the additional formation that the homotopical model for this is indeed X^{hV}. A philosophical and/or practialcal question: Why doesn't the Dwyer-Wilkerson formula give more aid in the proof??

3. Lecture 3: Homological algebra of the p-orbit category and the collection of fixed-points $\{X^P\}_{G/P\in\mathbf{O}_p(G)}$

We have seen that fixedpoints and homotopy fixedpoints relate nicely when the group is a p-group, but hardly at all when the group is not a p-group. We may however very well still want to study spaces up to hG-equivalence for arbitary (finite) G. The way to do this is to examine fixed-points under all p-subgroups, viewed together as a functor on the p-orbit category. The first-level results will only be up to p-completion, which can however later be used to get integral results, via the Sullivan arithmetic square.

We want to go a step further, and try to make naive *algebraic models* for hG-spaces.

For non-equivariant spaces a very naive model, namely $\tilde{C}_*(X;\mathbb{F}_p)$. This will in this case just be equal to giving $H^*(X;\mathbb{F}_p)$. We can try to do the same model non-equivariantly....

3.1. **The model theorem.**

Theorem 3.1 (Compact model theorem, G-Smith)**.** *Let G be a finite group and X be a finite G-CW-complex.*

Then the functor $\mathbb{X} : G/P \mapsto \tilde{C}_*(X^P; \mathbb{F}_p)$ in $Ch_*(\mathbb{F}_p\mathbf{O}_p(G)^{\mathrm{op}} - mod)$ is quasi-isomorphic to a finite chain complex of projective $\mathbb{F}_p\mathbf{O}_p(G)^{\mathrm{op}}$-modules.

(Also with $\mathbb{Z}_p$ and $\mathbb{Z}_{(p)}$ coefficients.)

Theorem 3.2 (Compact model theorem, G-Smith)**.** *Let G be a finite group and X a G-CW-complex with $H_*(X; \mathbb{F}_p)$ finite.*

Then the functor $\mathbb{X} : G/P \mapsto \tilde{C}_*(X^{hP}; \mathbb{F}_p)$ in $Ch_*(\mathbb{F}_p\mathbf{O}_p(G)^{\mathrm{op}} - mod)$ is quasi-isomorphic to a finite chain complex of projective $\mathbb{F}_p\mathbf{O}_p(G)^{\mathrm{op}}$-modules.

(X^{hP} is enterpreted as profinite space — or assume fixed-points simply connected, or $H_*(X)$ nice e.g., like sphere.)

Note this second version is significantly more powerful, since only depends on the hG-homotopy type, and we have good tools for calculating the homotopy fixed-points that occur, as explained last time.

Unique minimal model, up to *isomorphism*.

Same type of argument as $H^*_{hP}(X, X^P)$ finite, $H^*_{hP}(X, X^{hP})$ finite.

- What does this model look like?
- What does it tell us about X? Note, if G is the trivial group, this model is enough to determine sphere, up to p-completion, but not a complete invariant very much beyond this! (though still useful...)

Notation: We call such an object an *algebraic sphere* if $H(\mathbb{X}(G/e))$ is one-dimensional.

Warmup question number 1a: What happens when $X = S^0$? It says that the constant functor on $\mathbb{F}_p\mathbf{O}_p(G)^{\mathrm{op}}$ has a finite projective resolution, i.e. that for any functor F

$$\lim_{\mathbb{F}_p\mathbf{O}_p(G)^{\mathrm{op}}} {}^*F = \mathrm{Ext}^*_{\mathbb{F}_p\mathbf{O}_p(G)^{\mathrm{op}}}(\mathbb{F}_p, F)$$

vanishes in large degrees. This is a celebrated result of Jackowski-McClure-Oliver. So, what is this projective resolution. There is obviously some relevant homological algebra to be understood here. To be continued....

3.2. **Basic homological algebra.**

What does the category $\mathbb{F}_p\mathbf{O}_p(G)^{\mathrm{op}}$ look like?

Simple objects: Non-zero on only one G/P, where the value is a simple $\mathbb{F}_pN_G(P)/P$-module.

Projective objects: Let S be a simple $\mathbb{F}_pN_G(P)/P$-module with projective cover P_S. Then the projective functors are sums of

$$G/Q \mapsto \mathbb{F}_p \operatorname{map}_G(G/Q, G/P) \otimes_{\mathbb{F}_pN_G(P)/P} P_S$$

I.e. finitely many simples, finitely many projectives, we can write them down in concrete cases.

Example 3.3 ($G = C_p$)**.** The opposite p-orbit category looks like

$$\begin{tikzcd} G/G \arrow[r] & G/e \arrow[loop above] \end{tikzcd}$$

The two simple functors are the atomic functors $S_1 = (k \to 0)$ and $S_2 = (0 \to k)$, which have projective covers $P_1 = P_{S_1} = (k \to k)$ and $P_2 = P_{S_2} = (0 \to kC_p)$ respectively.

We easily see that any algebraic sphere has a minimal model with the following structure

$$\cdots \to 0 \to P_2 \to P_2 \to \cdots \to P_2 \to P_1 \to 0 \to \cdots$$

Here P_1 is in homological degree $d(G/G)$ and the last P_2 is in homological degree $d(G/e)$. If p is odd, then the number of P_2, which equals $d(G/e) - d(G/G)$, has to be even.

Smith theory revisited: $G = \mathbb{Z}/p$ If $H(\mathbb{X}(G/e)) = 0$. Then $\mathbb{X} = 0$, so fixed-points also contractible. Look at top dimension. If $H^*(\mathbb{X}(G/e)) = F_p$. Then

(1) $\mathbb{X}(G/G) = \mathbb{F}_p$
(2) In lower dimension
(3) if p odd difference congruent to 0 mod 2

Could do $\mathbb{Z}/2 \times \mathbb{Z}/2$ and one easily get the "Borel" conditions.

Together, these are called the Borel-Smith conditions.

Note also: The sphere is determined by it's dimension.

Example 3.4 ($G = \Sigma_3$, $p = 2$, $char(k) = 2$)**.** The opposite p-orbit category looks like

$$\begin{tikzcd} G/D \arrow[r, Rightarrow] & G/e \arrow[loop above] \end{tikzcd}$$

There are three simple functors, namely $S_1 = (k \to 0)$, $S_2 = (0 \to k)$, and $S_3 = (0 \to \mathrm{St})$. These have projective covers $P_1 = (k \to k[G/D])$, $P_2 = (0 \to k[G/C_3])$, and $P_3 = (0 \to \mathrm{St})$.

We see that any minimal model for an algebraic G-sphere looks like

$$\cdots \to 0 \to P_2 \to P_2 \to \cdots P_2 \to P_2 \oplus P_3 \to P_1 \to 0 \to \cdots$$

where the number of P_2's can be any non-negative number. In terms of the dimension function $d(G/D)$ counts the homological degree of P_1 and $d(G/e) - d(G/D)$ counts the number of copies of P_2. Note that the homological dimension equals the dimension of the minimal model, except in the case where the algebraic sphere is trivial (i.e., except in the case where the dimension function has no jumps).

One can easily go through the obstruction theory in this case, and see that vanish.

Non-equivariantly p-complete spheres are determined by their dimension i.e., mod p cohomology.

Theorem 3.5 (G-Smith)**.** *Let G be a finite group. Then*
$\{p$-complete G-sphere$\} \to \{$Algebraic G-sphere$\} \to \{$oriented dimension function$\}$
completely determines p-complete G-spheres up to hG-equivalence.

If X and Y are p-complete G-spaces with the mod $\tilde{H}^*(X;\mathbb{F}_p) = \tilde{H}^*(Y;\mathbb{F}_p) = \mathbb{F}_p$. Then X and Y are hG-equivalent if and only if they have the same oriented dimension function.

Oriented just means that one has to remember the action of G on $H^*(X\mathbb{F}_p) \cong \mathbb{F}_p$ when p is odd.

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