# Holomorphic Day 2021

## Program, abstracts and practical information

November 19, 2021

University of Copenhagen, Copenhagen, Denmark

Version of November 4, 2021

The Conference is sponsored by the Danish Council for Independent Research | Natural Sciences via grant DFF – 1026-00267B.

## Practical information

The meeting is held at the Department of Mathematical Sciences of the University of Copenhagen

> Department of Mathematical Sciences
> Universitetsparken 5
> Auditorium 10
> 2100 Copenhagen Ø

The lectures take place in Auditorium 10. The room is equipped with blackboards, and a beamer if you want to use a laptop. The coffee breaks take place outside the auditorium.

You may access the eduroam or KU-guest networks.

We plan a common dinner in the evening. More information will given during the day.

## Program

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| 09:30 – 10:00 | Coffee |
| 10:00 – 10:50 | Torben Krüger<br>Systems of randomly coupled differential equations |
| 11:00 – 11:50 | Aron Wennman<br>Asymptotics of weighted Carleman polynomials |
| 11:50 – 13:30 | Lunch |
| 13:30 – 14:20 | Christian Berg<br>A family of entire functions connecting the Bessel function $J_1$ and the Lambert $W$ function |
| 14:20 – 15:00 | Coffee |
| 15:00 – 15:50 | John Erik Fornæss<br>Variations of the complex structure on complex Euclidean space $\mathbb{C}^2$ |
| 16:00 – 16:50 | Christian Henriksen<br>TBA |
| 18:30 – | Common dinner |

# Abstracts

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## A family of entire functions connecting the Bessel function $J_1$ and the Lambert $W$ function

CHRISTIAN BERG

UNIVERSITY OF COPENHAGEN, DENMARK

Consider the family of holomorphic functions

$$h_\alpha(z) := (1+1/z)^{\alpha z} := \exp(\alpha z \operatorname{Log}(1+1/z)), \quad \alpha \in \mathbb{C},$$

defined in the cut plane $\mathcal{A} := \mathbb{C}\setminus ]-\infty, 0]$. Here Log is the principal logarithm holomorphic in $\mathcal{A}$ and real on the interval $]0, \infty[$.

In [1] we found a family $\varphi_\alpha$, $\alpha \in \mathbb{C}$ of entire functions such that

$$f_\alpha(x) := e^\alpha - h_\alpha(x) = \int_0^\infty e^{-sx} \varphi_\alpha(s)\, ds, \quad x > 0.$$

Each function $\varphi_\alpha$ has an expansion in power series

$$\varphi_\alpha(z) = e^\alpha \sum_{n=0}^{\infty} (-1)^n p_{n+1}(\alpha) \frac{z^n}{n!}, \quad \alpha, z \in \mathbb{C},$$

and $p_n$ is a sequence of recursively defined polynomials of degree $n$.

The expansion leads to several properties of the functions $\varphi_\alpha$, which turn out to be related to the well known Bessel function $J_1$, when $\alpha$ is large, and to the Lambert $W$ function, when $\alpha$ is small.

It is possible to calculate the zeros of $\varphi_\alpha$ when $\alpha > 0$ and to obtain a very precise approximation of a number $\alpha^*$ such that $\varphi_\alpha(s) \geq 0$ when $s > 0$ precisely for $0 \leq \alpha \leq \alpha^*$. In [1] we found $\alpha^* \approx 2.29965\,64432\,53461\,30332$.

By a theorem of Bernstein $f_\alpha$ is completely monotonic precisely if $0 \leq \alpha \leq \alpha^*$.

It follows that the derivative $h'_\alpha$ is completely monotonic for these values of $\alpha$, and in particular $h'_2$ is completely monotonic.

In [2] we examined if $h_2'$ is an infinitely divisible completely monotonic function, i.e., if any positive power of $h_2'$ is also completely monotonic. This led us to introduce a new class of Bernstein functions, which we call **Horn-Bernstein** functions, namely the non-negative $C^\infty$ functions $h$ on $]0,\infty[$ for which $h'$ is an infinitely divisible completely monotonic function. We found a number $2 < \beta^* < \alpha^*$ such that $h_\alpha$ is a Horn-Bernstein function if and only if $0 \leq \alpha \leq \beta^*$, and we calculated $\beta^*$ with 20 decimals.

[1] C. Berg, E. Massa and A. P. Peron, *A family of entire functions connecting the Bessel function $J_1$ and the Lambert W function*. Constructive Approximation **53** (2021), 121–154.

[2] C. Berg and H.L. Pedersen, *A family of Horn-Bernstein functions*. To appear in Experimental Mathematics.

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## Variations of the complex structure on complex Euclidean space $\mathbb{C}^2$

JOHN ERIK FORNÆSS

NTNU, NORWAY

I will talk about some variations of the complex structures on two dimensional complex Euclidean space, $\mathbb{C}^2$. These arise in complex function theory and complex dynamics. Mainly I will discuss Short and Long $\mathbb{C}^2$.

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## TBA

CHRISTIAN HENRIKSEN

DTU, DENMARK

TBA

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## Systems of randomly coupled differential equations

Torben Krüger

University of Erlangen, Germany and University of Copenhagen, Denmark

Systems of coupled differential equations play an important role in the study of dynamical properties of high dimensional networks, e.g. neural nets and food webs. In many applications the interaction between agents in the network can be modelled as random, while each agent's individual activity decays over time if no input from other agents is received. In this setup the system displays three possible types of long time behaviour, depending on the interaction strength. Strongly coupled systems show chaotic dynamics, while the activity in weakly coupled systems decays exponentially. The critically tuned networks in between these two regimes exhibit a universal power law decay of the network activity with exponent $-1/4$. The details of the asymptotic analysis are determined by the analytic extension of the expected resolvent associated to the system's non-Hermitian connectivity matrix to the boundary of its pseudo-spectrum.

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## Asymptotics of weighted Carleman polynomials

Aron Wennman

Stockholm University, Sweden

In the 1920s, Carleman found a remarkably precise asymptotic formula for large degree orthogonal polynomials on Jordan domains with real-analytic boundaries. Similarly to Szegő's theorem on asymptotics of orthogonal polynomials on Jordan curves, the formula involves the exterior conformal mapping.

Later work in the direction initiated by Carleman has focused on domains with less regular boundaries and weighted settings, and the leading order behaviour is known in large generality. However, corrections beyond the main term have remained mysterious even in smooth settings.

In this talk, we will discuss a full asymptotic expansion of weighted Carleman polynomials, obtained in recent joint work with Hedenmalm. The asymptotic expansion can be thought of as truncation of a Neumann series for an explicit operator. The central idea is that of ′disintegrating′ the two-dimensional orthogonality relations into orthogonality along a curve family, which are better understood following Szegő.