An introduction to stochastic integration with respect to continuous semimartingales

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ISBN 978-87-7078-953-0

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Preface

This monograph concerns itself with the theory of continuous-time martingales with continuous paths and the theory of stochastic integration with respect to continuous semimartingales.

To set the scene for the theory to be developed, we consider an example. Assume given a probability space $(\Omega, \mathcal{F}, P)$ endowed with a Brownian motion W, and consider a continuous mapping $f : [0, \infty) \to \mathbb{R}$. We would like to understand whether it is possible to define an integral $\int_0^t f(s) \, \mathrm{d}W_s$ in a fashion analogous to ordinary Lebesgue integrals. In general, there is a correspondence between bounded signed measures on $[0, t]$ and mappings of finite variation on $[0, t]$. Therefore, if we seek to define the integral with respect to Brownian motion in a pathwise sense, that is, by defining $\int_0^t f(s) \, \mathrm{d}W(\omega)_s$ for each ω separately, by reference to ordinary Lebesgue integration theory, it is necessary that the sample paths $W(\omega)$ have finite variation. However, Brownian motion has the property that its sample paths are almost surely of infinite variation on all compact intervals. Our conclusion is that the integral of f with respect to W cannot in general be defined pathwisely by immediate reference to Lebesgue integration theory.

We are thus left to seek an alternate manner of defining the integral. A natural starting point is to consider Riemann sums of the form $\sum_{k=1}^{2^n} f(t_{k-1}^n)(W_{t_k^n} - W_{t_{k-1}^n})$, where $t_k^n = kt2^{-n}$, and attempt to prove their convergence in some sense, say, in $\mathcal{L}^2$. By the completeness of $\mathcal{L}^2$, it suffices to prove that the sequence of Riemann sums constitute a Cauchy sequence in $\mathcal{L}^2$. In order to show this, put $\eta(\beta) = \sup\{|f(s) - f(u)| \mid s, u \in [0, t], |s - u| \leq \beta\}$. $\eta(\beta)$ is called the modulus of continuity for f over $[0, t]$. As f is continuous, f is uniformly continuous on $[0, t]$, and therefore $\eta(\beta)$ tends to zero as β tends to zero. Now note that for $m \geq n$, with $\xi_k^{mn} = f(t_{k-1}^m) - f(t_{k'-1}^n)$ where k' is such that $t_{k'-1}^n \leq t_{k-1}^m < t_{k'}^n$, we find by the independent and normally distributed increments of Brownian motion that

$$\begin{aligned}
& E\left(\sum_{k=1}^{2^m} f(t^m_{k-1})(W_{t^m_k} - W_{t^m_{k-1}}) - \sum_{k=1}^{2^n} f(t^n_{k-1})(W_{t^n_k} - W_{t^n_{k-1}})\right)^2 \\
= \quad & E\left(\sum_{k=1}^{2^m} \xi^{mn}_k (W_{t^m_k} - W_{t^m_{k-1}})\right)^2 = \sum_{k=1}^{2^m} E\left(\xi^{mn}_k (W_{t^m_k} - W_{t^m_{k-1}})\right)^2 \\
\leq \quad & \eta(t2^{-n})^2 \sum_{k=1}^{2^m} E(W_{t^m_k} - W_{t^m_{k-1}})^2 = t\eta(t2^{-n})^2,
\end{aligned}$$

which tends to zero as n tends to infinity. We conclude that as m and n tend to infinity, the $\mathcal{L}^2$ distance between the corresponding Riemann sums tend to zero, and so the sequence of Riemann sums $\sum_{k=1}^{2^n} f(t^n_{k-1})(W_{t^n_k} - W_{t^n_{k-1}})$ is a Cauchy sequence in $\mathcal{L}^2$. Therefore, by completeness, the sequence converges in $\mathcal{L}^2$ to some limit. Thus, while we cannot in general obtain pathwise convergence of the Riemann sums, we can in fact obtain convergence in $\mathcal{L}^2$, and may then define the stochastic integral $\int_0^t f(s)\,\mathrm{d}W_s$ as the limit.

Our conclusion from the above deliberations is that we cannot in general define the stochastic integral with respect to a Brownian motion using ordinary Lebesgue integration theory, but in certain circumstances, we may define the integral using an alternate limiting procedure. This provides evidence that a theory of stochastic integration may be feasible. In the following chapters, we will develop such a theory.

The structure of what is to come is as follows. In Chapter 1, we will develop the basic tools of continuous-time martingale theory, as well as develop the general concepts used in the theory of continuous-time stochastic processes. Using these results, we will in Chapter 2 define the stochastic integral $\int_0^t H_s\,\mathrm{d}X_s$ for all processes such that X belongs to the class of processes known as continuous semimartingales, which in particular includes continuous martingales and processes with continuous paths of finite variation, and H is a process satisfying certain measurability and integrability conditions. In this chapter, we also prove some basic properties of the stochastic integral, such as the dominated convergence theorem and Itô's formula, which is the stochastic version of the fundamental theorem of analysis.

As regards the prerequisites for this text, the reader is assumed to have a reasonable grasp of basic analysis, measure theory and discrete-time martingale theory, as can be obtained through the books Carothers (2000), Ash (2000) and Rogers & Williams (2000a).

Finally, many warm thanks go to Niels Richard Hansen, Benjamin Falkeborg and Peter Uttenthal for their editorial comments on the original manuscript. Also, I would like in particular to thank Ernst Hansen and Martin Jacobsen for teaching me probability theory.

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København, September 2014

Chapter 1

Continuous-time stochastic processes

In this chapter, we develop the fundamental results of stochastic processes in continuous time, covering mostly some basic measurability results and the theory of continuous-time continuous martingales.

Section 1.1 is concerned with stopping times and various measurability properties for processes in continuous time. In Section 1.2, we introduce continuous martingales in continuous time. We define the spaces $\mathbf{c}\mathcal{M}$, $\mathbf{c}\mathcal{M}^u$ and $\mathbf{c}\mathcal{M}^b$ consisting of continuous martingales, uniformly integrable martingales and bounded martingales, respectively, all with initial value zero. Mainly by discretization and reference to the classical results from discrete-time martingale theory, we show that the main theorems of discrete-time martingale theory carry over almost verbatim to the continuous-time case.

In Section 1.3, we introduce the space $\mathbf{c}\mathcal{M}^2$ of continuous martingales bounded in $\mathcal{L}^2$ with initial value zero. Analogously to the special properties of $\mathcal{L}^2$ among the spaces $\mathcal{L}^p$ for $p \geq 1$, the space $\mathbf{c}\mathcal{M}^2$ has some particularly pleasant properties. We prove results on convergence properties of martingales in $\mathbf{c}\mathcal{M}^2$, we show a completeness property of $\mathbf{c}\mathcal{M}^2$ and a type of Riesz representation theorem for $\mathbf{c}\mathcal{M}^2$, and we use these results to demonstrate the existence of a process, the quadratic variation process, for elements of $\mathbf{c}\mathcal{M}^b$, which will be essential to our development of the stochastic integral.

Finally, in Section 1.4, we introduce the space $\mathbf{c}\mathcal{M}_\ell$ of continuous local martingales with initial value zero. We prove the basic stability properties of the space of local martingales and extend the notion of quadratic variation and quadratic covariation to continuous local martingales.

1.1 Measurability and stopping times

We begin by reviewing basic results on continuous-time stochastic processes. We will work in the context of a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. Here, Ω denotes some set, $\mathcal{F}$ is a σ-algebra on Ω, P is a probability measure on $(\Omega, \mathcal{F})$ and $(\mathcal{F}_t)_{t\geq 0}$ is a family of σ-algebras such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ whenever $0 \leq s \leq t$ and such that $\mathcal{F}_t \subseteq \mathcal{F}$ for all $t \geq 0$. We refer to $(\mathcal{F}_t)_{t\geq 0}$ as the filtration of the probability space. We define $\mathcal{F}_\infty = \sigma(\cup_{t\geq 0}\mathcal{F}_t)$. We will require that the filtered probability space satisfies certain regularity properties given in the following definition. Recall that a P null set of $\mathcal{F}$ is a set $F \subseteq \Omega$ with the property that there exists $G \in \mathcal{F}$ with $P(G) = 0$ such that $F \subseteq G$.

Definition 1.1.1. *A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is said to satisfy the usual conditions if it holds that the filtration is right-continuous in the sense that $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$ for all $t \geq 0$, and for all $t \geq 0$, $\mathcal{F}_t$ contains all P null sets of $\mathcal{F}$. In particular, all P null sets of $\mathcal{F}$ are $\mathcal{F}$ measurable.*

We will always assume that the usual conditions hold. Note that because of this permanent assumption, our results a priori only hold for such filtered probability spaces. Therefore, we also need to ensure that the usual conditions may be assumed in practical cases, for example when dealing with Brownian motion. These issues are considered in Section A.4.

A stochastic process is a family $(X_t)_{t\geq 0}$ of $\mathbb{R}$-valued random variables. The sample paths of the stochastic process X are the functions $t \mapsto X_t(\omega)$ for $\omega \in \Omega$. We refer to X_0 as the initial value of X. In particular, we say that X has initial value zero if X_0 is zero.

In the following, $\mathcal{B}$ denotes the Borel-σ-algebra on $\mathbb{R}$. We put $\mathbb{R}_+ = [0, \infty)$ and let $\mathcal{B}_+$ denote the Borel-σ-algebra on $\mathbb{R}_+$, and we let $\mathcal{B}_t$ denote the Borel-σ-algebra on $[0, t]$. We say that two processes X and Y are versions of each other if $P(X_t = Y_t) = 1$ for all $t \geq 0$. In this case, we say that Y is a version of X and vice versa. We say that two processes X and Y are indistinguishable if their sample paths are almost surely equal, in the sense that the set where X and Y are not equal is a null set, meaning that the set $\{\omega \in \Omega \mid \exists\, t \geq 0 : X_t(\omega) \neq Y_t(\omega)\}$

is a null set. We then say that X is a modification of Y and vice versa. We call a process evanescent if it is indistinguishable from the zero process, and we call a set $A \in \mathcal{B}_+ \otimes \mathcal{F}$ evanescent if the process 1_A is evanescent. We say that a result holds up to evanescence, or up to indistinguishability, if it holds except perhaps on an evanescent set.

We have the following three measurability concepts for stochastic processes.

Definition 1.1.2. *Let X be a stochastic process. We say that X is adapted if X_t is $\mathcal{F}_t$ measurable for all $t \geq 0$. We say that X is measurable if $(t, \omega) \mapsto X_t(\omega)$ is $\mathcal{B}_+ \otimes \mathcal{F}$ measurable. We say that X is progressive if $X_{|[0,t]\times\Omega}$, the restriction of X to $[0, t]\times\Omega$, is $\mathcal{B}_t \otimes \mathcal{F}_t$ measurable for $t \geq 0$.*

If a process X has sample paths which are all continuous, we say that X is continuous. Note that we require that all paths of X are continuous, not only that X has continuous paths almost surely. Next, we introduce the progressive σ-algebra Σ^π and consider its basic properties.

Lemma 1.1.3. *Let Σ^π be the family of sets $A \in \mathcal{B}_+ \otimes \mathcal{F}$ such that $A \cap [0, t] \times \Omega \in \mathcal{B}_t \otimes \mathcal{F}_t$ for all $t \geq 0$. Then Σ^π is a σ-algebra, and a process X is progressive if and only if it is Σ^π measurable.*

Proof. We first show that Σ^π is a σ-algebra. It holds that Σ^π contains $\mathbb{R}_+ \times \Omega$. If $A \in \Sigma^\pi$, we have $A \cap [0, t] \times \Omega \in \mathcal{B}_t \otimes \mathcal{F}_t$ for all $t \geq 0$. As $A^c \cap [0, t] \times \Omega = ([0, t] \times \Omega) \setminus (A \cap [0, t] \times \Omega)$, $A^c \cap [0, t] \times \Omega$ is the complement of $A \cap [0, t] \times \Omega$ relative to $[0, t] \times \Omega$. Therefore, as $\mathcal{B}_t \otimes \mathcal{F}_t$ is stable under complements, we find that $A^c \cap [0, t] \times \Omega$ is in $\mathcal{B}_t \otimes \mathcal{F}_t$ as well for all $t \geq 0$. Thus, Σ^π is stable under taking complements. Analogously, we find that Σ^π is stable under taking countable unions, and so Σ^π is a σ-algebra. As regards the statement on measurability, we first note for any $A \in \mathcal{B}$ the equality

$$\{(s, \omega) \in \mathbb{R}_+ \times \Omega \mid X(s, \omega) \in A\} \cap [0, t] \times \Omega = \{(s, \omega) \in [0, t] \times \Omega \mid X_{|[0,t]\times\Omega}(s, \omega) \in A\}.$$

Now assume that X is progressive. Fix a set $A \in \mathcal{B}$. From the above, we then obtain $\{(t, \omega) \in \mathbb{R}_+ \times \Omega \mid X(t, \omega) \in A\} \cap [0, t] \times \Omega \in \mathcal{B}_t \otimes \mathcal{F}_t$, so that X is Σ^π measurable. In order to obtain the converse implication, assume that X is Σ^π measurable. The above then shows $\{(t, \omega) \in [0, t] \times \Omega \mid X_{|[0,t]\times\Omega}(t, \omega) \in A\} \in \mathcal{B}_t \otimes \mathcal{F}_t$. Thus, being progressive is equivalent to being Σ^π measurable. □

Lemma 1.1.3 in particular shows that being progressive is the same as being measurable with respect to the progressive σ-algebra, which we also refer to as being progressively measurable.

Lemma 1.1.4. *Let X be adapted. If X has left-continuous paths, then X is progressive. If X has right-continuous paths, then X is progressive. In particular, if X is a continuous, adapted process, then X is progressive.*

Proof. First consider the case where X is adapted and has left-continuous paths. In this case, $X_t = \lim_n X^n_t$ pointwise, where X^n is the process $X^n_t = \sum_{k=0}^\infty X_{k2^{-n}} 1_{[k2^{-n},(k+1)2^{-n})}(t)$. Therefore, using the result from Lemma 1.1.3 that being progressive means measurability with respect to the σ-algebra Σ^π, we find that in order to show the result, it suffices to show that the process $t \mapsto X_{k2^{-n}} 1_{[k2^{-n},(k+1)2^{-n})}(t)$ is progressive for any $n \geq 1$ and $k \geq 0$, since in this case, X inherits measurability with respect to Σ^π as a limit of Σ^π measurable maps. In order to show that $t \mapsto X_{k2^{-n}} 1_{[k2^{-n},(k+1)2^{-n})}(t)$ is progressive, let $A \in \mathcal{B}$ with $0 \notin A$. For any $t \geq 0$, we then have

$$\begin{aligned} & \{(s,\omega) \in [0,t] \times \Omega \mid X_{k2^{-n}}(\omega) 1_{[k2^{-n},(k+1)2^{-n})}(s) \in A\} \\ = \ & [k2^{-n},(k+1)2^{-n}) \cap [0,t] \times (X_{k2^{-n}} \in A). \end{aligned}$$

If $k2^{-n} > t$, this is empty and so in $\mathcal{B}_t \otimes \mathcal{F}_t$, and if $k2^{-n} \leq t$, this is in $\mathcal{B}_t \otimes \mathcal{F}_t$ as a product of a set in $\mathcal{B}_t$ and a set in $\mathcal{F}_t$. Thus, in both cases, we obtain an element of $\mathcal{B}_t \otimes \mathcal{F}_t$, and from this we conclude that the restriction of $t \mapsto X_{k2^{-n}} 1_{[k2^{-n},(k+1)2^{-n})}(t)$ to $[0,t] \times \Omega$ is $\mathcal{B}_t \otimes \mathcal{F}_t$ measurable, demonstrating that the process is progressive. This shows that X is progressive.

Next, consider the case where X is adapted and has right-continuous paths. In this case, we fix $t \geq 0$ and define, for $0 \leq s \leq t$, $X^n_s = X_0 1_{\{0\}}(s) + \sum_{k=0}^{2^n-1} X_{t(k+1)2^{-n}} 1_{(tk2^{-n},t(k+1)2^{-n}]}(t)$. By right-continuity, $X_s = \lim_n X^n_s$ pointwise for $0 \leq s \leq t$. Also, each term in the sum defining X^n is $\mathcal{B}_t \otimes \mathcal{F}_t$ measurable, and therefore, X^n is $\mathcal{B}_t \otimes \mathcal{F}_t$ measurable. As a consequence, the restriction of X to $[0,t] \times \Omega$ is $\mathcal{B}_t \otimes \mathcal{F}_t$, and so X is progressive. This concludes the proof. □

Lemma 1.1.5. *Let X be continuous. If X_t is almost surely zero for all $t \geq 0$, X is evanescent.*

Proof. We claim that $\{\omega \in \Omega \mid \forall\, t \geq 0 : X_t(\omega) = 0\} = \cap_{q \in \mathbb{Q}_+} \{\omega \in \Omega \mid X_q(\omega) = 0\}$. The inclusion towards the right is obvious. In order to show the inclusion towards the left, assume that ω is such that $X_q(\omega)$ is zero for all $q \in \mathbb{Q}_+$. Let $t \in \mathbb{R}_+$. Since $\mathbb{Q}_+$ is dense in $\mathbb{R}_+$, there is a sequence (q_n) in $\mathbb{Q}_+$ converging to t. As X has continuous paths, $X(\omega)$ is continuous, and so $X_t(\omega) = \lim_n X_{q_n}(\omega) = 0$. This proves the inclusion towards the left. Now, as a countable intersection of almost sure sets again is an almost sure set, we find that $\cap_{q \in \mathbb{Q}_+} \{\omega \in \Omega \mid X_q(\omega) = 0\}$ is an almost sure set. Therefore, $\{\omega \in \Omega \mid \forall\, t \geq 0 : X_t(\omega) = 0\}$ is an almost sure set, showing that X is evanescent. □

Lemma 1.1.6. *Let X be progressive. Then X is measurable and adapted.*

Proof. By definition $\Sigma^\pi \subseteq \mathcal{B}_+ \otimes \mathcal{F}$. Therefore, as X is progressive, we have for any $A \in \mathcal{B}$ that $\{(t,\omega) \in \mathbb{R}_+ \times \Omega \mid X_t(\omega) \in A\} \in \Sigma^\pi \subseteq \mathcal{B}_+ \otimes \mathcal{F}$. This proves that X is measurable. To show that X is adapted, note that when X is progressive, $X_{|[0,t]\times\Omega}$ is $\mathcal{B}_t \otimes \mathcal{F}_t$ measurable, and therefore $\omega \mapsto X_t(\omega)$ is $\mathcal{F}_t$ measurable. □

Next, we define stopping times in continuous time and consider their interplay with measurability concepts on $\mathbb{R}_+ \times \Omega$. A stopping time is a random variable $T : \Omega \to [0,\infty]$ such that $(T \leq t) \in \mathcal{F}_t$ for any $t \geq 0$. We say that T is finite if T maps into $\mathbb{R}_+$. We say that T is bounded if T maps into a bounded subset of $\mathbb{R}_+$. If X is a stochastic process and T is a stopping time, we denote by X^T the process $X^T_t = X_{T\wedge t}$ and call X^T the process stopped at T. Furthermore, we define the stopping time σ-algebra $\mathcal{F}_T$ of events determined at T by putting $\mathcal{F}_T = \{A \in \mathcal{F} \mid A \cap (T \leq t) \in \mathcal{F}_t \text{ for all } t \geq 0\}$. Clearly, $\mathcal{F}_T$ is a σ-algebra, and if T is constant, the stopping time σ-algebra is the same as the filtration σ-algebra, in the sense that $\{A \in \mathcal{F} \mid A \cap (s \leq t) \in \mathcal{F}_t \text{ for all } t \geq 0\} = \mathcal{F}_s$.

Our first goal is to develop some basic results on stopping times and their interplay with stopping time σ-algebras. In the following, we use the notation that $S \wedge T = \min\{S,T\}$ and $S \vee T = \max\{S,T\}$.

Lemma 1.1.7. *The following statements hold about stopping times:*

1. *Any constant in $[0,\infty]$ is a stopping time.*
2. *A nonnegative variable T is a stopping time if and only if $(T < t) \in \mathcal{F}_t$ for $t \geq 0$.*
3. *If S and T are stopping times, so are $S \wedge T$, $S \vee T$ and $S + T$.*
4. *If T is a stopping time and $F \in \mathcal{F}_T$, then $T_F = T1_F + \infty 1_{F^c}$ is a stopping time as well.*
5. *If $S \leq T$, then $\mathcal{F}_S \subseteq \mathcal{F}_T$.*

Proof. **Proof of (1).** Let c be a constant in $\mathbb{R}_+$. Then $(c \leq t)$ is either $\emptyset$ or Ω, both of which are in $\mathcal{F}_t$ for any $t \geq 0$. Therefore, any constant c in $\mathbb{R}_+$ is a stopping time.

Proof of (2). Assume first that T is a stopping time. Then $(T < t) = \cup_{n=1}^\infty (T \leq t - \frac{1}{n}) \in \mathcal{F}_t$, since $(T \leq t - \frac{1}{n}) \in \mathcal{F}_{t-\frac{1}{n}}$. Conversely, assume $(T < t) \in \mathcal{F}_t$ for all $t \geq 0$. We then obtain

$(T \leq t) = \cap_{k=n}^{\infty}(T < t + \frac{1}{k})$ for all n. This shows $(T \leq t) \in \mathcal{F}_{t+\frac{1}{n}}$ for all $n \geq 1$. Since $(\mathcal{F}_t)$ is increasing and n is arbitrary, we find $(T \leq t) \in \cap_{n=1}^{\infty}\mathcal{F}_{t+\frac{1}{n}} = \cap_{n=1}^{\infty} \cap_{s \geq t+\frac{1}{n}} \mathcal{F}_s = \cap_{s>t}\mathcal{F}_s$. By right-continuity of the filtration, $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$, so we conclude $(T \leq t) \in \mathcal{F}_t$, proving that T is a stopping time.

Proof of (3). Assume that S and T are stopping times and let $t \geq 0$. We then have $(S \wedge T \leq t) = (S \leq t) \cup (T \leq t) \in \mathcal{F}_t$, so $S \wedge T$ is a stopping time. Likewise, we obtain $(S \vee T \leq t) = (S \leq t) \cap (T \leq t) \in \mathcal{F}_t$, so $S \vee T$ is a stopping time as well. Finally, consider the sum $S + T$. Let $n \geq 1$ and fix ω. If $S(\omega)$ and $T(\omega)$ are finite, there are $q, q' \in \mathbb{Q}_+$ such that $q \leq S(\omega) \leq q + \frac{1}{n}$ and $q' \leq T(\omega) \leq q' + \frac{1}{n}$. In particular, $q + q' \leq S(\omega) + T(\omega)$ and $S(\omega) + T(\omega) \leq q + q' + \frac{2}{n}$. Next, if $S(\omega) + T(\omega) \leq t$, it holds in particular that both $S(\omega)$ and $T(\omega)$ are finite. Therefore, with $\Theta_t = \{q, q' \in \mathbb{Q}_+ \mid q + q' \leq t\}$, we find

$$(S + T \leq t) = \cap_{n=1}^{\infty} \cup_{(q,q') \in \Theta_t} (S \leq q + \tfrac{1}{n}) \cap (T \leq q' + \tfrac{1}{n}).$$

Now, the sequence of sets $\cup_{(q,q')\in\Theta_t}(S \leq q + \frac{1}{n}) \cap (T \leq q' + \frac{1}{n})$ is decreasing in n, and therefore we have for any $k \geq 1$ that $(S + T \leq t) = \cap_{n=k}^{\infty} \cup_{(q,q')\in\Theta_t} (S \leq q + \frac{1}{n}) \cap (T \leq q' + \frac{1}{n}) \in \mathcal{F}_{t+\frac{1}{k}}$. In particular, $(S + T \leq t) \in \mathcal{F}_s$ for any $s > t$, and so, by right-continuity of the filtration, $(S + T \leq t) \in \cap_{s>t}\mathcal{F}_s = \mathcal{F}_t$, proving that $S + T$ is a stopping time.

Proof of (4). Let T be a stopping time and let $F \in \mathcal{F}_T$. Then $(T_F \leq t) = (T \leq t) \cap F \in \mathcal{F}_t$, as was to be proven.

Proof of (5). Let $A \in \mathcal{F}_S$, so that $A \cap (S \leq t) \in \mathcal{F}_t$ for all $t \geq 0$. Since $S \leq T$, we have $(T \leq t) \subseteq (S \leq t)$ and so $A \cap (T \leq t) = A \cap (S \leq t) \cap (T \leq t) \in \mathcal{F}_t$, yielding $A \in \mathcal{F}_T$. $\square$

For the next results, we recall that for any $A \subseteq \mathbb{R}$, it holds that $\inf A < t$ if and only if there is $s \in A$ such that $s < t$.

Lemma 1.1.8. *Let (T_n) be a sequence of stopping times, then $\sup_n T_n$ and $\inf_n T_n$ are stopping times as well.*

Proof. Assume that T_n is a stopping time for each n. Fix $t \geq 0$, we then then have that $(\sup_n T_n \leq t) = \cap_{n=1}^{\infty}(T_n \leq t) \in \mathcal{F}_t$, so $\sup_n T_n$ is a stopping time as well. Likewise, using the second statement of Lemma 1.1.7, we find $(\inf_n T_n < t) = \cup_{n=1}^{\infty}(T_n < t) \in \mathcal{F}_t$, so $\inf_n T_n$ is a stopping time as well. $\square$

Lemma 1.1.9. *Let X be a continuous adapted process, and let U be an open set in $\mathbb{R}$. Define $T = \inf\{t \geq 0 \mid X_t \in U\}$. Then T is a stopping time.*

Proof. Note that if $s \geq 0$ and (s_n) is a sequence converging to s, we have by continuity that X_{s_n} converges to X_s and so, since U is open, $(X_s \in U) \subseteq \cup_{n=1}^\infty (X_{s_n} \in U)$. Using that $\mathbb{Q}_+$ is dense in $\mathbb{R}_+$, we find

$$\begin{aligned}(T < t) &= (\ \exists\, s \in \mathbb{R}_+ : s < t \text{ and } X_s \in U)\\ &= (\ \exists\, s \in \mathbb{Q}_+ : s < t \text{ and } X_s \in U) = \cup_{s \in \mathbb{Q}_+, s<t}(X_s \in U),\end{aligned}$$

and since X is adapted, we have $(X_s \in U) \in \mathcal{F}_s \subseteq \mathcal{F}_t$ whenever $s < t$, proving that $(T < t) \in \mathcal{F}_t$. By Lemma 1.1.7, this implies that T is a stopping time. □

Lemma 1.1.10. *Let X be a continuous adapted process, and let F be a closed set in $\mathbb{R}$. Define $T = \inf\{t \geq 0 \mid X_t \in F\}$. Then T is a stopping time.*

Proof. Define $U_n = \{x \in \mathbb{R} \mid \exists y \in F : |x - y| < \frac{1}{n}\}$. We claim that U_n is open, that U_n decreases to F and that $\overline{U}_{n+1} \subseteq U_n$, where $\overline{U}_{n+1}$ denotes the closure of U_{n+1}. As we have that $U_n = \cup_{y \in F}\{x \in \mathbb{R} \mid |x - y| < \frac{1}{n}\}$, U_n is open as a union of open sets. We have $F \subseteq U_n$ for all n, and conversely, if $x \in \cap_{n=1}^\infty U_n$, we have that there is a sequence (y_n) in F such that $|y_n - x| \leq \frac{1}{n}$ for all n. In particular, y_n tends to x, and as F is closed, we conclude $x \in F$. Thus, $F = \cap_{n=1}^\infty U_n$. Furthermore, if x is in the closure $\overline{U}_{n+1}$, there is a sequence (x_k) in U_{n+1} such that $|x_k - x| \leq \frac{1}{k}$, and there is a sequence (y_k) in F such that $|x_k - y_k| < \frac{1}{n+1}$, showing that $|x - y_k| < \frac{1}{k} + \frac{1}{n+1}$. Taking k so large that $\frac{1}{k} + \frac{1}{n+1} \leq \frac{1}{n}$, we see that $\overline{U}_{n+1} \subseteq U_n$.

Now note that whenever $t > 0$, we have

$$(T < t) = (\ \exists\, s \in \mathbb{R}_+ : s < t \text{ and } X_s \in F) = \cup_{n=1}^\infty (\ \exists\, s \in \mathbb{R}_+ : s \leq t - \tfrac{1}{n} \text{ and } X_s \in F),$$

so by Lemma 1.1.7, it suffices to prove that $(\ \exists\, s \in \mathbb{R}_+ : s \leq t \text{ and } X_s \in F)$ is $\mathcal{F}_t$ measurable for all $t > 0$. We claim that

$$(\ \exists\, s \in \mathbb{R}_+ : s \leq t \text{ and } X_s \in F) = \cap_{k=1}^\infty (\ \exists\, q \in \mathbb{Q}_+ : q \leq t \text{ and } X_q \in U_k).$$

To see this, first consider the inclusion towards the right. If there is $s \in \mathbb{R}_+$ with $s \leq t$ and $X_s \in F$, then we also have $X_s \in U_k$ for all k. As U_k is open, there is $\varepsilon > 0$ such that the ball of size ε around X_s is in U_k. In particular, there is $q \in \mathbb{Q}_+$ with $q \leq t$ such that $X_q \in U_k$. This proves the inclusion towards the right. In order to obtain the inclusion towards the left, assume that for all k, there is $q_k \in \mathbb{Q}_+$ with $q_k \leq t$ such that $X_{q_k} \in U_k$. As $[0, t]$ is compact, there exists $s \in \mathbb{R}_+$ with $s \leq t$ such that for some subsequence, $\lim_m q_{k_m} = s$. By continuity, $X_s = \lim_m X_{q_{k_m}}$. As $X_{q_{k_m}} \in U_{k_m}$, we have for any i that $X_{q_{k_m}} \in U_i$ for m large enough. Therefore, we conclude $X_s \in \cap_{i=1}^\infty \overline{U}_i \subseteq \cap_{i=1}^\infty U_i = F$, proving the other inclusion. This shows the desired result. □

Lemma 1.1.11. *Let X be any continuous adapted process with initial value zero. Defining $T_n = \inf\{t \geq 0 \mid |X_t| > n\}$, (T_n) is a sequence of stopping times increasing pointwise to infinity, and the process X^{T_n} is bounded by n.*

Proof. By Lemma 1.1.9, (T_n) is a sequence of stopping times. We prove that X^{T_n} is bounded by n. If T_n is infinite, $X_t \leq n$ for all $t \geq 0$, so on $(T_n = \infty)$, X^{T_n} is bounded by n. If T_n is finite, note that for all $\varepsilon > 0$, there is $t \geq 0$ with $T_n \leq t < T_n + \varepsilon$ such that $|X_t| > n$. Therefore, by continuity, $|X_{T_n}| \geq n$. In particular, as X has initial value zero, T_n cannot take the value zero. Therefore, there is $t \geq 0$ with $t < T_n$. For all such t, $|X_t| \leq n$. Therefore, again by continuity, $|X_{T_n}| \leq n$, and we conclude that in this case as well, X^{T_n} is bounded by n. Note that we have also shown that $|X_{T_n}| = n$ whenever T_n is finite.

It remains to show that T_n converges almost surely to infinity. To obtain this, note that as X is continuous, X is bounded on compacts. If for some samle path we have that $T_n \leq a$ for all n, we would have $|X_{T_n}| = n$ for all n and so X would be unbounded on $[0, a]$. This is a contradiction, since X has continuous sample paths and therefore is bounded on compact sets. Therefore, (T_n) is unbounded for every sample path. As T_n is increasing, this shows that T_n converges to infinity pointwise. □

Lemma 1.1.12. *Let X be progressively measurable, and let T be a stopping time. Then $X_T 1_{(T<\infty)}$ is $\mathcal{F}_T$ measurable and X^T is progressively measurable.*

Proof. We first prove that the stopped process X^T is progressively measurable. Fix $t \geq 0$, we need to show that $X^T_{|[0,t]\times\Omega}$ is $\mathcal{B}_t \otimes \mathcal{F}_t$ measurable, which means that we need to show that the mapping from $[0,t] \times \Omega$ to $\mathbb{R}$ given by $(s, \omega) \mapsto X_{T(\omega)\wedge s}(\omega)$ is $\mathcal{B}_t \otimes \mathcal{F}_t$-$\mathcal{B}$ measurable. To this end, note that whenever $0 \leq s \leq t$,

$$\{(u, \omega) \in [0, t] \times \Omega \mid T(\omega) \wedge u \leq s\} \quad = \quad ([0, t] \times (T \leq s)) \cup ([0, s] \times \Omega) \in \mathcal{B}_t \otimes \mathcal{F}_t,$$

so the mapping from $[0,t] \times \Omega$ to $[0,t]$ given by $(s, \omega) \mapsto T(\omega) \wedge s$ is $\mathcal{B}_t \otimes \mathcal{F}_t$-$\mathcal{B}_t$ measurable. And as the mapping from $[0,t] \times \Omega$ to Ω given by $(s, \omega) \mapsto \omega$ is $\mathcal{B}_t \otimes \mathcal{F}_t$-$\mathcal{F}_t$ measurable, we conclude that the mapping from $[0,t] \times \Omega$ to $[0,t] \times \Omega$ given by $(s, \omega) \mapsto (T(\omega) \wedge s, \omega)$ is $\mathcal{B}_t \otimes \mathcal{F}_t$-$\mathcal{B}_t \otimes \mathcal{F}_t$ measurable, since it has measurable coordinates. As X is progressive, the mapping from $[0,t] \times \Omega$ to $\mathbb{R}$ given by $(s, \omega) \mapsto X_s(\omega)$ is $\mathcal{B}_t \otimes \mathcal{F}_t$-$\mathcal{B}$ measurable. Therefore, the composite mapping from $[0,t] \times \Omega$ to $\mathbb{R}$ given by $(s, \omega) \mapsto X_{T(\omega)\wedge s}(\omega)$ is $\mathcal{B}_t \otimes \mathcal{F}_t$-$\mathcal{B}$ measurable. This shows that X^T is progressively measurable.

In order to prove that X_T is $\mathcal{F}_T$ measurable, we note that for any $B \in \mathcal{B}$, we have that $(X_T 1_{(T<\infty)} \in B) \cap (T \leq t) = (X^T_t \in B) \cap (T \leq t)$. Now, X^T_t is $\mathcal{F}_t$ measurable since X^T is progressive and therefore adapted by Lemma 1.1.6, and $(T \leq t) \in \mathcal{F}_t$ since T is a stopping time. Thus, $(X_T 1_{(T<\infty)} \in B) \cap (T \leq t) \in \mathcal{F}_t$, and we conclude $(X_T 1_{(T<\infty)} \in B) \in \mathcal{F}_T$. $\square$

Lemma 1.1.13. *Let T and S be stopping times. Assume that Z is $\mathcal{F}_S$ measurable. It then holds that both $Z1_{(S<T)}$ and $Z1_{(S\leq T)}$ are $\mathcal{F}_{S\wedge T}$ measurable.*

Proof. We first show $(S < T) \in \mathcal{F}_{S\wedge T}$. To prove the result, it suffices to show that the set $(S < T) \cap (S \wedge T \leq t)$ is in $\mathcal{F}_t$ for all $t \geq 0$. To this end, we begin by noting that $(S < T) \cap (S \wedge T \leq t) = (S < T) \cap (S \leq t)$. Consider some $\omega \in \Omega$ such that $S(\omega) < T(\omega)$ and $S(\omega) \leq t$. If $t < T(\omega)$, $S(\omega) \leq t < T(\omega)$. If $T(\omega) \leq t$, there is some $q \in \mathbb{Q} \cap [0, t]$ such that $S(\omega) \leq q < T(\omega)$. We thus obtain $(S < T) \cap (S \wedge T \leq t) = \cup_{q\in\mathbb{Q}\cap[0,t]\cup\{t\}}(S \leq q) \cap (q < T)$, which is in $\mathcal{F}_t$, showing $(S < T) \in \mathcal{F}_{S\wedge T}$. We next show that $Z1_{(S<T)}$ is $\mathcal{F}_{S\wedge T}$ measurable. Let $B \in \mathcal{B}$ with B not containing zero. As this type of sets generate $\mathcal{B}$, it will suffice to show that $(Z1_{(S<T)} \in B) \cap (S \wedge T \leq t) \in \mathcal{F}_t$ for all $t \geq 0$. To obtain this, we rewrite

$$\begin{aligned}(Z1_{(S<T)} \in B) \cap (S \wedge T \leq t) &= (Z \in B) \cap (S < T) \cap (S \wedge T \leq t) \\ &= (Z \in B) \cap (S < T) \cap (S \leq t).\end{aligned}$$

Since Z is $\mathcal{F}_S$ measurable, $(Z \in B) \cap (S \leq t) \in \mathcal{F}_t$. And by what we have already shown, $(S < T) \in \mathcal{F}_S$, so $(S < T) \cap (S \leq t) \in \mathcal{F}_t$. Thus, the above is in $\mathcal{F}_t$, as desired.

Finally, we show that $Z1_{(S\leq T)}$ is $\mathcal{F}_{S\wedge T}$ measurable. Let $B \in \mathcal{B}$ with B not containing zero. As above, it suffices to show that for any $t \geq 0$, $(Z1_{(S\leq T)} \in B) \cap (S \wedge T \leq t) \in \mathcal{F}_t$. To obtain this, we first write

$$\begin{aligned}(Z1_{(S\leq T)} \in B) \cap (S \wedge T \leq t) &= (Z \in B) \cap (S \leq T) \cap (S \wedge T \leq t) \\ &= (Z \in B) \cap (S \leq t) \cap (S \leq T) \cap (S \wedge T \leq t).\end{aligned}$$

Since $Z \in \mathcal{F}_S$, we find $(Z \in B) \cap (S \leq t) \in \mathcal{F}_t$. And since we know $(T < S) \in \mathcal{F}_{T\wedge S}$, $(S \leq T) = (T < S)^c \in \mathcal{F}_{S\wedge T}$, so $(S \leq T) \cap (S \wedge T \leq t) \in \mathcal{F}_t$. This demonstrates $(Z1_{(S\leq T)} \in B) \cap (S \wedge T \leq t) \in \mathcal{F}_t$, as desired. $\square$

1.2 Continuous-time martingales

In this section, we consider continuous martingales in continuous time. We say that a process M is a continuous-time martingale if for any $0 \leq s \leq t$, $E(M_t|\mathcal{F}_s) = M_s$ almost surely. In the same manner, if for any $0 \leq s \leq t$, $E(M_t|\mathcal{F}_s) \leq M_s$ almost surely, we say that M is a supermartingale, and if for any $0 \leq s \leq t$, $E(M_t|\mathcal{F}_s) \geq M_s$ almost surely, we say that M is a submartingale. We are interested in transferring the results known from discrete-time martingales to the continuous-time setting, mainly the criteria for almost sure convergence, $\mathcal{L}^1$ convergence and the optional sampling theorem. The classical results from discrete-time martingale theory are reviewed in Appendix A.3.

We will for the most part only take interest in martingales M whose initial value is zero, $M_0 = 0$, in order to simplify the exposition. We denote the space of continuous martingales in continuous time with initial value zero by $\mathbf{c}\mathcal{M}$. By $\mathbf{c}\mathcal{M}^u$, we denote the elements of $\mathbf{c}\mathcal{M}$ which are uniformly integrable, and by $\mathbf{c}\mathcal{M}^b$, we denote the elements of $\mathbf{c}\mathcal{M}^u$ which are bounded in the sense that there exists $c > 0$ such that $|M_t| \leq c$ for all $t \geq 0$. Clearly, $\mathbf{c}\mathcal{M}$ and $\mathbf{c}\mathcal{M}^b$ are both vector spaces, and by Lemma A.2.4, $\mathbf{c}\mathcal{M}^u$ is a vector space as well.

We begin by presenting our most basic example of a continuous martingale in continuous time, the p-dimensional $\mathcal{F}_t$ Brownian motion. Recall from Appendix A.4 that a p-dimensional Brownian motion is a continuous process W with values in $\mathbb{R}^p$ such that the increments are independent over disjoint intervals, and for $0 \leq s \leq t$, $W_t - W_s$ follows a p-dimensional normal distribution with mean zero and variance $(t-s)I_p$, where I_p is the identity matrix of order p. Furthermore, as in Definition A.4.4, a p-dimensional $\mathcal{F}_t$ Brownian motion is a process W with values in $\mathbb{R}^p$ adapted to $(\mathcal{F}_t)$ such that for any $t \geq 0$, the distribution of $s \mapsto W_{t+s} - W_t$ is an p-dimensional Brownian motion independent of $\mathcal{F}_t$. The difference between a plain p-dimensional Brownian motion and a p-dimensional $\mathcal{F}_t$ Brownian motion is that the p-dimensional $\mathcal{F}_t$ Brownian motion possesses a certain regular relationship with the filtration. The following basic result shows that the martingales associated with ordinary Brownian motions reoccur when considering $\mathcal{F}_t$ Brownian motions.

Theorem 1.2.1. *Let W be a p-dimensional $\mathcal{F}_t$ Brownian motion. For $i \leq p$, W^i and $(W^i_t)^2 - t$ are martingales, where W^i denotes the i'th coordinate of W. For $i, j \leq p$ with $i \neq j$, $W^i_t W^j_t$ is a martingale.*

Proof. Let $i \leq p$ and let $0 \leq s \leq t$. W^i is then an $\mathcal{F}_t$ Brownian motion, so $W^i_t - W^i_s$ is normally distributed with mean zero and variance $t - s$ and independent of $\mathcal{F}_s$. Therefore, we obtain $E(W^i_t|\mathcal{F}_s) = E(W^i_t - W^i_s|\mathcal{F}_s) + W^i_s = E(W^i_t - W^i_s) + W^i_s = W^i_s$, proving that W^i is a martingale. Furthermore, we find

$$\begin{aligned} E((W^i_t)^2 - t|\mathcal{F}_s) &= E((W^i_t - W^i_s)^2 - (W^i_s)^2 + 2W^i_s W^i_t|\mathcal{F}_s) - t \\ &= E((W^i_t - W^i_s)^2|\mathcal{F}_s) - (W^i_s)^2 + 2W^i_s E(W^i_t|\mathcal{F}_s) - t = (W^i_s)^2 - s, \end{aligned}$$

so $(W^i_t)^2 - t$ is a martingale. Next, let $i, j \leq p$ with $i \neq j$. We then obtain that for $0 \leq s \leq t$, using independence and the martingale property,

$$\begin{aligned}
E(W_t^i W_t^j|\mathcal{F}_s) &= E(W_t^i W_t^j - W_s^i W_s^j|\mathcal{F}_s) + W_s^i W_s^j \\
&= E(W_t^i W_t^j - W_t^i W_s^j + W_t^i W_s^j - W_s^i W_s^j|\mathcal{F}_s) + W_s^i W_s^j \\
&= E(W_t^i (W_t^j - W_s^j) - W_s^j (W_t^i - W_s^i)|\mathcal{F}_s) + W_s^i W_s^j \\
&= E(W_t^i (W_t^j - W_s^j)|\mathcal{F}_s) + W_s^i W_s^j \\
&= E((W_t^i - W_s^i)(W_t^j - W_s^j)|\mathcal{F}_s) + E(W_s^i (W_t^j - W_s^j)|\mathcal{F}_s) + W_s^i W_s^j \\
&= W_s^i W_s^j,
\end{aligned}$$

where we have used that the variables $E(W_s^j(W_t^i - W_s^i)|\mathcal{F}_s)$, $E((W_t^i - W_s^i)(W_t^j - W_s^j)|\mathcal{F}_s)$ and $E(W_s^i(W_t^j - W_s^j)|\mathcal{F}_s)$ are all equal to zero, because $t \mapsto W_{t+s} - W_s$ is independent of $\mathcal{F}_s$ and has the distribution of a p-dimensional Brownian motion. Thus, $W^i W^j$ is a martingale. □

Furthermore, when W is a p-dimensional $\mathcal{F}_t$ Brownian motion, W^i has the distribution of a Brownian motion, so all ordinary distributional results for Brownian motions transfer verbatim to $\mathcal{F}_t$ Brownian motions, for example that the following results hold almost surely:

$$\limsup_{t\to\infty} \frac{W_t^i}{\sqrt{2t\log\log t}} = 1, \liminf_{t\to\infty} \frac{W_t^i}{\sqrt{2t\log\log t}} = -1, \lim_{t\to\infty} \frac{W_t^i}{t} = 0.$$

After introducing this central example, we will in the remainder of this section work on transferring the results of discrete-time martingale theory to continuous-time martingale theory. The main lemma for doing so is the following.

Lemma 1.2.2. *Let M be a continous-time martingale, supermartingale or submartingale, and let (t_n) be an increasing sequence in $\mathbb{R}_+$. Then $(\mathcal{F}_{t_n})_{n\geq 1}$ is a discrete-time filtration, and the process $(M_{t_n})_{n\geq 1}$ is a discrete-time martingale, supermartingale or submartingale, respectively, with respect to the filtration $(\mathcal{F}_{t_n})_{n\geq 1}$.*

Proof. This follows immediately from the definition of continuous-time and discrete-time martingales, supermartingales and submartingales. □

Lemma 1.2.3 (Doob's upcrossing lemma)**.** *Let Z be a continuous supermartingale bounded in $\mathcal{L}^1$. Define $U(Z,a,b) = \sup\{n \mid \exists\, 0 \leq s_1 < t_1 < \cdots s_n < t_n : Z_{s_k} < a, Z_{t_k} > b, k \leq n\}$ for any $a, b \in \mathbb{R}$ with $a < b$. We refer to $U(Z,a,b)$ as the number of upcrossings from a to b by Z. Then $U(Z,a,b)$ is $\mathcal{F}$ measurable and it holds that*

$$EU(Z,a,b) \leq \frac{|a| + \sup_t E|Z_t|}{b-a}.$$

Proof. We will prove the result by reducing to the case of upcrossings relative to a countable number of timepoints and applying Lemma 1.2.2 and the discrete-time upcrossing result of Lemma A.3.1. For any $D \subseteq \mathbb{R}$, we define

$$U(Z,a,b,D) = \sup\{m \mid \exists\, 0 \le s_1 < t_1 < \cdots s_m < t_m : s_i, t_i \in D, Z_{s_i} < a, Z_{t_i} > b, i \le m\},$$

and we refer to $U(Z,a,b,D)$ as the number of upcrossings from a to b at the timepoints in D. Define $\mathbb{D}_+ = \{k2^{-n} \mid k \ge 0, n \ge 1\}$, we refer to $\mathbb{D}_+$ as the dyadic nonnegative rationals. It holds that $\mathbb{D}_+$ is dense in $\mathbb{R}_+$. Now as Z is continuous, we find that for any finite sequence $0 \le s_1 < t_1 < \cdots s_m < t_m$ such that $s_i, t_i \in \mathbb{R}_+$ with $Z_{s_i} < a$ and $Z_{t_i} > b$ for $i \le m$, there exists $0 \le p_1 < q_1 < \cdots p_m < q_m$ such that $p_i, q_i \in \mathbb{D}_+$ with $Z_{p_i} < a$ and $Z_{q_i} > b$ for $i \le m$. Therefore, $U(Z,a,b) = U(Z,a,b,\mathbb{D}_+)$. In other words, it suffices to consider upcrossings at dyadic nonnegative rational timepoints. In order to use this to prove that $U(Z,a,b)$ is $\mathcal{F}$ measurable, note that for any $m \ge 1$, we have

$$\begin{aligned} &(\,\exists\, 0 \le s_1 < t_1 < \cdots s_m < t_m : s_i, t_i \in \mathbb{D}_+, Z_{s_i} < a, Z_{t_i} > b, i \le m) \\ = \;& \cup\{(Z_{s_i} < a, Z_{t_i} > b \text{ for all } i \le m) \mid 0 \le s_1 < t_1 < \cdots s_m < t_m : s_i, t_i \in \mathbb{D}_+\}, \end{aligned}$$

which is in $\mathcal{F}$, as $(Z_{s_i} < a, Z_{t_i} > b$ for all $i \le m)$ is $\mathcal{F}$ measurable, and all subsets of $\cup_{n=1}^\infty \mathbb{D}_+^n$ are countable. Here, $\mathbb{D}_+^n$ denotes the n-fold product of $\mathbb{D}_+$. From these observations, we conclude that the set $(\,\exists\, 0 \le s_1 < t_1 < \cdots s_m < t_m : s_i, t_i \in \mathbb{D}_+, Z_{s_i} < a, Z_{t_i} > b, i \le m)$ is $\mathcal{F}$ measurable. Denote this set by A_m, we then have $U(Z,a,b)(\omega) = \sup\{m \in \mathbb{N} \mid \omega \in A_m\}$, so that in particular $(U(Z,a,b) \ge m) = \cup_{k=m}^\infty A_k \in \mathcal{F}$ and so $U(Z,a,b)$ is $\mathcal{F}$ measurable.

It remains to prove the bound for the mean of $U(Z,a,b)$. Putting $t_k^n = k2^{-n}$ and defining $D_n = \{t_k^n \mid k \ge 0\}$, we obtain $\mathbb{D}_+ = \cup_{n=1}^\infty D_n$. We then have

$$\begin{aligned} & \sup\{m \mid \exists\, 0 \le s_1 < t_1 < \cdots s_m < t_m : s_i, t_i \in \mathbb{D}_+, Z_{s_i} < a, Z_{t_i} > b, i \le m\} \\ = \;& \sup \cup_{n=1}^\infty \{m \mid \exists\, 0 \le s_1 < t_1 < \cdots s_m < t_m : s_i, t_i \in D_n, Z_{s_i} < a, Z_{t_i} > b, i \le m\} \\ = \;& \sup_n \sup\{m \mid \exists\, 0 \le s_1 < t_1 < \cdots s_m < t_m : s_i, t_i \in D_n, Z_{s_i} < a, Z_{t_i} > b, i \le m\}, \end{aligned}$$

so $U(Z,a,b,\mathbb{D}_+) = \sup_n U(Z,a,b,D_n)$. Now fix $n \in \mathbb{N}$. As $(t_k^n)_{k \ge 0}$ is an increasing sequence, Lemma 1.2.2 shows that $(Z_{t_k^n})_{k \ge 0}$ is a discrete-time supermartingale with respect to the filtration $(\mathcal{F}_{t_k^n})_{k \ge 0}$. As $(Z_t)_{t \ge 0}$ is bounded in $\mathcal{L}^1$, so is $(Z_{t_k^n})_{k \ge 0}$. Therefore, Lemma A.3.1 yields

$$EU(Z,a,b,D_n) \le \frac{|a| + \sup_k E|Z_{t_k^n}|}{b-a} \le \frac{|a| + \sup_t E|Z_t|}{b-a}.$$

As (D_n) is increasing, $U(Z,a,b,D_n)$ is increasing, so the monotone convergence theorem and our previous results yield

$$\begin{aligned} EU(Z,a,b) &= EU(Z,a,b,\mathbb{D}_+) = E\sup_n U(Z,a,b,D_n) \\ &= E\lim_n U(Z,a,b,D_n) = \lim_n EU(Z,a,b,D_n) \le \frac{|a| + \sup_t E|Z_t|}{b-a}. \end{aligned}$$

This concludes the proof of the lemma. □

Theorem 1.2.4 (Doob's supermartingale convergence theorem)**.** *Let Z be a continuous supermartingale. If Z is bounded in $\mathcal{L}^1$, then Z is almost surely convergent to an integrable limit. If Z is uniformly integrable, then Z also converges in $\mathcal{L}^1$, and the limit Z_∞ satisfies that for all $t \ge 0$, $E(Z_\infty|\mathcal{F}_t) \le Z_t$ almost surely. If Z is a martingale, the inequality may be exchanged with an equality.*

Proof. Assume that Z is bounded in $\mathcal{L}^1$. Fix $a, b \in \mathbb{Q}$ with $a < b$. By Lemma 1.2.3, the number of upcrossings from a to b made by Z has finite expectation, in particular it is almost surely finite. As $\mathbb{Q}$ is countable, we conclude that it almost surely holds that the number of upcrossings from a to b made by Z is finite for any $a, b \in \mathbb{Q}$. Therefore, Lemma A.1.16 shows that Z is almost surely convergent to a limit in $[-\infty, \infty]$. Using Fatou's lemma, we obtain $E|Z_\infty| = E\liminf_t |Z_t| \le \liminf_t E|Z_t| \le \sup_{t\ge 0} E|Z_t|$, which is finite, so we conclude that the limit Z_∞ is integrable.

Assume next that Z is uniformly integrable. In particular, Z is bounded in $\mathcal{L}^1$, so Z_t converges almost surely to some variable Z_∞. Then Z_t also converges in probability, so Lemma A.2.5 shows that Z_t converges to Z_∞ in $\mathcal{L}^1$. We then find that for any $t \ge 0$ that, using Jensen's inequality, $E|E(Z_\infty|\mathcal{F}_t) - E(Z_s|\mathcal{F}_t)| \le E|Z_\infty - Z_s|$, so $E(Z_s|\mathcal{F}_t)$ tends to $E(Z_\infty|\mathcal{F}_t)$ in $\mathcal{L}^1$ as s tends to infinity, and we get $E(Z_\infty|\mathcal{F}_t) = \lim_{s\to\infty} E(Z_s|\mathcal{F}_t) \le Z_t$. This proves the results on supermartingales.

In order to obtain the results for the martingale case, next assume that Z is a continuous submartingale bounded in $\mathcal{L}^1$. Then $-Z$ is a continuous supermartingale bounded in $\mathcal{L}^1$. From what we already have proved, $-Z$ is almost surely convergent to a finite limit, yielding that Z is almost surely convergent to a finite limit. If Z is uniformly integrable, so is $-Z$, and so we obtain convergence in $\mathcal{L}^1$ as well for $-Z$ and therefore also for Z. Also, we have $E(-Z_\infty|\mathcal{F}_t) \le -Z_t$, so $E(Z_\infty|\mathcal{F}_t) \ge Z_t$. Finally, assume that Z is a martingale. Then Z is both a supermartingale and a submartingale, and the result follows. □

Theorem 1.2.5 (Uniformly integrable martingale convergence theorem)**.** *Let $M \in \mathbf{c}\mathcal{M}$. The following are equivalent:*

1. *M is uniformly integrable.*
2. *M is convergent almost surely and in $\mathcal{L}^1$.*
3. *There is some integrable variable ξ such that $M_t = E(\xi|\mathcal{F}_t)$ almost surely for $t \geq 0$.*

In the affirmative, with M_∞ denoting the limit of M_t almost surely and in $\mathcal{L}^1$, we have for all $t \geq 0$ that $M_t = E(M_\infty|\mathcal{F}_t)$ almost surely, and $M_\infty = E(\xi|\mathcal{F}_\infty)$, where $\mathcal{F}_\infty = \sigma(\cup_{t\geq 0}\mathcal{F}_t)$.

Proof. We show that (1) implies (2), that (2) implies (3) and that (3) implies (1).

Proof that (1) **implies** (2)**.** Assume that M is uniformly integrable. By Lemma A.2.3, M is bounded in $\mathcal{L}^1$, and Theorem 1.2.4 shows that M converges almost surely. In particular, M is convergent in probability, and so Lemma A.2.5 allows us to conclude that M is convergent in $\mathcal{L}^1$.

Proof that (2) **implies** (3)**.** Assume now that M is convergent almost surely and in $\mathcal{L}^1$. Let M_∞ be the limit. Fix $F \in \mathcal{F}_s$ for some $s \geq 0$. As M_t converges to M_∞ in $\mathcal{L}^1$, $1_F M_t$ converges to $1_F M_\infty$ in $\mathcal{L}^1$ as well, and we then obtain

$$E1_F M_\infty = \lim_{t\to\infty} E1_F M_t = \lim_{t\to\infty} E1_F E(M_t|\mathcal{F}_s) = E1_F M_s,$$

proving that $E(M_\infty|\mathcal{F}_s) = M_s$ almost surely for any $s \geq 0$.

Proof that (3) **implies** (1)**.** Finally, assume that there is some integrable variable ξ such that $M_t = E(\xi|\mathcal{F}_t)$. By Lemma A.2.6, M is uniformly integrable.

It remains to prove that in the affirmative, with M_∞ denoting the limit, it holds that for all $t \geq 0$, $M_t = E(M_\infty|\mathcal{F}_t)$ almost surely, and $M_\infty = E(\xi|\mathcal{F}_\infty)$. By what was already shown, in the affirmative case, $M_t = E(M_\infty|\mathcal{F}_t)$. We thus have $E(M_\infty|\mathcal{F}_t) = E(\xi|\mathcal{F}_t)$ almost surely for all $t \geq 0$. In particular, for any $F \in \cup_{t\geq 0}\mathcal{F}_t$, we have $EM_\infty 1_F = EE(\xi|\mathcal{F}_\infty)1_F$. Now let $\mathbb{D} = \{F \in \mathcal{F} | EM_\infty 1_F = EE(\xi|\mathcal{F}_\infty)1_F\}$. We then have that $\mathbb{D}$ is a Dynkin class containing $\cup_{t\geq 0}\mathcal{F}_t$, and $\cup_{t\geq 0}\mathcal{F}_t$ is a generating class for $\mathcal{F}_\infty$, stable under intersections. Therefore, Lemma A.1.19 shows that $\mathcal{F}_\infty \subseteq \mathbb{D}$, so that $EM_\infty 1_F = EE(\xi|\mathcal{F}_\infty)1_F$ for all $F \in \mathcal{F}_\infty$. Since M_∞ is $\mathcal{F}_\infty$ measurable as the almost sure limit of $\mathcal{F}_\infty$ measurable variables, this implies $M_\infty = E(\xi|\mathcal{F}_\infty)$ almost surely, proving the result. □

Lemma 1.2.6. *If Z is a continuous martingale, supermartingale or submartingale, and $c \geq 0$, then the stopped process Z^c is also a continuous martingale, supermartingale or* *submartingale, respectively. Z^c is always convergent almost surely and in $\mathcal{L}^1$ to Z_c. In the martingale case, Z^c is a uniformly integrable martingale.*

Proof. Fix $c \geq 0$. Z^c is adapted and continuous. Let $0 \leq s \leq t$ and consider the supermartingale case. If $c \leq s$, we also have $c \leq t$ and the adaptedness of Z allows us to conclude $E(Z^c_t|\mathcal{F}_s) = E(Z_{t\wedge c}|\mathcal{F}_s) = E(Z_c|\mathcal{F}_s) = Z_c = Z^c_s$. If instead $c \geq s$, the supermartingale property yields $E(Z^c_t|\mathcal{F}_s) = E(Z_{t\wedge c}|\mathcal{F}_{s\wedge c}) \leq Z_{s\wedge c} = Z^c_s$. This shows that Z^c is a supermartingale. From this, it follows that the submartingale and martingale properties are preserved by stopping at c as well. Also, as Z^c is constant from a deterministic point onwards, Z^c converges almost surely and in $\mathcal{L}^1$ to Z_c. If Z is a martingale, Theorem 1.2.5 shows that Z is uniformly integrable. □

Theorem 1.2.7 (Optional sampling theorem). *Let Z be a continuous supermartingale, and let S and T be two stopping times with $S \leq T$. If Z is uniformly integrable, then Z is almost surely convergent, Z_S and Z_T are integrable, and $E(Z_T|\mathcal{F}_S) \leq Z_S$. If Z is nonnegative, then Z is almost surely convergent as well and $E(Z_T|\mathcal{F}_S) \leq Z_S$. If instead S and T are bounded, $E(Z_T|\mathcal{F}_S) \leq Z_S$ holds as well, where Z_S and Z_T are integrable. Finally, if Z is a martingale in the uniformly integrable case or the case of bounded stopping times, the inequality may be exchanged with an equality.*

Proof. Assume that Z is a supermartingale which is convergent almost surely and in $\mathcal{L}^1$, and let $S \leq T$ be two stopping times. We will prove $E(Z_T|\mathcal{F}_S) \leq Z_S$ in this case and obtain the other cases from this. To prove the result in this case, we will use a discretization procedure along with Lemma 1.2.2 and Theorem A.3.5 to obtain the result. First, define a mapping S_n by putting $S_n = \infty$ whenever $S = \infty$, and $S_n = k2^{-n}$ when $(k-1)2^{-n} \leq S < k2^{-n}$. We then find

$$\begin{aligned}
(S_n \leq t) &= \cup_{k=0}^{\infty}(S_n = k2^{-n}) \cap (k2^{-n} \leq t) \\
&= \cup_{k=0}^{\infty}((k-1)2^{-n} \leq S < k2^{-n}) \cap (k2^{-n} \leq t),
\end{aligned}$$

which is in $\mathcal{F}_t$, as $((k-1)2^{-n} \leq S < k2^{-n})$ is in $\mathcal{F}_t$ when $k2^{-n} \leq t$. Therefore, S_n is a stopping time. Furthermore, we have $S \leq S_n$ with S_n converging downwards to S, in the sense that S_n is decreasing and converges to S. We define (T_n) analogously, such that (T_n) is a sequence of stopping times converging downwards to T, and $(T_n = k2^{-n}) = ((k-1)2^{-n} \leq T < k2^{-n})$. We then obtain that $(T_n = k2^{-n}) = ((k-1)2^{-n} \leq T < k2^{-n}) \subseteq (S < k2^{-n}) = (S_n \leq k2^{-n})$, from which we conclude $S_n \leq T_n$.

We would like to apply the discrete-time optional sampling theorem to the stopping times S_n and T_n. To this end, first note that with $t_k^n = k2^{-n}$, we obtain that by Lemma 1.2.2, $(Z_{t_k^n})_{k\geq 0}$ is a discrete-time supermartingale with respect to the filtration $(\mathcal{F}_{t_k^n})_{k\geq 0}$. As Z is convergent almost surely and in $\mathcal{L}^1$, so is $(Z_{t_k^n})_{k\geq 0}$, and then Lemma A.2.5 shows that $(Z_{t_k^n})_{k\geq 0}$ is uniformly integrable. Therefore, $(Z_{t_k^n})_{k\geq 0}$ satisfies the requirements in Theorem A.3.5. Furthermore, it holds that $Z_{t_k^n}$ converges to Z_∞. Putting $K_n = S_n 2^n$, K_n takes its values in $\mathbb{N}\cup\{\infty\}$ and $(K_n \leq k) = (S_n \leq k2^{-n}) \in \mathcal{F}_{t_k^n}$, so K_n is a discrete-time stopping time with respect to $(\mathcal{F}^n_{t_k^n})_{k\geq 0}$. As regards the discrete-time stopping time σ-algebra, we have

$$\begin{aligned}
\mathcal{F}_{t^n_{K_n}} &= \{F \in \mathcal{F} \mid F \cap (K_n \leq k) \in \mathcal{F}_{t_k^n} \text{ for all } k \geq 0\} \\
&= \{F \in \mathcal{F} \mid F \cap (S_n \leq t_k^n) \in \mathcal{F}_{t_k^n} \text{ for all } k \geq 0\} \\
&= \{F \in \mathcal{F} \mid F \cap (S_n \leq t) \in \mathcal{F}_t \text{ for all } t \geq 0\} = \mathcal{F}_{S_n},
\end{aligned}$$

where $\mathcal{F}_{t^n_{K_n}}$ denotes the stopping time σ-algebra of the discrete filtration $(\mathcal{F}_{t_k^n})_{k\geq 0}$. Putting $L_n = T_n 2^n$, we find analogous results for the sequence (L_n). Also, since $S_n \leq T_n$, we obtain $K_n \leq L_n$. Therefore, we may now apply Theorem A.3.5 with the uniformly integrable discrete-time supermartingale $(Z_{t_k^n})_{k\geq 0}$ to conclude that Z_{S_n} and Z_{T_n} are integrable and that $E(Z_{T_n}|\mathcal{F}_{S_n}) = E(Z_{t^n_{L_n}}|\mathcal{F}_{t^n_{K_n}}) \leq Z_{t^n_{K_n}} = Z_{S_n}$.

Next, we show that Z_{T_n} converges almost surely and in $\mathcal{L}^1$ to Z_T. This will in particular show that Z_T is integrable. As before, $(Z_{t_k^{n+1}})_{k\geq 0}$ is a discrete-time supermartingale satisfying the requirements in Theorem A.3.5. Also, $(2L_n \leq k) = (2T_n 2^n \leq k) = (T_n \leq k2^{-(n+1)})$, which is in $\mathcal{F}_{t_k^{n+1}}$, so $2L_n$ is a discrete-time stopping time with respect to $(\mathcal{F}_{t_k^{n+1}})_{k\geq 0}$, and $L_{n+1} = T_{n+1}2^{n+1} \leq T_n 2^{n+1} = 2L_n$. Therefore, applying Theorem A.3.5 to the stopping times $2L_n$ and L_{n+1}, we obtain $E(Z_{T_n}|\mathcal{F}_{T_{n+1}}) = E(Z_{t^{n+1}_{2L_n}}|\mathcal{F}_{t^{n+1}_{L_{n+1}}}) \leq Z_{t^{n+1}_{L_{n+1}}} = Z_{T_{n+1}}$. Iterating this relationship, we find that for $n \geq k$, $Z_{T_n} \geq E(Z_{T_k}|\mathcal{F}_{T_n})$. Thus, (Z_{T_n}) is a backwards submartingale with respect to $(\mathcal{F}_{T_n})_{n\geq 0}$. Therefore, $(-Z_{T_n})$ is a backwards supermartingale. Furthermore, as $Z_{T_n} \geq E(Z_{T_k}|\mathcal{F}_{T_n})$ for $n \geq k$, we have $EZ_{T_n} \geq EZ_{T_1}$, so $E(-Z_{T_n}) \leq E(-Z_{T_1})$. This shows that $\sup_{n\geq 1} E(-Z_{T_n})$ is finite, and so we may apply Theorem A.3.6 to conclude that $(-Z_{T_n})$, and therefore (Z_{T_n}), converges almost surely and in $\mathcal{L}^1$. By continuity, we know that Z_{T_n} also converges almost surely to Z_T. By uniqueness of limits, the convergence is in $\mathcal{L}^1$ as well, which in particular implies that Z_T is integrable. Analogously, Z_{S_n} converges to Z_S almost surely and in $\mathcal{L}^1$.

Now fix $F \in \mathcal{F}_S$. As $S \leq S_n$, we have $\mathcal{F}_S \subseteq \mathcal{F}_{S_n}$. Using the convergence of Z_{T_n} to Z_T and Z_{S_n} to Z_S in $\mathcal{L}^1$, we find that $1_F Z_{T_n}$ converges to $1_F Z_T$ and $1_F Z_{S_n}$ converges to $1_F Z_S$ in $\mathcal{L}^1$, so that $E1_F Z_T = \lim_n E1_F Z_{T_n} = \lim_n E1_F E(Z_{T_n}|\mathcal{F}_{S_n}) \leq \lim_n E1_F Z_{S_n} = E1_F Z_S$, and therefore, we conclude $E(Z_T|\mathcal{F}_S) \leq Z_S$, as desired.

This proves that the optional sampling result holds in the case where Z is a supermartingale which is convergent almost surely and in $\mathcal{L}^1$ and $S \leq T$ are two stopping times. We will now obtain the remaining cases from this case.

If Z is a uniformly integrable supermartingale, it is in particular convergent almost surely and in $\mathcal{L}^1$, so we find that the result holds in this case as well. Next, consider the case where we merely assume that Z is a supermartingale and that $S \leq T$ are bounded stopping times. Letting $c \geq 0$ be a bound for S and T, Lemma 1.2.6 shows that Z^c is a supermartingale, and it is convergent almost surely and in $\mathcal{L}^1$. Therefore, as $Z_T = Z^c_T$, we find that Z_T is integrable and that $E(Z_T|\mathcal{F}_S) = E(Z^c_T|\mathcal{F}_S) \leq Z^c_S = Z_S$, proving the result in this case as well.

Finally, consider the case where Z is nonnegative and $S \leq T$ are any two stopping times. We then find that $E|Z_t| = EZ_t \leq EZ_0$, so Z is bounded in $\mathcal{L}^1$. Therefore, Theorem 1.2.4 shows that Z is almost surely convergent and so Z_T is well-defined. From what we already have shown, $Z_{T\wedge n}$ is integrable and $E(Z_{T\wedge n}|\mathcal{F}_{S\wedge n}) \leq Z_{S\wedge n}$. For any $F \in \mathcal{F}_S$, we find $F \cap (S \leq n) \in \mathcal{F}_{S\wedge n}$ for any n by Lemma 1.1.13. Therefore, we obtain

$$\begin{aligned} E1_F Z_{T\wedge n} &= E1_F 1_{(S\leq n)} Z_{T\wedge n} + E1_F 1_{(S>n)} Z_{T\wedge n} \\ &\leq E1_F 1_{(S\leq n)} Z_{S\wedge n} + E1_F 1_{(S>n)} Z_{S\wedge n} = E1_F Z_{S\wedge n}, \end{aligned}$$

and so, by Lemma A.1.14, $E(Z_{T\wedge n}|\mathcal{F}_S) \leq Z_{S\wedge n}$. Applying Fatou's lemma for conditional expectations, we obtain

$$E(Z_T|\mathcal{F}_S) = E(\liminf_n Z_{T\wedge n}|\mathcal{F}_S) \leq \liminf_n E(Z_{T\wedge n}|\mathcal{F}_S) \leq \liminf_n Z_{S\wedge n} = Z_S,$$

as was to be shown. We have now proved all of the supermartingale statements in the theorem. The martingale results follow immediately from the fact that a martingale is both a supermartingale and a submartingale. □

Lemma 1.2.8. *Let T be a stopping time. If Z is a supermartingale, then Z^T is a supermartingale as well. In particular, if $M \in \mathbf{c}\mathcal{M}$, then $M^T \in \mathbf{c}\mathcal{M}$ as well, and if $M \in \mathbf{c}\mathcal{M}^u$, then $M^T \in \mathbf{c}\mathcal{M}^u$ as well.*

Proof. Let a supermartingale Z be given, and let T be some stopping time. Fix two timepoints $0 \leq s \leq t$, we need to prove $E(Z^T_t|\mathcal{F}_s) \leq Z^T_s$ almost surely, and to this end, it suffices to show that $E1_F Z^T_t \leq E1_F Z^T_s$ for any $F \in \mathcal{F}_s$. Let $F \in \mathcal{F}_s$ be given. By Lemma 1.1.13, $F \cap (s \leq T)$ is $\mathcal{F}_{s\wedge T}$ measurable, and so Theorem 1.2.7 applied with the two bounded stopping times $T \wedge s$ and $T \wedge t$ yields

$$\begin{aligned} E1_F Z_t^T &= E1_{F\cap(s\leq T)} Z_{T\wedge t} + E1_{F\cap(s>T)} Z_{T\wedge t} \\ &\leq E1_{F\cap(s\leq T)} Z_{T\wedge s} + E1_{F\cap(s>T)} Z_{T\wedge t} \\ &= E1_{F\cap(s\leq T)} Z_{T\wedge s} + E1_{F\cap(s>T)} Z_{T\wedge s} \\ &= E1_F Z_s^T. \end{aligned}$$

Thus, $E(Z_t^T|\mathcal{F}_s) \leq Z_s^T$ and so Z^T is a supermartingale. From this it follows in particular that if $M \in \mathbf{c}\mathcal{M}$, it holds that $M^T \in \mathbf{c}\mathcal{M}$ as well. And if $M \in \mathbf{c}\mathcal{M}^u$, we find that $M^T \in \mathbf{c}\mathcal{M}$ from what was already shown. Then, by Theorem 1.2.5, $M_t^T = E(M_\infty|\mathcal{F}_{T\wedge t})$, so by Lemma A.2.6, M^T is uniformly integrable, and so $M^T \in \mathbf{c}\mathcal{M}^u$. □

We end the section with two extraordinarily useful results, first a criterion for determining when a process is a martingale or a uniformly integrable martingale, and secondly a result showing that a particular class of continuous martingales consists of the zero process only.

Lemma 1.2.9 (Komatsu's lemma)**.** *Let M be a continuous adapted process with initial value zero. It holds that $M \in \mathbf{c}\mathcal{M}$ if and only if M_T is integrable with $EM_T = 0$ for any bounded stopping time T. If the limit $\lim_{t\to\infty} M_t$ exists almost surely, it holds that $M \in \mathbf{c}\mathcal{M}^u$ if and only if M_T is integrable with $EM_T = 0$ for any stopping time T.*

Proof. We first consider the case where we assume that the limit $\lim_{t\to\infty} M_t$ exists almost surely. By Theorem 1.2.7, we have that if $M \in \mathbf{c}\mathcal{M}^u$, M_T is integrable and $EM_T = 0$ for any for any stopping time T. Conversely, assume that M_T is integrable and $EM_T = 0$ for any stopping time T. We will prove that $M_t = E(M_\infty|\mathcal{F}_t)$ for any $t \geq 0$. To this end, let $F \in \mathcal{F}_t$ and note that by Lemma 1.1.7, t_F is a stopping time, where $t_F = t1_F + \infty 1_{F^c}$, taking only the values t and infinity. We obtain $EM_{t_F} = E1_F M_t + E1_{F^c} M_\infty$, and we also have $EM_\infty = E1_F M_\infty + E1_{F^c} M_\infty$. By our assumptions, both of these are zero, and so $E1_F M_t = E1_F M_\infty$. As M_t is $\mathcal{F}_t$ measurable by assumption, this proves $M_t = E(M_\infty|\mathcal{F}_t)$. From this, we see that M is in $\mathbf{c}\mathcal{M}$, and by Theorem 1.2.5, M is in $\mathbf{c}\mathcal{M}^u$.

Consider next the case where we merely assume that M is a continuous process with initial value zero. If $M \in \mathbf{c}\mathcal{M}$, Theorem 1.2.7 shows that M_T is integrable with $EM_T = 0$ for any bounded stopping time. Assume instead that M_T is integrable and $EM_T = 0$ for any bounded stopping time T. From what we already have shown, we then find that M^t is in $\mathbf{c}\mathcal{M}^u$ for any $t \geq 0$ and therefore, $M \in \mathbf{c}\mathcal{M}$. □

For the statement of the final result, we say that a process X is of finite variation if it has sample paths which are functions of finite variation, see Appendix A.1 for a review of the properties of functions of finite variation. If the process X has finite variation, we denote the variation over $[0,t]$ by $(V_X)_t$, such that $(V_X)_t = \sup \sum_{k=1}^n |X_{t_k} - X_{t_{k-1}}|$, where the supremum is taken over partitions $0 = t_0 < \cdots < t_n = t$ of $[0,t]$.

Lemma 1.2.10. *Let X be adapted and continuous with finite variation. Then the variation process V_X is adapted and continuous as well.*

Proof. By Lemma A.1.4, V_X is continuous. As for proving that V_X is adapted, note that from Lemma A.1.9, we have $(V_X)_t = \sup \sum_{k=1}^n |X_{q_k} - X_{q_{k-1}}|$, where the supremum is taken over partitions of $[0,t]$ with elements in $\mathbb{Q}_+ \cup \{t\}$. As $\cup_{n=1}^\infty (\mathbb{Q}_+ \cup \{t\})^n$ is countable, there are only countably many such partitions, and so we find that $(V_X)_t$ is $\mathcal{F}_t$ measurable, since X_q is $\mathcal{F}_t$ measurable whenever $q \le t$. Therefore, V_X is adapted. □

Lemma 1.2.11. *Let X be adapted and continuous with finite variation. Then $(V_X)^T = V_{X^T}$.*

Proof. Fix $\omega \in \Omega$. With the supremum being over all partitions of $[0, T(\omega) \wedge t]$, we have

$$\begin{aligned}(V_X)^T_t(\omega) &= (V_X)_{T(\omega)\wedge t}(\omega) = \sup \sum_{k=1}^n |X_{t_k}(\omega) - X_{t_{k-1}}(\omega)| \\ &= \sup \sum_{k=1}^n |X^T_{t_k}(\omega) - X^T_{t_{k-1}}(\omega)| = (V_{X^T})_{t\wedge T(\omega)}(\omega) \le (V_{X^T})_t(\omega).\end{aligned}$$

Conversely, with the supremum being over all partitions of $[0,t]$, we also have

$$\begin{aligned}(V_{X^T})_t(\omega) &= \sup \sum_{k=1}^n |X^T_{t_k}(\omega) - X^T_{t_{k-1}}(\omega)| \\ &= \sup \sum_{k=1}^n |X_{t_k\wedge T(\omega)}(\omega) - X_{t_{k-1}\wedge T(\omega)}(\omega)| \le (V_X)_{t\wedge T(\omega)}(\omega).\end{aligned}$$

Combining our conclusions, the result follows. □

Lemma 1.2.12. *If $M \in \mathbf{c}\mathcal{M}$ with paths of finite variation, then M is evanescent.*

Proof. We first consider the case where $M \in \mathbf{c}\mathcal{M}^b$ and the variation process V_M is bounded, $(V_M)_t$ being the variation of M over $[0,t]$. Fix $t \ge 0$ and let $t_k^n = kt2^{-n}$. Now note that by the martingale property, $EM_{t_{k-1}^n}(M_{t_k^n} - M_{t_{k-1}^n}) = EM_{t_{k-1}^n}E(M_{t_k^n} - M_{t_{k-1}^n}|\mathcal{F}_{t_{k-1}^n}) = 0$, and by rearrangement, $M_{t_k^n}^2 - M_{t_{k-1}^n}^2 = 2M_{t_{k-1}^n}(M_{t_k^n} - M_{t_{k-1}^n}) + (M_{t_k^n} - M_{t_{k-1}^n})^2$. Therefore, we obtain

$$\begin{aligned} EM_t^2 &= E\sum_{k=1}^{2^n}(M_{t_k^n}^2 - M_{t_{k-1}^n}^2) = 2E\sum_{k=1}^{2^n} M_{t_{k-1}^n}(M_{t_k^n} - M_{t_{k-1}^n}) + E\sum_{k=1}^{2^n}(M_{t_k^n} - M_{t_{k-1}^n})^2 \\ &= E\sum_{k=1}^{2^n}(M_{t_k^n} - M_{t_{k-1}^n})^2 \leq E(V_M)_t \max_{k\leq 2^n}|M_{t_k^n} - M_{t_{k-1}^n}|. \end{aligned}$$

Now, as M is continuous, $(V_M)_t \max_{k\leq n}|M_{t_k^n} - M_{t_{k-1}^n}|$ tends pointwisely to zero as n tends to infinity. The boundedness of M and V_M then allows us to apply the dominated convergence theorem and obtain

$$EM_t^2 \leq \lim_{n\to\infty} E(V_M)_t \max_{k\leq 2^n}|M_{t_k} - M_{t_{k-1}}| \leq E\lim_{n\to\infty}(V_M)_t \max_{k\leq 2^n}|M_{t_k} - M_{t_{k-1}}| = 0,$$

so that M_t is almost surely zero by Lemma A.1.15, and so by Lemma 1.1.5, M is evanescent. In the case of a general $M \in \mathbf{c}\mathcal{M}$, define $T_n = \inf\{t \geq 0 \mid (V_M)_t > n\}$. By Lemma 1.1.11, (T_n) is a sequence of stopping times increasing almost surely to infinity, and $(V_M)^{T_n}$ is bounded by n. By Lemma A.1.6 and Lemma 1.2.11, $|M_t^{T_n}| \leq |(V_{M^{T_n}})_t = |(V_M)_t^{T_n}| \leq n$ for all $t \geq 0$. Therefore, M^{T_n} is a bounded martingale with bounded variation, so our previous results show that M^{T_n} is evanescent. Letting n tend to infinity, T_n tends to infinity, and so we almost surely obtain $M_t = \lim_n M_t^{T_n} = 0$, allowing us to conclude by Lemma 1.1.5 that M is evanescent. □

1.3 Square-integrable martingales

In this section, we consider the properties of square-integrable martingales, and we apply these properties to prove the existence of the quadratic variation process for bounded continuous martingales. We say that a continuous martingale M is square-integrable if $\sup_{t\geq 0} EM_t^2$ is finite. The space of continuous square-integrable martingales with initial value zero is denoted by $\mathbf{c}\mathcal{M}^2$. We note that $\mathbf{c}\mathcal{M}^2$ is a vector space. For any $M \in \mathbf{c}\mathcal{M}^2$, we put $M_t^* = \sup_{s\leq t}|M_s|$ and $M_\infty^* = \sup_{t\geq 0}|M_t|$. We use the notational convention that $M_t^{*2} = (M_t^*)^2$, and likewise $M_\infty^{*2} = (M_\infty^*)^2$.

Theorem 1.3.1. *Let $M \in \mathbf{c}\mathcal{M}^2$. Then, there exists a square-integrable variable M_∞ such that $M_t = E(M_\infty|\mathcal{F}_t)$ for all $t \geq 0$. Furthermore, M_t converges to M_∞ almost surely and in $\mathcal{L}^2$, and $EM_\infty^{*2} \leq 4EM_\infty^2$.*

Proof. As M is bounded in $\mathcal{L}^2$, M is in particular uniformly integrable by Lemma A.2.4, so by Theorem 1.2.5, M_t converges almost surely and in $\mathcal{L}^1$ to some variable M_∞, which is integrable and satisfies that $M_t = E(M_\infty|\mathcal{F}_t)$ almost surely for $t \geq 0$. It remains to prove that M_∞ is square-integrable, that we have convergence in $\mathcal{L}^2$ and that $EM_\infty^{*2} \leq 4EM_\infty^2$ holds.

Put $t_k^n = k2^{-n}$ for $n, k \geq 0$. Then $(M_{t_k^n})_{k\geq 0}$ is a discrete-time martingale for $n \geq 0$ with $\sup_{k\geq 0} EM_{t_k^n}^2$ finite. By Lemma A.3.4, $M_{t_k^n}$ converges almost surely and in $\mathcal{L}^2$ to some square-integrable limit as k tends to infinity. By uniqueness of limits, the limit is M_∞, so we conclude that M_∞ is square-integrable. Lemma A.3.4 also yields $E\sup_{k\geq 0} M_{t_k^n}^2 \leq 4EM_\infty^2$. We then obtain by the monotone convergence theorem and the continuity of M that

$$EM_\infty^{*2} = E\lim_{n\to\infty}\sup_{k\geq 0} M_{t_k^n}^2 = \lim_{n\to\infty} E\sup_{k\geq 0} M_{t_k^n}^2 \leq 4EM_\infty^2.$$

This proves the inequality $EM_\infty^{*2} \leq 4EM_\infty^2$. It remains to show that M_t converges to M_∞ in $\mathcal{L}^2$. To this end, note that as we have $(M_t - M_\infty)^2 \leq (2M_\infty^*)^2 = 4M_\infty^{*2}$, which is integrable, the dominated convergence theorem yields $\lim_t E(M_t - M_\infty)^2 = E\lim_t (M_t - M_\infty)^2 = 0$, so M_t also converges in $\mathcal{L}^2$ to M_∞, as desired. □

Lemma 1.3.2. *Assume that $M \in \mathbf{c}\mathcal{M}^2$. Then $M^T \in \mathbf{c}\mathcal{M}^2$ as well.*

Proof. By Lemma 1.2.8, M^T is a martingale. Furthermore, we have

$$\sup_{t\geq 0} E(M^T)_t^2 \leq E\sup_{t\geq 0}(M_t^T)^2 \leq E\sup_{t\geq 0} M_t^2 = EM_\infty^{*2},$$

and this is finite by Theorem 1.3.1, proving that $M^T \in \mathbf{c}\mathcal{M}^2$. □

Theorem 1.3.3. *Assume that (M^n) is a sequence in $\mathbf{c}\mathcal{M}^2$ such that (M_∞^n) is convergent in $\mathcal{L}^2$ to a limit M_∞. Then there is some $M \in \mathbf{c}\mathcal{M}^2$ such that for all $t \geq 0$, $M_t = E(M_\infty|\mathcal{F}_t)$. Furthermore, $E\sup_{t\geq 0}(M_t^n - M_t)^2$ tends to zero.*

Proof. The difficulty in the proof lies in demonstrating that the martingale M obtained by putting $M_t = E(M_\infty|\mathcal{F}_t)$ has a continuous version. First note that $M^n - M^m \in \mathbf{c}\mathcal{M}^2$ for all n and m, so for any $\delta > 0$ we may apply Chebyshev's inequality and Theorem 1.3.1 to obtain, using $(x+y)^2 \leq 2x^2 + 2y^2$,

$$\begin{aligned} P((M^n - M^m)_\infty^* \geq \delta) &\leq \delta^{-2} E(M^n - M^m)_\infty^{*2} \\ &\leq 4\delta^{-2} E(M_\infty^n - M_\infty^m)^2 \\ &\leq 8\delta^{-2}(E(M_\infty^n - M_\infty)^2 + E(M_\infty - M_\infty^m)^2) \\ &\leq 16\delta^{-2} \sup_{k\geq m\wedge n} E(M_\infty^k - M_\infty)^2. \end{aligned}$$

Now let (n_i) be a strictly increasing sequence of naturals such that for each i, it holds that

$$16(2^{-i})^{-2} \sup_{k \geq n_i} E(M^k_\infty - M_\infty)^2 \leq 2^{-i},$$

this is possible as $\sup_{k \geq n} E(M^k_\infty - M_\infty)^2$ tends to zero as n tends to infinity. In particular, $P((M^{n_{i+1}} - M^{n_i})^*_\infty > 2^{-i}) \leq 2^{-i}$ for all $i \geq 1$. Then $\sum_{i=1}^\infty P((M^{n_{i+1}} - M^{n_i})^*_\infty > 2^{-i})$ is finite, so therefore, by the Borel-Cantelli Lemma, the event that $(M^{n_{i+1}} - M^{n_i})^*_\infty > 2^{-i}$ infinitely often has probability zero. Therefore, $(M^{n_{i+1}} - M^{n_i})^*_\infty \leq 2^{-i}$ from a point onwards almost surely. In particular, it almost surely holds that for any two numbers $k \leq m$ large enough that

$$(M^{n_m} - M^{n_k})^*_\infty \leq \sum_{i=k+1}^m (M^{n_i} - M^{n_{i-1}})^*_\infty \leq \sum_{i=k+1}^\infty 2^{-i} = 2^{-k}.$$

Thus, it holds almost surely that M^{n_i} is Cauchy in the uniform norm on $\mathbb{R}_+$, and therefore almost surely uniformly convergent to some continuous limit. Define M to be the uniform limit when it exists and zero otherwise, M is then a process with continuous paths. With F being the null set where we do not have uniform convergence, our assumption that the usual conditions hold allows us to conclude that $F \in \mathcal{F}_t$ for all $t \geq 0$. As uniform convergence implies pointwise convergence, we have $M_t = 1_{F^c} \lim_{i \to \infty} M^{n_i}_t$, so M is also adapted. We now claim that $M \in \mathbf{c}\mathcal{M}^2$. To see this, note that by Jensen's inequality, we have

$$\begin{aligned} E(M^n_t - E(M_\infty|\mathcal{F}_t))^2 &= E(E(M^n_\infty|\mathcal{F}_t) - E(M_\infty|\mathcal{F}_t))^2 = EE(M^n_\infty - M_\infty|\mathcal{F}_t)^2 \\ &\leq EE((M^n_\infty - M_\infty)^2|\mathcal{F}_t) = E(M^n_\infty - M_\infty)^2, \end{aligned}$$

which tends to zero, so for any $t \geq 0$, M^n_t tends to $E(M_\infty|\mathcal{F}_t)$ in $\mathcal{L}^2$. As $M^{n_i}_t$ tends to M_t almost surely, we conclude that $M_t = E(M_\infty|\mathcal{F}_t)$ almost surely by uniqueness of limits. This shows that M is a martingale, and as $EM^2_t \leq EE(M^2_\infty|\mathcal{F}_t) = EM^2_\infty$, which is finite, we conclude that M is bounded in $\mathcal{L}^2$. As M has initial value zero, we then obtain $M \in \mathbf{c}\mathcal{M}^2$. Finally, $\limsup_n E\sup_{t\geq 0}(M^n_t - M_t)^2 \leq 4 \lim_n E(M^n_\infty - M_\infty)^2 = 0$ by Theorem 1.3.1, yielding the desired convergence of M^n to M. □

We now introduce a seminorm $\|\cdot\|_2$ on the space $\mathbf{c}\mathcal{M}^2$ by putting $\|M\|_2 = (EM^2_\infty)^{\frac{1}{2}}$. Note that this is possible as we have ensured in Theorem 1.3.1 that for any $M \in \mathbf{c}\mathcal{M}^2$, $M_t = E(M_\infty|\mathcal{F}_t)$ for some almost surely unique square-integrable M_∞, so that the limit determines the entire martingale. Note that $\|\cdot\|_2$ is generally not a norm, only a seminorm, in the sense that $\|M\|_2 = 0$ does not imply that M is zero, only that M is evanescent.

Theorem 1.3.4. *The space $\mathbf{c}\mathcal{M}^2$ is complete under the seminorm $\|\cdot\|_2$, in the sense that any Cauchy sequence in $\mathbf{c}\mathcal{M}^2$ has a limit.*

Proof. Assume that (M^n) is a Cauchy sequence in $\mathbf{c}\mathcal{M}^2$. By our definition of the seminorm on $\mathbf{c}\mathcal{M}^2$, we have $(E(M^n_\infty - M^m_\infty)^2)^{\frac{1}{2}} = \|M^n - M^m\|_2$, and so (M^n_∞) is a Cauchy sequence in $\mathcal{L}^2$. As $\mathcal{L}^2$ is complete, there exists M_∞ such that M^n_∞ converges in $\mathcal{L}^2$ to M_∞. By Theorem 1.3.3, there exists $M \in \mathbf{c}\mathcal{M}^2$ such that for any $t \geq 0$, $M_t = E(M_\infty|\mathcal{F}_t)$ almost surely. Therefore, M^n tends to M in $\mathbf{c}\mathcal{M}^2$. □

Theorem 1.3.5 (Riesz' representation theorem for $\mathbf{c}\mathcal{M}^2$)**.** *Let $M \in \mathbf{c}\mathcal{M}^2$. Then, the mapping $\varphi : \mathbf{c}\mathcal{M}^2 \to \mathbb{R}$ defined by $\varphi(N) = EM_\infty N_\infty$ is linear and continuous. Conversely, assume that $\varphi : \mathbf{c}\mathcal{M}^2 \to \mathbb{R}$ is linear and continuous. Then, there exists $M \in \mathbf{c}\mathcal{M}^2$, unique up to indistinguishability, such that $\varphi(N) = EM_\infty N_\infty$ for all $N \in \mathbf{c}\mathcal{M}^2$.*

Proof. First consider $M \in \mathbf{c}\mathcal{M}^2$ and define $\varphi : \mathbf{c}\mathcal{M}^2 \to \mathbb{R}$ by putting $\varphi(N) = EM_\infty N_\infty$. φ is then linear, and $|\varphi(N - N')| = |EM_\infty(N_\infty - N'_\infty)| \leq \|M\|_2 \|N - N'\|_2$ for all $N, N' \in \mathbf{c}\mathcal{M}^2$ by the Cauchy-Schwartz inequality, showing that φ is Lipschitz with Lipschitz constant $\|M\|_2$, therefore continuous.

Conversely, assume given any $\varphi : \mathbf{c}\mathcal{M}^2 \to \mathbb{R}$ which is linear and continuous, we need to find $M \in \mathbf{c}\mathcal{M}^2$ such that $\varphi(N) = EM_\infty N_\infty$ for all $N \in \mathbf{c}\mathcal{M}^2$. If φ is identically zero, this is trivially satisfied with M being the zero martingale. Therefore, assume that φ is not identically zero. In this case, there is $M' \in \mathbf{c}\mathcal{M}^2$ such that $\varphi(M') \neq 0$. Define the set $C \subseteq \mathbf{c}\mathcal{M}^2$ by $C = \{L \in \mathbf{c}\mathcal{M}^2 \mid \varphi(L) = \|M'\|_2\}$. As φ is continuous, C is closed. And as φ is linear, C is convex.

We claim that there is $M'' \in C$ such that such that $\|M''\|_2 = \inf_{L \in C} \|L\|_2$. To prove this, it suffices to put $\alpha = \inf_{L \in C} \|L\|_2^2$ and identify $M'' \in C$ such that $\|M''\|_2^2 = \alpha$. Take a sequence (L^n) in C such that $\|L^n\|_2^2$ converges to α. Since $\frac{1}{2}(L^m + L^n) \in C$ by convexity, we have

$$\begin{aligned} \|L^m - L^n\|_2^2 &= 2\|L^m\|_2^2 + 2\|L^n\|_2^2 - \|L^m + L^n\|_2^2 \\ &= 2\|L^m\|_2^2 + 2\|L^n\|_2^2 - 4\|\tfrac{1}{2}(L^m + L^n)\|_2^2 \\ &\leq 2\|L^m\|_2^2 + 2\|L^n\|_2^2 - 4\alpha. \end{aligned}$$

As m and n tend to infinity, $\|L^m\|_2^2$ and $\|L^n\|_2^2$ tend to α, so $\|L^m - L^n\|_2^2$ tends to zero. Therefore, (L^n) is Cauchy. By Theorem 1.3.4, (L^n) is convergent towards some M''. As C is closed, $M'' \in C$, and we furthermore find $\|M''\|_2^2 = \lim_n \|L^n\|_2^2 = \alpha$. Thus, M'' is an element of C satisfying that $\|M''\|_2 = \inf_{L \in C} \|L\|_2$.

We next claim that for any $N \in \mathbf{c}\mathcal{M}^2$ with $\varphi(N) = 0$, $EM''_\infty N_\infty = 0$. This is true if N is evanescent, assume therefore that N is not evanescent, so that $\|N\|_2 \neq 0$. By linearity,

$\varphi(M''-tN)=\varphi(M'')$ for any $t\in\mathbb{R}$, so that $M''-tN\in C$. We then find that

$$\|M''\|_2^2 \quad = \quad \inf_{L\in C}\|L\|_2^2\leq\inf_{t\in\mathbb{R}}\|M''-tN\|_2^2\leq\|M''\|_2^2,$$

so that $\|M''\|_2^2$ is the minimum of the mapping $t\mapsto\|M''-tN\|_2^2$, attained at zero. However, $\|M''-tN\|_2^2=t^2\|N\|_2^2-2tEM''_\infty N_\infty+\|M''\|_2^2$, so $t\mapsto\|M''_\infty-tN_\infty\|_2^2$ is a quadratic polynomial, and as $\|N\|_2\neq 0$, it attains its unique minimum at $\|N\|_2^{-2}EM''_\infty N_\infty$. As we also know that the minimum is attained at zero, we conclude $EM''_\infty N_\infty=0$.

We have now proven the existence of a process M'' in $\mathbf{c}\mathcal{M}^2$ which is nonzero and satisfies $EM''_\infty N_\infty=0$ whenever $\varphi(N)=0$. We then note for any $N\in\mathbf{c}\mathcal{M}^2$ that, using the linearity of φ, $\varphi(\varphi(M'')N-\varphi(N)M'')=\varphi(M'')\varphi(N)-\varphi(N)\varphi(M'')=0$, yielding the relationship $0=EM''_\infty(\varphi(M'')N_\infty-\varphi(N)M''_\infty)=\varphi(M'')EM''_\infty N_\infty-\varphi(N)\|M''\|_2^2$, so that we finally obtain the relation

$$\varphi(N)=\|M''\|_2^{-2}\varphi(M'')EM''_\infty N_\infty=E\left(\frac{\varphi(M'')M''_\infty}{\|M''\|_2^2}N_\infty\right),$$

which proves the desired result using the element $(\varphi(M'')M'')\|M''\|_2^{-2}$ of $\mathbf{c}\mathcal{M}^2$. It remains to prove uniqueness. Assume therefore that $M,M'\in\mathbf{c}\mathcal{M}^2$ such that $EM_\infty N_\infty=EM'_\infty N_\infty$ for all $N\in\mathbf{c}\mathcal{M}^2$. Then $E(M_\infty-M'_\infty)N_\infty=0$ for all $N\in\mathbf{c}\mathcal{M}^2$, in particular we have $E(M_\infty-M'_\infty)^2=0$ so that $M_\infty=M'_\infty$ almost surely and so M and M' are indistinguishable. This completes the proof. □

Finally, we apply our results on $\mathbf{c}\mathcal{M}^2$ to prove the existence of the quadratic variation process for continuous bounded martingales. In the next section on continuous local martingales, we will then extend the quadratic variation process to continuous local martingales and introduce the quadratic covariation process as well.

We say that a nonnegative process X is increasing if its sample paths are increasing. In this case, the limit of X_t exists almost surely as a variable with values in $[0,\infty]$ and is denoted by X_∞. We say that an increasing process is integrable if its limit X_∞ is integrable. In this case, X_∞ is in particular almost surely finite. We denote by $\mathbf{c}\mathcal{A}^i$ the set of stochastic processes with initial value zero which are continuous, adapted, increasing and integrable.

Theorem 1.3.6. *Let $M\in\mathbf{c}\mathcal{M}^b$. There exists a process $[M]$ in $\mathbf{c}\mathcal{A}^i$, unique up to indistinguishability, such that $M^2-[M]\in\mathbf{c}\mathcal{M}^2$. We call $[M]$ the quadratic variation process of M.*

Proof. We first consider uniqueness. Assume that A and B are two processes in $\mathbf{c}\mathcal{A}^i$ such that M^2-A and M^2-B are in $\mathbf{c}\mathcal{M}^2$. In particular, $A-B$ is in $\mathbf{c}\mathcal{M}^2$ and has paths of finite variation, so Lemma 1.2.12 shows that $A - B$ is evanescent, such that A and B are indistinguishable. This proves uniqueness.

Next, we consider the existence of the process. Let $t^n_k = k2^{-n}$ for $n, k \geq 0$, we then find

$$M_t^2 = \sum_{k=1}^{\infty} M^2_{t \wedge t^n_k} - M^2_{t \wedge t^n_{k-1}} = 2\sum_{k=1}^{\infty} M_{t \wedge t^n_{k-1}}(M_{t \wedge t^n_k} - M_{t \wedge t^n_{k-1}}) + \sum_{k=1}^{\infty}(M_{t \wedge t^n_k} - M_{t \wedge t^n_{k-1}})^2,$$

where the terms in the sum are zero from a point onwards, namely for such k that $t^n_{k-1} \geq t$. Define $N^n_t = 2\sum_{k=1}^{\infty} M_{t \wedge t^n_{k-1}}(M_{t \wedge t^n_k} - M_{t \wedge t^n_{k-1}})$. Our plan for the proof is to show that N^n is in $\mathbf{c}\mathcal{M}^2$ and that $(N^n_\infty)_{n \geq 1}$ is bounded in $\mathcal{L}^2$. This will allow us to apply Lemma A.2.7 in order to obtain some $N \in \mathbf{c}\mathcal{M}^2$ which is the limit of appropriate convex combinations of the (N^n). We then show that by putting $[M] = M^2 - N$, we obtain a process with the desired qualities.

We first show that N^n is a martingale by applying Lemma 1.2.9. As N^n is continuous and adapted with initial value zero, it suffices to prove that N^n_T is integrable and that $EN^n_T = 0$ for all bounded stopping times T. To this end, note that as M is bounded, there is $c > 0$ such that $|M_t| \leq c$ for all $t \geq 0$. Therefore, for any k, $|2M_{t \wedge t^n_{k-1}}(M_{t \wedge t^n_k} - M_{t \wedge t^n_{k-1}})| \leq 4c^2$. As T is also bounded, N^n_T is integrable, as it is the sum of finitely many terms bounded by $4c^2$, and we have

$$\begin{aligned} EN^n_T &= 2E\sum_{k=1}^{\infty} M_{T \wedge t^n_{k-1}}(M_{T \wedge t^n_k} - M_{T \wedge t^n_{k-1}}) \\ &= 2\sum_{k=1}^{\infty} EM^T_{t^n_{k-1}}(M^T_{t^n_k} - M^T_{t^n_{k-1}}) = 2\sum_{k=1}^{\infty} EM^T_{t^n_{k-1}} E(M^T_{t^n_k} - M^T_{t^n_{k-1}} | \mathcal{F}_{t^n_{k-1}}), \end{aligned}$$

where the interchange of summation and expectation is allowed, as the only nonzero terms in the sum are for those k such that $t^n_{k-1} \leq T$, and there are only finitely many such terms. As M^T is a martingale by Lemma 1.2.8, $E(M^T_{t^n_k} - M^T_{t^n_{k-1}} | \mathcal{F}_{t^n_{k-1}}) = 0$, so the above is zero and N^n is a martingale by Lemma 1.2.9. Next, we show that N^n is bounded in $\mathcal{L}^2$. Fix $k \geq 1$, we first consider a bound for the second moment of $N^n_{t^n_k}$. To obtain this, note that for $i < j$,

$$\begin{aligned} & E(M_{t^n_{i-1}}(M_{t^n_i} - M_{t^n_{i-1}}))(M_{t^n_{j-1}}(M_{t^n_j} - M_{t^n_{j-1}})) \\ =\; & E(M_{t^n_{i-1}}(M_{t^n_i} - M_{t^n_{i-1}}) E(M_{t^n_{j-1}}(M_{t^n_j} - M_{t^n_{j-1}}) | \mathcal{F}_{t^n_{j-1}})) \\ =\; & E(M_{t^n_{i-1}}(M_{t^n_i} - M_{t^n_{i-1}}) M_{t^n_{j-1}} E(M_{t^n_j} - M_{t^n_{j-1}} | \mathcal{F}_{t^n_{j-1}})), \end{aligned}$$

which is zero, as $E((M_{t^n_j} - M_{t^n_{j-1}}) | \mathcal{F}_{t^n_{j-1}}) = 0$, and by the same type of argument, we obtain

$E(M_{t_i^n} - M_{t_{i-1}^n})(M_{t_j^n} - M_{t_{j-1}^n}) = 0$. Therefore, we obtain

$$\begin{aligned} E(N^n_{t^n_k})^2 &= 4E\left(\sum_{i=1}^k M_{t^n_{i-1}}(M_{t^n_i} - M_{t^n_{i-1}})\right)^2 = 4\sum_{i=1}^k E\left(M_{t^n_{i-1}}(M_{t^n_i} - M_{t^n_{i-1}})\right)^2 \\ &\leq 4c^2\sum_{i=1}^k E(M_{t^n_i} - M_{t^n_{i-1}})^2 = 4c^2 E\left(\sum_{i=1}^k M_{t^n_i} - M_{t^n_{i-1}}\right)^2 = 4c^2 E M^2_{t^n_k} \leq 4c^4. \end{aligned}$$

Finally, note that for any $0 \leq s \leq t$, $E(N^n_s)^2 = E(E(N^n_t|\mathcal{F}_s)^2) \leq E(N^n_t)^2$ by Jensen's inequality, so $t \mapsto E(N^n_t)^2$ is increasing, and thus $\sup_{t\geq 0} E(N^n_t)^2 = \sup_{k\geq 1} E(N^n_{t^n_k})^2 \leq 4c^4$. Therefore, $N^n \in \mathbf{c}\mathcal{M}^2$, and in particular, $E(N^n_\infty)^2 = \lim_t E(N^n_t)^2 \leq 4c^4$, so $(N^n_\infty)_{n\geq 1}$ is bounded in $\mathcal{L}^2$.

Now, by Lemma A.2.7, there exists a sequence of naturals (K_n) with $K_n \geq n$ and for each n a finite sequence of reals $\lambda^n_n, \ldots, \lambda^n_{K_n}$ in the unit interval summing to one, such that $\sum_{i=n}^{K_n} \lambda^n_i N^i_\infty$ is convergent in $\mathcal{L}^2$ to some variable N_∞. By Theorem 1.3.3, it then holds that there is $N \in \mathbf{c}\mathcal{M}^2$ such that $E\sup_{t\geq 0}(N_t - \sum_{i=n}^{K_n}\lambda^n_i N^i_t)^2$ tends to zero. Define $A = M^2 - N$, we claim that there is a modification of A satisfying the criteria of the theorem.

To prove this, first note that as M^2 and N are continuous and adapted, so is A. We will prove that A is almost surely increasing. To this end, define $\mathbb{D}_+ = \{k2^{-n} | k \geq 0, n \geq 1\}$. Then $\mathbb{D}_+$ is dense in $\mathbb{R}_+$. Let $p, q \in \mathbb{D}_+$ with $p \leq q$, we will show that $A_p \leq A_q$ almost surely. There exists $j \geq 1$ and naturals $n_p \leq n_q$ such that $p = n_p 2^{-j}$ and $q = n_q 2^{-j}$. We then find, with the limits being in $\mathcal{L}^2$,

$$A_p = M^2_p - N_p = \lim_{n\to\infty}\sum_{i=n}^{K_n}\lambda^n_i(M^2_p - N^i_p) = \lim_{n\to\infty}\sum_{i=n}^{K_n}\lambda^n_i\sum_{k=1}^\infty (M_{p\wedge t^i_k} - M_{p\wedge t^i_{k-1}})^2.$$

Now note that for $i \geq j$, we have $p \wedge t^i_k = n_p 2^{-j} \wedge k2^{-i} = n_p 2^{i-j} 2^{-i} \wedge k 2^{-i} = (n_p 2^{i-j} \wedge k)2^{-i}$ and analogously for $q \wedge t^i_k$, so we obtain that almost surely, by Lemma A.2.2,

$$\begin{aligned} \lim_{n\to\infty}\sum_{i=n}^{K_n}\lambda^n_i\sum_{k=1}^\infty (M_{p\wedge t^i_k} - M_{p\wedge t^i_{k-1}})^2 &= \lim_{n\to\infty}\sum_{i=n}^{K_n}\lambda^n_i\sum_{k=1}^{n_p 2^{i-j}}(M_{t^i_k} - M_{t^i_{k-1}})^2 \\ &\leq \lim_{n\to\infty}\sum_{i=n}^{K_n}\lambda^n_i\sum_{k=1}^{n_q 2^{i-j}}(M_{t^i_k} - M_{t^i_{k-1}})^2 \\ &= \lim_{n\to\infty}\sum_{i=n}^{K_n}\lambda^n_i\sum_{k=1}^\infty (M_{q\wedge t^i_k} - M_{q\wedge t^i_{k-1}})^2, \end{aligned}$$

allowing us to make the same calculations in reverse and conclude that $A_p \leq A_q$ almost surely. As $\mathbb{D}_+$ is countable, we conclude that A is increasing on $\mathbb{D}_+$ almost surely, and by continuity, we conclude that A is increasing almost surely. Furthermore, as $A_\infty = M_\infty^2 - N_\infty$ and both M_∞^2 and N_∞ are integrable, we conclude that A_∞ is integrable.

Finally, let F be the null set where A is not increasing and put $[M] = A1_{F^c}$. As we have assumed that all null sets are in $\mathcal{F}_t$ for $t \geq 0$, $[M]$ is adapted as A is adapted. Furthermore, $[M]$ is continuous, increasing and $[M]_\infty$ exists and is integrable. As $M^2 - [M] = N + A1_F$, where $A1_F$ is evanescent and continuous and therefore in $\mathbf{c}\mathcal{M}^2$, the theorem is proven. □

1.4 Local martingales

In this section, we consider continuous local martingales, which is a convenient extension of the concept of martingales. When we define the stochastic integral with respect to a local martingale, we will see that the integral of a progressive process with respect to a martingale is not necessarily a martingale, but the integral of a progressive process with respect to a local martingale is always a local martingale. This stability property makes local martingales a natural type of integrator.

We say that an increasing sequence of stopping times tending almost surely to infinity is a localising sequence. We then say that a process M is a continous local martingale if M is continuous and adapted and there is a localising sequence (T_n) such that M^{T_n} is a continuous martingale for all n, and in this case, we say that (T_n) is a localising sequence for M. The space of continuous local martingales with initial value zero is denoted by $\mathbf{c}\mathcal{M}_\ell$.

Lemma 1.4.1. *It holds that* $\mathbf{c}\mathcal{M}^b \subseteq \mathbf{c}\mathcal{M}^2 \subseteq \mathbf{c}\mathcal{M}^u \subseteq \mathbf{c}\mathcal{M} \subseteq \mathbf{c}\mathcal{M}_\ell$.

Proof. That $\mathbf{c}\mathcal{M}^b \subseteq \mathbf{c}\mathcal{M}^2$ is immediate. By Lemma A.2.4, $\mathbf{c}\mathcal{M}^2 \subseteq \mathbf{c}\mathcal{M}^u$, and by construction we have $\mathbf{c}\mathcal{M}^u \subseteq \mathbf{c}\mathcal{M}$. If $M \in \mathbf{c}\mathcal{M}$, $M^T \in \mathbf{c}\mathcal{M}$ for any stopping time by Lemma 1.2.8, and so $\mathbf{c}\mathcal{M} \subseteq \mathbf{c}\mathcal{M}_\ell$, using the localising sequence (T_n) with $T_n = n$. □

Lemma 1.4.2. *Let (S_n) and (T_n) be localising sequences. Then $(S_n \wedge T_n)$ is a localising sequence as well. If $M, N \in \mathbf{c}\mathcal{M}_\ell$, with localising sequences (S_n) and (T_n), then $(S_n \wedge T_n)$ is a locasing sequence for both M and N.*

Proof. As $S_n \wedge T_n$ is a stopping time by Lemma 1.1.7 and $S_n \wedge T_n$ tends almost surely to infinity, $(S_n \wedge T_n)$ is a localising sequence. Now assume that $M, N \in \mathbf{c}\mathcal{M}_\ell$ with localising sequences (T_n) and (S_n), respectively. Then $M^{T_n \wedge S_n} = (M^{T_n})^{S_n}$ is a martingale by Lemma

1.2.8, and so $(T_n \wedge S_n)$ is a localising sequence for M. Analogously, $(T_n \wedge S_n)$ is also a localising sequence for N. □

Lemma 1.4.3. *$\mathbf{c}\mathcal{M}_\ell$ is a vector space. If T is any stopping time and $M \in \mathbf{c}\mathcal{M}_\ell$, then $M^T \in \mathbf{c}\mathcal{M}_\ell$ as well. If $F \in \mathcal{F}_0$ and $M \in \mathbf{c}\mathcal{M}_\ell$, then $1_F M$ is in $\mathbf{c}\mathcal{M}_\ell$ as well, where $1_F M$ denotes the process $(1_F M)_t = 1_F M_t$.*

Proof. Let $M, N \in \mathbf{c}\mathcal{M}_\ell$ and let $\alpha, \beta \in \mathbb{R}$. Using Lemma 1.4.2, let (T_n) be a localising sequence for both M and N. Then $(\alpha M + \beta N)^{T_n} = \alpha M^{T_n} + \beta N^{T_n}$ is a martingale, so $\alpha M + \beta N \in \mathbf{c}\mathcal{M}_\ell$ and $\mathbf{c}\mathcal{M}_\ell$ is a vector space. As regards the stopped process, let $M \in \mathbf{c}\mathcal{M}_\ell$ and let T be any stopping time. Let (T_n) be a localising sequence for M. As $M^{T_n} \in \mathbf{c}\mathcal{M}$, we obtain that $(M^T)^{T_n} = (M^{T_n})^T \in \mathbf{c}\mathcal{M}$, proving that (T_n) is also a localising sequence for M^T, so that $M^T \in \mathbf{c}\mathcal{M}_\ell$. Finally, let $M \in \mathbf{c}\mathcal{M}_\ell$ and $F \in \mathcal{F}_0$. Let (T_n) be a localising sequence such that $M^{T_n} \in \mathbf{c}\mathcal{M}$. For any bounded stopping time T, $E1_F M_T^{T_n} = E1_F(EM_T^{T_n}|\mathcal{F}_0) = 0$ by Theorem 1.2.7, so by Lemma 1.2.9, $1_F M^{T_n}$ is a martingale. As $(1_F M)^{T_n} = 1_F M^{T_n}$, $1_F M$ is in $\mathbf{c}\mathcal{M}_\ell$. □

Lemma 1.4.4. *Let $M \in \mathbf{c}\mathcal{M}_\ell$. There exists a localising sequence (T_n) such that M^{T_n} is a bounded continuous martingale for all n. In particular, $M^{T_n} \in \mathbf{c}\mathcal{M}^2$ and $M^{T_n} \in \mathbf{c}\mathcal{M}^u$.*

Proof. Let S_n be a localising sequence for M and let $U_n = \inf\{t \geq 0 \mid |M_t| > n\}$. By Lemma 1.1.11, (U_n) is a localising sequence and M^{U_n} is bounded. Defining $T_n = S_n \wedge U_n$, (T_n) is a localising sequence by Lemma 1.4.2. By definition, M^{S_n} is in $\mathbf{c}\mathcal{M}$. As $M^{T_n} = (M^{S_n})^{U_n}$ and M^{S_n} is in $\mathbf{c}\mathcal{M}$, Lemma 1.2.8 shows that M^{T_n} is in $\mathbf{c}\mathcal{M}$ as well. And as $M^{T_n} = (M^{U_n})^{S_n}$ and M^{U_n} is bounded, M^{T_n} is bounded as well. Thus, M^{T_n} is a bounded continuous martingale. M^{T_n} is then also bounded in $\mathcal{L}^2$, so we also obtain $M^{T_n} \in \mathbf{c}\mathcal{M}^2$ and $M^{T_n} \in \mathbf{c}\mathcal{M}^u$. □

Lemma 1.4.5. *If $M \in \mathbf{c}\mathcal{M}_\ell$ with paths of finite variation, then M is evanescent.*

Proof. Using Lemma 1.4.4, let (T_n) be a localising sequence for M such that $M^{T_n} \in \mathbf{c}\mathcal{M}$. Then M^{T_n} also has paths of finite variation, so by Lemma 1.2.12, M^{T_n} is evanescent. As T_n tends to infinity, we conclude that M is evanescent as well. □

Next, in order to extend the notion of the quadratic variation process from bounded continuous martingales to $\mathbf{c}\mathcal{M}_\ell$, we introduce some further spaces of stochastic processes. By $\mathbf{c}\mathcal{A}$, we denote the space of continuous, adapted and increasing processes with initial value zero, and by $\mathbf{c}\mathcal{V}$, we denote the space of continuous, adapted processes with finite variation and initial value zero. If $A \in \mathbf{c}\mathcal{V}$, we have that $V_A \in \mathbf{c}\mathcal{A}$. Clearly, $\mathbf{c}\mathcal{A}^i$ are the elements of $\mathbf{c}\mathcal{A}$ such that A_∞ is integrable. We also define $\mathbf{c}\mathcal{V}^i$ as the elements of $\mathbf{c}\mathcal{V}$ such that $V_A \in \mathbf{c}\mathcal{A}^i$, implying that $(V_A)_\infty$ is integrable.

Lemma 1.4.6. *If $A \in \mathbf{c}\mathcal{A}$, there is a localising sequence (T_n) such that $A^{T_n} \in \mathbf{c}\mathcal{A}^i$, and if $A \in \mathbf{c}\mathcal{V}$, there is a localising sequence such that $A^{T_n} \in \mathbf{c}\mathcal{V}^i$.*

Proof. Consider first $A \in \mathbf{c}\mathcal{A}$, such that A is continuous, adapted and increasing. Define $T_n = \inf\{t \geq 0 \mid A_t > n\}$. By Lemma 1.1.11, (T_n) is a localising sequence such that A^{T_n} is bounded, so $A^{T_n}_\infty$ is integrable, and so $A^{T_n} \in \mathbf{c}\mathcal{A}^i$, as desired. Considering the case $A \in \mathbf{c}\mathcal{V}$, we may instead define $T_n = \inf\{t \geq 0 \mid (V_A)_t > n\}$ and obtain the same result. □

Theorem 1.4.7. *Let $M \in \mathbf{c}\mathcal{M}_\ell$. There exists a process $[M] \in \mathbf{c}\mathcal{A}$, unique up to indistinguishability, such that $M^2 - [M] \in \mathbf{c}\mathcal{M}_\ell$. If $M, N \in \mathbf{c}\mathcal{M}_\ell$, there exists a process $[M,N] \in \mathbf{c}\mathcal{V}$, unique up to indistinguishability, such that $MN - [M,N] \in \mathbf{c}\mathcal{M}_\ell$. We call $[M]$ the quadratic variation of M, and we call $[M,N]$ the quadratic covariation of M and N.*

Proof. Consider first the case of the quadratic variation. As $M \in \mathbf{c}\mathcal{M}_\ell$, there exists by Lemma 1.4.4 a localising sequence (T_n) such that $M^{T_n} \in \mathbf{c}\mathcal{M}^b$. By Theorem 1.3.6, there exists a process $[M^{T_n}]$ in $\mathbf{c}\mathcal{A}^i$, unique up to indistinguishability, with the property that $(M^{T_n})^2 - [M^{T_n}] \in \mathbf{c}\mathcal{M}^2$. Note that $(M^{T_n})^2 - [M^{T_{n+1}}]^{T_n} = ((M^{T_{n+1}})^2 - [M^{T_{n+1}}])^{T_n}$, which is in $\mathbf{c}\mathcal{M}^2$ by Lemma 1.3.2. Therefore, by uniqueness, $[M^{T_{n+1}}]^{T_n} = [M^{T_n}]$ up to indistinguishability. In particular, $[M^{T_{n+1}}]1_{(t \leq T_n)}$ and $[M^{T_n}]1_{(t \leq T_n)}$ are indistinguishable. Now, with $T_0 = 0$, define

$$[M]_t = \sum_{k=1}^\infty [M^{T_k}]_t 1_{(T_{k-1} < t \leq T_k)}.$$

It then holds that, up to indistinguishability,

$$\begin{aligned} [M]^{T_n}_t &= [M]_{T_n} 1_{(t > T_n)} + \sum_{k=1}^n [M^{T_k}]_t 1_{(T_{k-1} < t \leq T_k)} \\ &= [M]_{T_n} 1_{(t > T_n)} + \sum_{k=1}^n [M^{T_n}]_t 1_{(T_{k-1} < t \leq T_k)} = [M^{T_n}]_t. \end{aligned}$$

As $[M^{T_n}]$ is in $\mathbf{c}\mathcal{A}^i$, this shows that $[M]$ is in $\mathbf{c}\mathcal{A}$. Also, $(M^2 - [M])^{T_n} = (M^{T_n})^2 - [M^{T_n}]$, which is in $\mathbf{c}\mathcal{M}^2$, yielding $M^2 - [M] \in \mathbf{c}\mathcal{M}_\ell$. This proves the existence of $[M]$. Uniqueness follows from Lemma 1.4.5.

As for the quadratic covariation, consider any two processes $M, N \in \mathbf{c}\mathcal{M}_\ell$. Recalling the polarization identity $4MN = (M+N)^2 - (M-N)^2$, we define $[M,N] = \frac{1}{4}([M+N] - [M-N])$.

We then obtain $MN-[M,N]=\frac{1}{4}((M+N)^2-[M+N])-\frac{1}{4}((M-N)^2-[M-N])$, so that $MN-[M,N]$ is in $\mathbf{c}\mathcal{M}_\ell$. Furthermore, as $[M+N]$ and $[M-N]$ are in $\mathbf{c}\mathcal{A}$, we find that $[M,N]\in\mathbf{c}\mathcal{V}$. This proves the existence of $[M,N]$. As in the case of the quadratic variation process, uniqueness follows from Lemma 1.4.5. □

The quadratic covariation process of Theorem 1.4.7 will be indispensable to our construction of the stochastic integral. As an essential example, we next consider the quadratic covariation process of the coordinates of a p-dimensional $\mathcal{F}_t$ Brownian motion.

Theorem 1.4.8. *Let W be a p-dimensional $\mathcal{F}_t$ Brownian motion. For $i\le p$, $[W^i]_t=t$ up to indistinguishability, and for $i,j\le p$ with $i\neq j$, $[W^i,W^j]$ is zero up to indistinguishability.*

Proof. By Theorem 1.2.1, it holds for $i\le p$ that $(W^i_t)^2-t$ is a martingale, in particular an element of $\mathbf{c}\mathcal{M}_\ell$, and so $[W^i]_t=t$ up to indistinguishability, by the characterization given in Theorem 1.4.7. Likewise, Theorem 1.2.1 shows that for $i,j\le p$ with $i\neq j$, $W^i_tW^j_t$ is a martingale, in particular an element of $\mathbf{c}\mathcal{M}_\ell$, so $[W^i,W^j]$ is zero up to indistinguishability by Theorem 1.4.7. □

Before ending the section, we prove some general properties of the quadratic covariation process.

Lemma 1.4.9. *Let M and N be in $\mathbf{c}\mathcal{M}_\ell$, and let T by any stopping time. The quadratic covariation satisfies the following properties up to indistinguishability.*

1. $[M,M]=[M]$.
2. $[\cdot,\cdot]$ *is symmetric and linear in both of its arguments.*
3. *For any $\alpha\in\mathbb{R}$, $[\alpha M]=\alpha^2[M]$.*
4. $[M+N]=[M]+2[M,N]+[N]$.
5. $[M,N]^T=[M^T,N]=[M,N^T]=[M^T,N^T]$.
6. $[M,N]$ *is zero if and only if $MN\in\mathbf{c}\mathcal{M}_\ell$.*
7. *M is evanescent if and only if $[M]$ is evanescent.*
8. *M is evanescent if and only if $[M,N]$ is zero for all $N\in\mathbf{c}\mathcal{M}_\ell$.*
9. *M is evanescent if and only if $[M,N]$ is zero for all $N\in\mathbf{c}\mathcal{M}^b$.*

10. If $F \in \mathcal{F}_0$, $1_F[M,N] = [1_F M, N] = [M, 1_F N] = [1_F M, 1_F N]$.

Proof. **Proof of (1).** We know that $[M]$ is in $\mathbf{c}\mathcal{A}$ and satisfies $M^2 - [M] \in \mathbf{c}\mathcal{M}_\ell$. Therefore, $[M]$ is in particular in $\mathbf{c}\mathcal{V}$, and therefore satisfies the requirements characterizing $[M,M]$. By uniqueness, we conclude that $[M,M] = [M]$ up to indistinguishability.

Proof of (2). As $MN - [M,N]$ is in $\mathbf{c}\mathcal{M}_\ell$ if and only if this holds for $NM - [M,N]$, we have that the quadratic covariation is symmetric in the sense that $[M,N] = [N,M]$ up to indistinguishability. In particular, it suffices to prove that the quadratic covariation is linear in its first coordinate. Fix $M, M' \in \mathbf{c}\mathcal{M}_\ell$ and $\alpha, \beta \in \mathbb{R}$, then

$$(\alpha M + \beta M')N - (\alpha[M,N] + \beta[M',N]) = \alpha(MN - [M,N]) + \beta(M'N - [M',N]),$$

so $(\alpha M + \beta M')N - (\alpha[M,N] + \beta[M',N])$ is in $\mathbf{c}\mathcal{M}_\ell$ and so by uniqueness, we have the linearity relationship $[\alpha M + \beta M', N] = \alpha[M,N] + \beta[M',N]$ up to indistinguishability.

Proof of (3). This is immediate from $[\alpha M] = [\alpha M, \alpha M] = \alpha^2[M,M] = \alpha^2[M]$, using the linearity properties already proven.

Proof of (4). This follows as

$$\begin{aligned} [M+N] &= [M+N, M+N] \\ &= [M,M] + [M,N] + [N,M] + [N,N] \\ &= [M] + 2[M,N] + [N], \end{aligned}$$

using the symmetry and linearity properties already proven.

Proof of (5). Note that as M^T and N^T are in $\mathbf{c}\mathcal{M}_\ell$, the conclusion is well-defined by Lemma 1.4.3. To prove the result, first note that by symmetry, it suffices to prove $[M,N]^T = [M^T, N]$, and this will be accomplished if we can show that $M^T N - [M,N]^T$ is in $\mathbf{c}\mathcal{M}_\ell$. We have $M^T N - [M,N]^T = (MN - [M,N])^T + M^T(N - N^T)$, where $(MN - [M,N])^T \in \mathbf{c}\mathcal{M}_\ell$ by Lemma 1.4.3. Therefore, it suffices to prove that $M^T(N - N^T)$ is in $\mathbf{c}\mathcal{M}_\ell$.

To this end, first consider the case of $N, M \in \mathbf{c}\mathcal{M}^b$. For any stopping time S, we then find $EM_S^T(N_S - N_S^T) = EM_{S\wedge T}E(N_S - N_{S\wedge T}|\mathcal{F}_{S\wedge T}) = 0$ by the optional sampling theorem. Therefore, by Lemma 1.2.9, $M^T(N - N^T) \in \mathbf{c}\mathcal{M}^u$. In the case where $M, N \in \mathbf{c}\mathcal{M}_\ell$, we may let (S_n) be a localising sequence such that M^{S_n} and N^{S_n} are in $\mathbf{c}\mathcal{M}^b$. As we have the relation $(M^T(N - N^T))^{S_n} = (M^{S_n})^T(N^{S_n} - (N^{S_n})^T)$, we conclude that $(M^T(N - N^T))^{S_n}$ is in $\mathbf{c}\mathcal{M}^u$, and so $M^T(N - N^T)$ is in $\mathbf{c}\mathcal{M}_\ell$, as desired.

Proof of (6). This is immediate from the definition of the quadratic covariation.

Proof of (7). If M is the zero process, then the zero process satisfies the requirements for being the quadratic variation of M. Conversely, assume that $[M]$ is evanescent. Then M^2 is in $\mathbf{c}\mathcal{M}_\ell$. Letting T_n be a localising sequence for M^2 such that $(M^2)^{T_n}$ is in $\mathbf{c}\mathcal{M}$, we find that $EM^2_{T_n\wedge t} = E(M^2)^{T_n}_t = 0$, so that $M^2_{T_n\wedge t}$ is almost surely zero. Therefore, M_t is almost surely zero as well. As $t \geq 0$ was arbitrary and M is continuous, we conclude that M is evanescent.

Proof of (8). Assume that M is evanescent. Then the zero process satisfies the requirements characterizing $[M,N]$ for all $N \in \mathbf{c}\mathcal{M}_\ell$, and so $[M,N]$ is evanescent for all $N \in \mathbf{c}\mathcal{M}_\ell$. Conversely, assume that $[M,N]$ is evanescent for all $N \in \mathbf{c}\mathcal{M}_\ell$. In particular, $[M,M]$ is evanescent, so by what was already shown, M is evanescent.

Proof of (9). If M is evanescent, the results already shown yield that $[M,N]$ is evanescent for all $N \in \mathbf{c}\mathcal{M}_\ell$ and in particular for all $N \in \mathbf{c}\mathcal{M}^b$. Conversely, assume that $[M,N]$ is evanescent for all $N \in \mathbf{c}\mathcal{M}^b$. Let $N \in \mathbf{c}\mathcal{M}_\ell$. By Lemma 1.4.4, there exists a localising sequence (T_n) such that $N^{T_n} \in \mathbf{c}\mathcal{M}^b$. We have $[M,N]^{T_n} = [M,N^{T_n}]$, which is zero, so we conclude that $[M,N]$ is evanescent, and so M is evanescent by the results above.

Proof of (10). Note that the conclusion is well-defined, as $1_F M$ is in $\mathbf{c}\mathcal{M}_\ell$ by Lemma 1.4.3. By the properties already proven for the quadratic covariation, it suffices to prove that for any $F \in \mathcal{F}_0$ and $M,N \in \mathbf{c}\mathcal{M}_\ell$, $1_F[M,N] = [1_F M,N]$. However, we know that $MN - [M,N]$ is in $\mathbf{c}\mathcal{M}_\ell$, and so by Lemma 1.4.3, $1_F MN - 1_F[M,N]$ is in $\mathbf{c}\mathcal{M}_\ell$. Therefore, by the characterisation of the quadratic covariation, $1_F[M,N]$ is the quadratic covariation process of $1_F M$ and N, meaning that $1_F[M,N] = [1_F M,N]$, as desired. □

Lemma 1.4.10. *Let $M \in \mathbf{c}\mathcal{M}_\ell$. $M \in \mathbf{c}\mathcal{M}^2$ if and only if $[M]_\infty$ is integrable, and in the affirmative, $M^2 - [M] \in \mathbf{c}\mathcal{M}^u$. If M and N are in $\mathbf{c}\mathcal{M}^2$, then $[M,N]$ is in $\mathbf{c}\mathcal{V}^i$, in particular $[M,N]_\infty$ exists and is integrable as well, and $MN - [M,N] \in \mathbf{c}\mathcal{M}^u$.*

Proof. First assume $M \in \mathbf{c}\mathcal{M}^2$. We know that $M^2 - [M] \in \mathbf{c}\mathcal{M}_\ell$. Using Lemma 1.4.4, let (T_n) be a localising sequence with $(M^2 - [M])^{T_n} \in \mathbf{c}\mathcal{M}^u$. By the optional sampling theorem and Theorem 1.3.1, $E[M]_{T_n} = E[M]^{T_n}_\infty = E(M^{T_n}_\infty)^2 = EM^2_{T_n} \leq 4EM^2_\infty$, and then, as $[M]$ is increasing, $E[M]_\infty = E\lim_n [M]_{T_n} = \lim_n E[M]_{T_n} \leq 4EM^2_\infty$ by the monotone convergence theorem, so that $[M]_\infty$ is integrable. Assume conversely that $[M]_\infty$ is integrable. Let (T_n) be a localising sequence with $M^{T_n} \in \mathbf{c}\mathcal{M}^2$ and $(M^2 - [M])^{T_n} \in \mathbf{c}\mathcal{M}^u$. Applying Theorem 1.3.1, $E\sup_{0\leq s\leq T_n} M^2_s = E((M^{T_n})^{*2}_\infty) \leq 4E(M^{T_n}_\infty)^2 = 4E[M]^{T_n}_\infty = 4E[M]_{T_n}$. Using the monotone convergence theorem, we then obtain $EM^{*2}_\infty \leq 4E[M]_\infty$, in particular $\sup_{t\geq 0} EM_t^2$ is finite and so $M \in \mathbf{c}\mathcal{M}^2$, as desired.

It remains to prove that in the case where $[M]_\infty$ is integrable and $M \in \mathbf{c}\mathcal{M}^2$, we have $M^2 - [M] \in \mathbf{c}\mathcal{M}^u$. We use Lemma 1.2.9. Note that $M^2 - [M]$ is continuous with initial value zero and convergent to an almost sure limit. Let T be any stopping time. As $[M]_\infty$ is integrable and $M \in \mathbf{c}\mathcal{M}^2$, we know that $M_T^2 - [M]_T$ is integrable as well, we need to show that $E(M_T^2 - [M]_T)$ is zero. To this end, let (T_n) be a localising sequence such that $(M^2 - [M])^{T_n} \in \mathbf{c}\mathcal{M}^u$. We then obtain

$$E[M]_T = E\lim_n [M]_{T\wedge T_n} = \lim_n E[M]_T^{T_n} = \lim_n E(M_T^2)^{T_n} = \lim_n EM^2_{T\wedge T_n}.$$

Now, as $(M_T - M_{T\wedge T_n})^2 \leq 4M^{*2}_\infty$, which is integrable by Theorem 1.3.1, and $M_{T\wedge T_n}$ converges almost surely to M_T, we find that $M_{T\wedge T_n}$ converges in $\mathcal{L}^2$ to M_T, so that $EM^2_{T\wedge T_n}$ tends to EM_T^2, finally allowing us to conclude that $E[M]_T = EM_T^2$ and so Lemma 1.2.9 shows that $M^2 - [M] \in \mathbf{c}\mathcal{M}^u$.

Finally, consider two elements M and N of $\mathbf{c}\mathcal{M}^2$. As $[M,N] = \frac{1}{2}([M+N] - [M] - [N])$, we find by our previous results that $[M,N]$ is in $\mathbf{c}\mathcal{V}^i$ and that the limit $[M,N]_\infty$ exists and is integrable. Noting that $MN - [M,N] = \frac{1}{2}((M+N)^2 - [M+N]) - \frac{1}{2}(M^2 - [M]) - \frac{1}{2}(N^2 - [N])$, we find that that $MN - [M,N]$ is in $\mathbf{c}\mathcal{M}^u$ as a linear combination of elements in $\mathbf{c}\mathcal{M}^u$. □

Lemma 1.4.11. *Let (M^n) be a sequence in $\mathbf{c}\mathcal{M}_\ell$ and let T be some stopping time. Then $(M^n)^*_T \xrightarrow{P} 0$ if and only if $[M^n]_T \xrightarrow{P} 0$.*

Proof. First note that $(M^n)^*_T = ((M^n)^T)^*_\infty$ and $[M^n]_T = [(M^n)^T]_\infty$, so it suffices to prove that $(M^n)^*_\infty \xrightarrow{P} 0$ if and only if $[M^n]_\infty \xrightarrow{P} 0$. First assume $(M^n)^*_\infty \xrightarrow{P} 0$. Fix $\varepsilon > 0$, we need to show $\lim_n P([M^n]_\infty > \varepsilon) = 0$. To do so, we take $\delta > 0$, and define the stopping time T_n by putting $T_n = \inf\{t \geq 0 \mid |M_t^n| > \delta\}$. We then have $(T_n < \infty) = ((M^n)^*_\infty > \delta)$. Markov's inequality shows

$$\begin{aligned} P([M^n]_\infty > \varepsilon) &= P([M^n]_\infty > \varepsilon, T_n < \infty) + P([M^n]_\infty > \varepsilon, T_n = \infty) \\ &\leq P(T_n < \infty) + P([M^n]_{T_n} > \varepsilon) \\ &\leq P((M^n)^*_\infty > \delta) + \tfrac{1}{\varepsilon}E([M^n]_{T_n}). \end{aligned}$$

Now note that $(M^n)^{T_n}$ is bounded by δ, in particular it is in $\mathbf{c}\mathcal{M}^2$, so by Lemma 1.4.10, $((M^n)^{T_n})^2 - [(M^n)^{T_n}]$ is in $\mathbf{c}\mathcal{M}^u$ and so $E[(M^n)]_{T_n} = E(M^n)^2_{T_n} \leq \delta^2$. We now conclude that for any $\delta > 0$, $P([M^n]_\infty > \varepsilon) \leq P((M^n)^*_\infty > \delta) + \frac{1}{\varepsilon}\delta^2$, so $\limsup_n P([M^n]_\infty > \varepsilon) \leq \frac{1}{\varepsilon}\delta^2$, and as $\delta > 0$ was arbitrary, we conclude that $[M^n]_\infty \xrightarrow{P} 0$, as desired. Consider the converse statement. We assume that $[M^n]_\infty \xrightarrow{P} 0$ and we wish to prove $(M^n)^*_\infty \xrightarrow{P} 0$. Let $\varepsilon > 0$. As before, we fix $\delta > 0$ and define $T_n = \inf\{t \geq 0 \mid [M^n]_t > \delta\}$, so that T_n is a stopping time and $(T_n < \infty) = ([M^n]_\infty > \delta)$. We then use Chebychev's inequality to see that

$$\begin{aligned}
P((M^n)^*_\infty > \varepsilon) &= P((M^n)^*_\infty > \varepsilon, T_n < \infty) + P((M^n)^*_\infty > \varepsilon, T_n = \infty) \\
&\leq P(T_n < \infty) + P((M^n)^*_{T_n} > \varepsilon) \\
&\leq P([M^n]_\infty > \delta) + \tfrac{1}{\varepsilon^2} E(M^n)^{*2}_{T_n}.
\end{aligned}$$

Now, as $[(M^n)^{T_n}]_\infty \leq \delta$, $[(M^n)^{T_n}]_\infty$ is in particular integrable, and so Lemma 1.4.10 shows that $(M^n)^{T_n}$ is in $\mathbf{c}\mathcal{M}^2$ and $((M^n)^{T_n})^2 - [(M^n)^{T_n}]$ is in $\mathbf{c}\mathcal{M}^u$. Therefore, Theorem 1.3.1 yields $E(M^n)^{*2}_{T_n} = E((M^n)^{T_n})^{*2}_\infty \leq 4E((M^n)^{T_n})^2_\infty = 4E[(M^n)^{T_n}]_\infty = 4E[M^n]_{T_n}$, which is less than 4δ, so all in all, we find $P((M^n)^*_\infty > \varepsilon) \leq P([M^n]_\infty > \delta) + \frac{4}{\varepsilon^2}\delta$. Thus, we may now conclude $\limsup_n P((M^n)^*_\infty > \varepsilon) \leq \frac{4}{\varepsilon^2}\delta$, and as $\delta > 0$ was arbitrary, we obtain $(M^n)^*_\infty \xrightarrow{P} 0$. □

We end the section with a proof of the Kunita-Watanabe inequality. Recall that integrals of the form $\int_0^t h(s) |\,\mathrm{d}f_s|$ denote integration with respect to the variation of f, see Appendix A.1 for the definition of the Lebesgue integral with respect to the variation of a mapping with finite variation.

Theorem 1.4.12 (Kunita-Watanabe)**.** *Let $M, N \in \mathbf{c}\mathcal{M}_\ell$, and let X and Y be measurable processes. Then it almost surely holds that*

$$\int_0^\infty |X_t Y_t| |\,\mathrm{d}[M,N]_t| \leq \left(\int_0^\infty X_t^2 \,\mathrm{d}[M]_t\right)^{\frac{1}{2}} \left(\int_0^\infty Y_t^2 \,\mathrm{d}[N]_t\right)^{\frac{1}{2}}.$$

Proof. First note that the result is well-defined for each ω, as $[M,N]$, $[M]$ and $[N]$ have paths of finite variation for each ω, and the mappings $|X_tY_t|$, X_t^2 and Y_t^2 from $\mathbb{R}_+$ to $\mathbb{R}$ are Borel measurable for each ω.

Applying Lemma A.1.11, it suffices to prove that we almost surely have the inequality $|[M,N]_t - [M,N]_s| \leq \sqrt{[M]_t - [M]_s}\sqrt{[N]_t - [N]_s}$ for all $0 \leq s \leq t$. As the processes are continuous, it suffices to prove the result almost surely for any pair of rational s and t. Fix any such pair, by Lemma A.1.10 it suffices to prove that for all $\lambda \in \mathbb{Q}$, we have the inequality $\lambda^2([M]_t - [M]_s) + 2\lambda([M,N]_t - [M,N]_s) + [N]_t - [N]_s \geq 0$. Thus, we need to prove that this inequality holds almost surely for rational s, t and λ with $0 \leq s \leq t$. To this end, note that $\lambda^2[M]_s + 2\lambda[M,N]_s + [N]_s = [\lambda M]_s + 2[\lambda M, N]_s + [N]_s = [\lambda M + N]_s$, and $[\lambda M + N]_s \leq [\lambda M + N]_t$, so by performing the same calculations in reverse, we obtain

$\lambda^2[M]_s + 2\lambda[M,N]_s + [N]_s \leq \lambda^2[M]_t + 2\lambda[M,N]_t + [N]_t$, yielding the desired conclusion. The theorem now follows from Lemma A.1.11. $\square$

1.5 Exercises

Exercise 1.1. Let S and T be two stopping times. Show that $\mathcal{F}_{S\vee T} = \sigma(\mathcal{F}_S, \mathcal{F}_T)$.

Exercise 1.2. Let X be a continuous and adapted stochastic process, and let $a \in \mathbb{R}$. Put $T = \inf\{t \geq 0 \mid X_t = a\}$. Prove that T is a stopping time and that $X_T = a$ whenever $T < \infty$.

Exercise 1.3. Let T be exponentially distributed and let $X_t = 1_{(t\geq T)}$. Argue that X does not have continuous sample paths, but for any $t \geq 0$, it holds that whenever (t_n) tends to t, $X_{t_n} \xrightarrow{P} X_t$ for $p \geq 1$.

Exercise 1.4. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be some mapping. A point t is said to be a local maximum for f if there exists $\varepsilon > 0$ such that $f(t) = \sup\{f(s) \mid s \in \mathbb{R}_+, |t-s| \leq \varepsilon\}$. Let X be a continuous process, let $t \geq 0$ and define $F = (t \text{ is a local maximum for } X)$. Show that $F \in \mathcal{F}_t$.

Exercise 1.5. Let (T_n) be a decreasing sequence of stopping times with limit T. Show that T is a stopping time and that $\mathcal{F}_T = \cap_{n=1}^\infty \mathcal{F}_{T_n}$.

Exercise 1.6. Assume that M is a continuous martingale with respect to the filtration $(\mathcal{F}_t)$. Consider a filtration $(\mathcal{G}_t)$ such that $\mathcal{G}_t \subseteq \mathcal{F}_t$ and such that M is also adapted to $(\mathcal{G}_t)$. Show that M is a continuous martingale with respect to $(\mathcal{G}_t)$.

Exercise 1.7. Let Z be a uniformly integrable continuous supermartingale. Show that Z is continuous in $\mathcal{L}_1$, in the sense that the mapping $t \mapsto Z_t$ from $\mathbb{R}_+$ to $\mathcal{L}_1$ is continuous.

Exercise 1.8. Let $M \in \mathbf{c}\mathcal{M}^b$ and let M_∞ be the limit of M_t almost surely and in $\mathcal{L}^1$. With $c \geq 0$ denoting a bound for M, show that $E|M_t - M_\infty| \leq 4c\sup_{F\in\mathcal{F}_\infty}\inf_{G\in\mathcal{F}_t} P(G\Delta F)$, where the symmetric difference $G\Delta F$ is defined by putting $G\Delta F = (G\cap F^c)\cup(F\cap G^c)$.

Exercise 1.9. Let $M \in \mathbf{c}\mathcal{M}_\ell$. Show that $M \in \mathbf{c}\mathcal{M}^u$ if and only if $(M_T)_{T\in\mathcal{C}}$ is uniformly integrable, where $\mathcal{C} = \{T | T \text{ is a bounded stopping time}\}$.

Exercise 1.10. Let $M \in \mathbf{c}\mathcal{M}_\ell$ and let $S \leq T$ be two stopping times. Show that if the equality $[M]_S = [M]_T$ holds almost surely, then $M^T = M^S$ almost surely.

Exercise 1.11. Assume that X is a continuous adapted process with initial value zero and that S and T are stopping times. Show that if X^T and X^S are in $\mathbf{c}\mathcal{M}^u$, then $X^{S\wedge T}$ and $X^{S\vee T}$ are in $\mathbf{c}\mathcal{M}^u$ as well.

Exercise 1.12. Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion. Put $M_t = W_t^2 - t$. Show that M is not uniformly integrable.

Exercise 1.13. Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion. Let $t \geq 0$ and define $t_k^n = kt2^{-n}$ for $k \leq 2^n$. Show that $\sum_{k=1}^{2^n}(W_{t_k^n} - W_{t_{k-1}^n})^2$ converges in $\mathcal{L}^2$ to t. Conclude that $\sum_{k=1}^{2^n} W_{t_{k-1}^n}(W_{t_k^n} - W_{t_{k-1}^n}) \xrightarrow{P} \frac{1}{2}W_t^2 - \frac{1}{2}t$ and $\sum_{k=1}^{2^n} W_{t_k^n}(W_{t_k^n} - W_{t_{k-1}^n}) \xrightarrow{P} \frac{1}{2}W_t^2 + \frac{1}{2}t$ as n tends to infinity.

Exercise 1.14. Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion. Let $a, b > 0$ and define $T = \inf\{t \geq 0 \mid W_t = -a \text{ or } W_t = b\}$. Show that T is a stopping time. Find the distribution of W_T. Show that if $a \neq b$, it holds that W^T and $-W^T$ have different distributions, while they have the same quadratic variation processes.

Exercise 1.15. Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion and let $\alpha \in \mathbb{R}$. Show that the process M^α defined by $M_t^\alpha = \exp(\alpha W_t - \frac{1}{2}\alpha^2 t)$ is a martingale. Let $a \in \mathbb{R}$ and define $T = \inf\{t \geq 0 \mid W_t = a\}$. Show that for any $\beta \geq 0$, $E\exp(-\beta T) = \exp(-|a|\sqrt{2\beta})$.

Exercise 1.16. Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion. Show by direct calculation that the processes $W_t^3 - 3tW_t$ and $W_t^4 - 6tW_t^2 + 3t^2$ are in $\mathbf{c}\mathcal{M}$.

Exercise 1.17. Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion and define T by putting $T = \inf\{t \geq 0 \mid W_t \geq a + bt\}$. Show that T is a stopping time and that for $a > 0$ and $b > 0$, it holds that $P(T < \infty) = \exp(-2ab)$.

Exercise 1.18. Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion. Let $a > 0$ and define $T = \inf\{t \geq 0 \mid W_t^2 \geq a(1-t)\}$. Show that T is a stopping time. Find ET and ET^2.

Exercise 1.19. Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion, and let $p > \frac{1}{2}$. Show that $\frac{1}{n^p}\sup_{0\leq s\leq n}|W_s| \xrightarrow{P} 0$.

Chapter 2

Stochastic integration

In this chapter, we define the stochastic integral and prove its basic properties. We define the stochastic integral with respect to continuous semimartingales, which are sums of continuous local martingales and continuous processes with paths of finite variation.

Section 2.1 concerns itself with the introduction of continuous semimartingales and their basic properties. In Section 2.2, we perform the construction of the stochastic integral. This is done by a separate construction for continuous local martingales and continuous finite variation processes. For the integral with respect to continuous finite variation processes, the construction is pathwise and the main difficulty is ensuring the proper measurability properties of the resulting process. For the integral with respect to continuous local martingales, the integral cannot be defined pathwise as in ordinary measure theory, since in general, continuous local martingales do not have paths of finite variation. The construction is instead made in a rather abstract fashion. In Section 2.3, however, we show that the integral may also be obtained as a limit in probability, demonstrating that the integral is the limit of a sequence of Riemann sums in a particular sense. In this section, we also develop a few of the general properties of the stochastic integral, ending with the proof of Itô's formula, the stochastic version of the fundamental theorem of analysis.

In Section 2.4, we discuss extensions of the theory and literature for further reading.

2.1 Continuous semimartingales

In this section, we introduce continuous semimartingales and consider their basic properties.

Definition 2.1.1. *A stochastic process X is said to be a continuous semimartingale if it can be decomposed as $X = X_0 + M + A$, where X_0 is an $\mathcal{F}_0$ measurable variable, $M \in \mathbf{c}\mathcal{M}_\ell$ and $A \in \mathbf{c}\mathcal{V}$, that is, M is a continuous local martingale with initial value zero, and A is continuous and adapted with finite variation and initial value zero. The space of continuous semimartingales is denoted by $\mathbf{c}\mathcal{S}$.*

Note that contrarily to most of our previous definitions, we allow semimartingales to have a nonzero initial value. In general, most issues regarding local martingales with nonzero initial value can easily be reduced to the case with initial zero value by simple subtraction. In contrast, there are many semimartingales whose properties are quite essentially related to their initial values, for example exponential martingales, or stationary solutions to stochastic differential equations. We could have chosen to include the constant as a nonzero initial value for the martingale or the finite variation parts, but this would introduce cumbersome issues regarding compatibility with previous results.

The following result shows that the decomposition in the definition of a continuous semimartingale is unique up to indistinguishability.

Lemma 2.1.2. *Let $X \in \mathbf{c}\mathcal{S}$. If $X = Z_0+M+A$ and $X = Y_0+N+B$ are two decompositions of X, where Z_0 and Y_0 are $\mathcal{F}_0$ measurable, M and N are in $\mathbf{c}\mathcal{M}_\ell$ and A and B are in $\mathbf{c}\mathcal{V}$, it holds that X_0 and Y_0 are equal, M and N are indistinguishable and A and B are indistinguishable.*

Proof. We have $Z_0 = X_0 = Y_0$, proving the first part of the uniqueness. We then find that $M + A = N + B$, so that $M - N = B - A$. Therefore, $M - N$ and $B - A$ are both continuous local martingales with initial value zero and with paths of finite variation. Lemma 1.4.5 then shows that M and N are indistinguishable and that A and B are indistinguishable. □

In the sequel, for any $X \in \mathbf{c}\mathcal{S}$, we will refer to the decomposition $X = X_0 + M + A$ where X_0 is $\mathcal{F}_0$ measurable, $M \in \mathbf{c}\mathcal{M}_\ell$ and $A \in \mathbf{c}\mathcal{V}$, as the decomposition of X. By Lemma 2.1.2, the variable X_0 and the processes M and A in the decomposition are almosts surely unique.

Lemma 2.1.3. *Let T be a stopping time. If A is in $\mathbf{c}\mathcal{A}$, $\mathbf{c}\mathcal{A}^i$, $\mathbf{c}\mathcal{V}$, or $\mathbf{c}\mathcal{V}^i$, then the same holds for A^T.*

Proof. If $A \in \mathbf{c}\mathcal{A}$, then A^T remains continuous, adapted and increasing, so $A^T \in \mathbf{c}\mathcal{A}$ as well. If $A \in \mathbf{c}\mathcal{V}$, A^T remains continuous, adapted and with paths of finite variation, so $A^T \in \mathbf{c}\mathcal{V}$. If $A \in \mathbf{c}\mathcal{A}^i$, we have $EA^T_\infty = EA_T \leq EA_\infty$, so that A^T is continuous, adapted, increasing and integrable, thus $A^T \in \mathbf{c}\mathcal{A}^i$. If $A \in \mathbf{c}\mathcal{V}^i$, we have $V_{A^T} = (V_A)^T$ by Lemma 1.2.11, so that we obtain $E(V_{A^T})_\infty = E(V_A)^T_\infty = E(V_A)_T \leq E(V_A)_\infty$. Thus, $(V_{A^T})_\infty$ is integrable and $A^T \in \mathbf{c}\mathcal{V}^i$. □

Lemma 2.1.4. *Let X be a continuous semimartingale and let T be a stopping time. Then X^T is a continuous semimartingale as well.*

Proof. This follows immediately from Lemma 2.1.3 and Lemma 1.4.3. □

Lemma 2.1.5. *Let X be a continuous semimartingale and let $F \in \mathcal{F}_0$. Then $1_F X$ is a continuous semimartingale as well.*

Proof. Let $X = X_0 + M + A$ be the decomposition of X. We claim that $1_F X$ is a continuous semimartingale with decomposition $X = 1_F X_0 + 1_F M + 1_F A$. We have that $1_F X_0$ is $\mathcal{F}_0$ measurable. By Lemma 1.4.3, $1_F M$ is in $\mathbf{c}\mathcal{M}_\ell$. $1_F A$ is continuous with paths of finite variation, and as it is also adapted, we obtain $1_F A \in \mathbf{c}\mathcal{V}$. Thus, $1_F X \in \mathbf{c}\mathcal{S}$. □

Lemma 2.1.6. *Let X be a continuous semimartingale. There exists a localising sequence (T_n) such that $X^{T_n} = X_0 + M^{T_n} + A^{T_n}$, where $M^{T_n} \in \mathbf{c}\mathcal{M}^b$ and A^{T_n} is an element of $\mathbf{c}\mathcal{V}$ with $V_{A^{T_n}}$ bounded.*

Proof. Let $X = X_0 + M + A$ be the decomposition of X into its initial, continuous martingale and continuous finite variation parts. By Lemma 1.4.4, we know that there is a localising sequence (S_n) such that M^{S_n} is in $\mathbf{c}\mathcal{M}^b$. Also, putting $U_n = \inf\{t \geq 0 \mid (V_A)_t > n\}$, we know by by Lemma 1.1.11 that (U_n) is a localising sequence, and $V_A^{U_n}$ is bounded. By Lemma 1.4.2, we may put $T_n = S_n \wedge U_n$ and obtain that (T_n) is a localising sequence. Furthermore, as $M^{T_n} = (M^{S_n})^{U_n}$, we find that M^{T_n} is in $\mathbf{c}\mathcal{M}^b$. Also, A^{T_n} is in $\mathbf{c}\mathcal{V}$ by Lemma 2.1.3, and as $V_{A^{T_n}} = V_A^{T_n} = (V_A^{U_n})^{S_n}$, we have that $V_{A^{T_n}}$ is bounded. As $X^{T_n} = X_0 + M^{T_n} + A^{T_n}$, the result follows. □

Next, we introduce the quadratic variation and quadratic covariation processes for semimartingales.

Definition 2.1.7. *Let $X \in \mathbf{c}\mathcal{S}$ with decomposition $X = X_0 + M + A$. We define the quadratic variation process of X as $[X] = [M]$. If Y is another semimartingale with decomposition $Y = Y_0 + N + B$, we define the quadratic covariation process of X and Y as $[X, Y] = [M, N]$.*

Note that from the uniqueness of the decomposition of continuous semimartingales given in Lemma 2.1.2, Definition 2.1.7 is well-formed. The following lemma is the semimartingale analogoue of Lemma 1.4.9.

Lemma 2.1.8. *Let X and Y be continuous semimartingales, and let T be any stopping time. The quadratic covariation satisfies the following properties.*

1. $[X, X] = [X]$.
2. $[\cdot, \cdot]$ *is symmetric and linear in both of its arguments.*
3. *For any* $\alpha \in \mathbb{R}$, $[\alpha X] = \alpha^2 [X]$.
4. $[X + Y] = [X] + 2[X, Y] + [Y]$.
5. $[X, Y]^T = [X^T, Y] = [X, Y^T] = [X^T, Y^T]$.
6. *If* $F \in \mathcal{F}_0$, $1_F [X, Y] = [1_F X, Y] = [X, 1_F Y] = [1_F X, 1_F Y]$.
7. *With* $A, B \in \mathbf{c}\mathcal{V}$, $[X + A, Y + B] = [X, Y]$.

Proof. These results follow immediately from Lemma 1.4.9, with the exception of the last property, which follows as X and $X + A$ have the same continuous martingale part, and so does Y and $Y + B$. □

2.2 Construction of the stochastic integral

In this section, we define the stochastic integral and consider its basic properties. In general, we will define the stochastic integral with respect to all processes $X \in \mathbf{c}\mathcal{S}$. For any $X \in \mathbf{c}\mathcal{S}$, there exists an almost surely unique decomposition $X = X_0 + M + A$, where $M \in \mathbf{c}\mathcal{M}_\ell$ and $A \in \mathbf{c}\mathcal{V}$. Given a stochastic process H satisfying suitable measurability and integrability conditions, we will define the integrals $\int_0^t H_s \,\mathrm{d}M_s$ and $\int_0^t H_s \,\mathrm{d}A_s$ separately.

We first consider the finite variation case. With $A \in \mathbf{c}\mathcal{V}$, we may use the results from Appendix A.1, in particular the correspondence between functions of finite variation and pairs of positive singular measures, to define the integral with respect to A. However, in order to obtain a useful integral, we need to ensure that the integral satisfies certain measurability properties. With $A \in \mathbf{c}\mathcal{V}$ and, say, H a bounded measurable process, we may always define a process $H \cdot A$ by putting $(H \cdot A)_t(\omega) = \int_0^t H_s(\omega) \, \mathrm{d}A(\omega)_s$, where the integral is defined as in Appendix A.1. While $H \cdot A$ thus defined will always have continuous paths of finite variation and initial value zero, we have no guarantee that it will be adapted, and adaptedness will be necessary for the integral to be useful in practice.

Thus, the difficulty in defining the stochastic integral with respect to processes $A \in \mathbf{c}\mathcal{V}$ is to identify what requirements on H are necessary to ensure that the resulting integral becomes adapted. Theorem 2.2.1 shows how to solve this problem. Recall from Appendix A.1 that for a mapping f of finite variation on $\mathbb{R}_+$ and a measurable function $h : \mathbb{R}_+ \to \mathbb{R}$, we say that h is integrable with respect to f if $\int_0^t |h(s)| |\, \mathrm{d}f_s|$ is finite for all $t \geq 0$, where $\int_0^t |h(s)| |\, \mathrm{d}f_s|$ denotes the integral of $|h|$ over $[0,t]$ with respect to the total variation measure of f.

Theorem 2.2.1. *Let $A \in \mathbf{c}\mathcal{V}$ and assume that H is progressive and that almost surely, H is integrable with respect to A. There is a process $H \cdot A \in \mathbf{c}\mathcal{V}$, unique up to indistinguishability, such that almost surely, $(H \cdot A)_t$ is the Lebesgue integral of H with respect to A over $[0,t]$ for all $t \geq 0$. If H is nonnegative and $A \in \mathbf{c}\mathcal{A}$, then $H \cdot A \in \mathbf{c}\mathcal{A}$.*

Proof. First note that as the requirements on $H \cdot A$ define the process pathwisely almost surely, $H \cdot A$ is unique up to indistinguishability. As for existence, we prove the result in three steps, first considering bounded A in $\mathbf{c}\mathcal{A}$, then general A in $\mathbf{c}\mathcal{A}$ and finally the case where we merely assume $A \in \mathbf{c}\mathcal{V}$.

Step 1. The case $A \in \mathbf{c}\mathcal{A}$, A bounded. First assume that $A \in \mathbf{c}\mathcal{A}$ and that A is bounded. Let F be the null set such that when $\omega \in F$, $H(\omega)$ is not integrable with respect to $A(\omega)$. By our assumptions on the filtration, $F \in \mathcal{F}_t$ for all $t \geq 0$, in particular we obtain that $\{(s,\omega) \in [0,t] \times \Omega \mid 1_F(\omega) = 1\} = [0,t] \times F \in \mathcal{B}_t \otimes \mathcal{F}_t$, and so the process $(t,\omega) \mapsto 1_F(\omega)$ is progressive. Therefore, the process $(t,\omega) \mapsto 1_{F^c}(\omega)$ is progressive as well. Thus, defining $K = H1_{F^c}$, K is progressive, and $K(\omega)$ is integrable with respect to $A(\omega)$ for all ω. We may then define a process Y by putting $Y_t(\omega) = \int_0^t K_s(\omega) \, \mathrm{d}A(\omega)_s$. We claim that Y satisfies the properties required of the process $H \cdot A$ in the statement of the lemma. It is immediate that Y_t almost surely is the Lebesgue integral of H with respect to A over $[0,t]$ for all $t \geq 0$. It remains to prove $Y \in \mathbf{c}\mathcal{V}$. As Y is a pathwise Lebesgue integral with respect to a nonnegative measure, Y has finite variation. We would like to prove that Y is continuous. As Y is zero on F, it suffices to show that Y is continuous on F^c. Let $\omega \in F^c$ and let $t \geq 0$. For $h \geq 0$, we obtain by Lemma A.1.7 that

$$|Y_{t+h}(\omega) - Y_t(\omega)| = \left| \int_t^{t+h} H_s(\omega) \, \mathrm{d}A(\omega)_s \right| \leq \int_t^{t+h} |H_s(\omega)| | \, \mathrm{d}A(\omega)_s|.$$

Since $A(\omega)$ accords zero measure to one-point sets, we may then apply the dominated convergence theorem with dominating function $s \mapsto |H_s(\omega)|1_{[t,t+\varepsilon]}(s)$ for some $\varepsilon > 0$ to obtain

$$\lim_{h \to 0} |Y_{t+h}(\omega) - Y_t(\omega)| \quad \leq \quad \lim_{h \to 0} \int_t^{t+h} |H_s(\omega)| | \, \mathrm{d}A(\omega)_s| = \int |H_t(\omega) 1_{\{t\}}| | \, \mathrm{d}A(\omega)_s| = 0,$$

showing that $Y(\omega)$ is right-continuous at t. Analogously, we find that $Y(\omega)$ is left-continuous at t for any $t > 0$, and so $Y(\omega)$ is continuous. This proves that Y has continuous paths. Furthermore, by construction, Y has initial value zero. Therefore, it only remains to prove that Y is adapted, meaning that Y_t is $\mathcal{F}_t$ measurable for all $t \geq 0$.

This is the case for t equal to zero, therefore, assume that $t > 0$. Let $\mu_A^t(\omega)$ be the restriction to $\mathcal{B}_t$ of the nonnegative measure induced by $A(\omega)$ according to Theorem A.1.5. We will show that $(\mu_A^t(\omega))_{\omega \in \Omega}$ is an $(\Omega, \mathcal{F}_t)$ Markov kernel on $([0,t], \mathcal{B}_t)$ in the sense that for any $B \in \mathcal{B}_t$, $\omega \mapsto \mu_A^t(\omega)(B)$ is $\mathcal{F}_t$ measurable. The family of $B \in \mathcal{B}_t$ for which this holds is a Dynkin class, and by Lemma A.1.19, it will therefore suffice to show the claim for intervals in $[0,t]$. Let $0 \leq a \leq b \leq t$. Then $\mu_A^t(\omega)[a,b] = A_b(\omega) - A_a(\omega)$, and by the adaptedness of A, this is $\mathcal{F}_t$ measurable. Therefore, $(\mu_A^t(\omega))_{\omega \in \Omega}$ is an $(\Omega, \mathcal{F}_t)$ Markov kernel on $([0,t], \mathcal{B}_t)$. Now, as K is progressive, the restriction of K to $[0,t] \times \Omega$ is $\mathcal{B}_t \otimes \mathcal{F}_t$ measurable. Theorem A.1.13 then yields that the integral $\int_0^t K_s(\omega) \, \mathrm{d}A(\omega)_s$ is $\mathcal{F}_t$ measurable, proving that Y_t is adapted. Finally, we conclude that $Y \in \mathbf{c}\mathcal{V}$.

Step 2. The case $A \in \mathbf{c}\mathcal{A}$. We now consider the case where $A \in \mathbf{c}\mathcal{A}$. By Lemma 1.1.11, there exists a localising sequence (T_n) such that A^{T_n} is in $\mathbf{c}\mathcal{A}$ and is bounded, and so, by what was already shown, there exists a process $H \cdot A^{T_n}$ in $\mathbf{c}\mathcal{V}$ such that almost surely, $(H \cdot A^{T_n})_t$ is the integral of H with respect to A^{T_n} over $[0,t]$ for all $t \geq 0$. Let F be the null set such that on F^c, T_n converges to infinity, H is integrable with respect to A and $(H \cdot A^{T_n})_t$ is the integral of H with respect to A^{T_n} over $[0,t]$ for all $t \geq 0$ and all n. As before, we put $K = H1_{F^c}$ and conclude that K is progressive, and defining $Y_t = \int_0^t K_s \, \mathrm{d}A_s$, we find that Y_t is almost surely the Lebesgue integral of H with respect to A over $[0,t]$ for all $t \geq 0$. Furthermore, whenever $\omega \in F^c$, we have

$$Y_t = \int_0^t H_s \, \mathrm{d}A_s = \lim_{n \to \infty} \int_0^{t \wedge T_n} H_s \, \mathrm{d}A_s = \lim_{n \to \infty} \int_0^t H_s \, \mathrm{d}A_s^{T_n} = \lim_{n \to \infty} (H \cdot A^{T_n})_t,$$

where the second equality follows from the dominated convergence theorem, as H is assumed to be integrable with respect to A on F^c. In particular, Y_t is the almost sure limit of a sequence of $\mathcal{F}_t$ measurable variables. Therefore, Y_t is itself $\mathcal{F}_t$ measurable. As Y is also continuous, of finite variation and has initial value zero, we conclude that $Y \in \mathbf{c}\mathcal{V}$ and so Y satisfies the requirements of the process $H \cdot A$ in the theorem.

Step 3. The case $A \in \mathbf{c}\mathcal{V}$. Finally, assume that $A \in \mathbf{c}\mathcal{V}$. Using Theorem A.1.5, we know that by putting $A^+_t = \frac{1}{2}((V_A)_t + A_t)$ and $A^-_t = \frac{1}{2}((V_A)_t - A_t)$, H is almost surely integrable with respect to A^+ and A^-. Also by Theorem A.1.5, A^+ and A^- are increasing with paths in $\mathbf{cFV}_0$. By Lemma A.1.9, we find that V_A is adapted and so A^+ and A^- are both adapted. Thus, A^+ and A^- are in $\mathbf{c}\mathcal{A}$. Therefore, by what was already shown, there are processes $H \cdot A^+$ and $H \cdot A^-$ in $\mathbf{c}\mathcal{V}$ such that almost surely, these processes at time t are the Lebesgue integrals of H with respect to A^+ and A^- over $[0,t]$ for all $t \geq 0$. The process $H \cdot A = H \cdot A^+ - H \cdot A^-$ then satisfies the requirements of the theorem. $\square$

Theorem 2.2.1 shows that given a progressively measurable process H and $A \in \mathbf{c}\mathcal{V}$ such that H is almost surely integrable with respect to A, we may define the integral pathwisely in such a manner as to obtain a process $H \cdot A \in \mathbf{c}\mathcal{V}$, where it holds for almost all ω that for any $t \geq 0$, $(H \cdot A)_t(\omega) = \int_0^t H_s(\omega)\,\mathrm{d}A(\omega)_s$. The stochastic integral of H with respect to A thus becomes another stochastic process.

Next, we consider defining the stochastic integral with respect to a continuous local martingale. The following lemma motivates our definition. To formulate the lemma, and for our later convenience, we first introduce the notion of stochastic intervals. Let S and T be two stopping times. We then define the subset $]\!]S,T]\!]$ of $\mathbb{R}_+ \times \Omega$ by putting

$$]\!]S,T]\!] = \{(t,\omega) \in \mathbb{R}_+ \times \Omega \mid S(\omega) < t \leq T(\omega)\}.$$

We define $[\![S,T]\!]$, $]\!]S,T[\![$ and $[\![S,T[\![$ in analogous manner as subsets of $\mathbb{R}_+ \times \Omega$. Note in particular that even if T is infinite, the sets $]\!]S,T]\!]$ and $[\![S,T]\!]$ do not contain infinity. Also note that by the properties of ordinary Lebesgue integration, if $H = \xi 1_{]\!]S,T]\!]}$ for ξ bounded and $\mathcal{F}_S$ measurable and $S \leq T$ two stopping times, we have $H \cdot A = \xi(A^T - A^S)$. The following lemma shows that if we consider an analogous construction with A exchanged by an element of $\mathbf{c}\mathcal{M}_\ell$, we obtain another element of $\mathbf{c}\mathcal{M}_\ell$.

Lemma 2.2.2. *Let $M \in \mathbf{c}\mathcal{M}_\ell$, let $S \leq T$ be stopping times, let ξ be bounded and $\mathcal{F}_S$ measurable, and let $H = \xi 1_{]\!]S,T]\!]}$. Then H is progressive. Defining $L = \xi(M^T - M^S)$, it holds that $L \in \mathbf{c}\mathcal{M}_\ell$, and for all $N \in \mathbf{c}\mathcal{M}_\ell$, $[L,N] = H \cdot [M,N]$.*

Proof. We first show that H is progressive. Note that for any $A \in \mathcal{B}$ with $0 \notin A$, it holds that $(H_t \in A) = (\xi \in A) \cap (S < t) \cap (t \leq T)$. By Lemma 1.1.13, $(\xi \in A) \cap (S < t) \in \mathcal{F}_t$, and as T is a stopping time, $(t \leq T) = (T < t)^c \in \mathcal{F}_t$ by Lemma 1.1.7. Thus, H is adapted. As it has left-continuous paths, Lemma 1.1.4 shows that it is progressive.

Next, we show that $L \in \mathbf{c}\mathcal{M}_\ell$. As $L_t = \xi 1_{(S \leq t)}(M^T_t - M^S_t)$, Lemma 1.1.13 shows that L is adapted. To prove the local martingale property, first consider the case where $M \in \mathbf{c}\mathcal{M}^2$. Fix any stopping time U, we then find $EL_U = E\xi(M^T_U - M^S_U) = E\xi E(M^U_T - M^U_S|\mathcal{F}_S) = 0$ by Theorem 1.2.7, so Lemma 1.2.9 shows that $L \in \mathbf{c}\mathcal{M}^u$. In the case where $M \in \mathbf{c}\mathcal{M}_\ell$, there is a localising sequence (T_n) such that $M^{T_n} \in \mathbf{c}\mathcal{M}^2$. Since we have the identity $L^{T_n} = \xi((M^{T_n})^T - (M^{T_n})^S)$, we conclude that $L^{T_n} \in \mathbf{c}\mathcal{M}^2$ and so $L \in \mathbf{c}\mathcal{M}_\ell$.

Next, we consider the quadratic covariation. Fix $N \in \mathbf{c}\mathcal{M}_\ell$. Note that as H is bounded and progressive, $H \cdot [M,N]$ is a well-defined process in $\mathbf{c}\mathcal{V}$ by Theorem 2.2.1. In order for $H \cdot [M,N]$ to be the quadratic covariation $[L,N]$, we need to show that $LN - H \cdot [M,N]$ is in $\mathbf{c}\mathcal{M}_\ell$. Consider the case where $M, N \in \mathbf{c}\mathcal{M}^2$. We will use Lemma 1.2.9 to prove that $LN - H\cdot[M,N] \in \mathbf{c}\mathcal{M}^u$. Fix any stopping time U, we first prove that $L_U N_U - (H\cdot[M,N])_U$ is well-defined and integrable. Note that by what we already proved, $L \in \mathbf{c}\mathcal{M}^u$. As $M \in \mathbf{c}\mathcal{M}^2$, we in fact have $L \in \mathbf{c}\mathcal{M}^2$, and so, by the Cauchy-Schwartz inequality,

$$E|L_U N_U| \leq (EL_U^2)^{\frac{1}{2}}(EN_U^2)^{\frac{1}{2}} \leq (EL_\infty^{*2})^{\frac{1}{2}}(EN_\infty^{*2})^{\frac{1}{2}},$$

which is finite by Theorem 1.3.1. Thus, $L_U N_U$ is integrable. As M and N are in $\mathbf{c}\mathcal{M}^2$, Lemma 1.4.10 shows that $[M,N]$ is in $\mathbf{c}\mathcal{V}^i$, and so in particular as H is bounded, $\int_0^\infty |H_s||\,\mathrm{d}[M,N]_s|$ is integrable. Therefore, the integral of H with respect to $[M,N]$ over $\mathbb{R}_+$ is almost surely well-defined, and $\int_0^U |H_s||\,\mathrm{d}[M,N]_s|$ is integrable. Lemma A.1.7 then shows that $(H\cdot[M,N])_U$ is integrable. Thus, $L_U N_U - (H\cdot[M,N])_U$ is well-defined and integrable. Now, by Lemma 1.1.13, $\xi 1_{(S\leq U)}$ is $\mathcal{F}_{S\wedge U}$ measurable, and so

$$\begin{aligned}
& EL_U N_U - (H\cdot[M,N])_U \\
= \quad & E\xi(M^T_U - M^S_U)N_U - E\xi([M,N]^T_U - [M,N]^S_U) \\
= \quad & E\xi 1_{(S\leq U)}(M^T_U - M^S_U)N_U - E\xi 1_{(S\leq U)}([M,N]^T_U - [M,N]^S_U) \\
= \quad & E\xi 1_{(S\leq U)}(M^T_U N_U - [M,N]^T_U) - E\xi 1_{(S\leq U)}(M^S_U N_U - [M,N]^S_U) \\
= \quad & E\xi 1_{(S\leq U)}E(M^T_U N_U - [M^T,N]_U|\mathcal{F}_{S\wedge U}) - E\xi 1_{(S\leq U)}E(M^S_U N_U - [M^S,N]_U|\mathcal{F}_{S\wedge U}) \\
= \quad & E\xi 1_{(S\leq U)}(M^T_{S\wedge U}N_{S\wedge U} - [M^T,N]_{S\wedge U}) - E\xi 1_{(S\leq U)}E(M^S_{S\wedge U}N_{S\wedge U} - [M^S,N]_{S\wedge U}),
\end{aligned}$$

which is zero. Therefore, by Lemma 1.2.9, $LN - H\cdot[M,N]$ is a uniformly integrable martingale. For the case of $M, N \in \mathbf{c}\mathcal{M}_\ell$, we know from Lemma 1.4.4 that there exists a localising sequence (T_n) such that $M^{T_n}, N^{T_n} \in \mathbf{c}\mathcal{M}^2$. We have

$$(LN - H\cdot[M,N])^{T_n} = \xi((M^{T_n})^T - (M^{T_n})^S)N^{T_n} - H\cdot[M^{T_n}, N^{T_n}],$$

where the latter is a uniformly integrable martingale by what we already have shown, so $LN - H \cdot [M,N]$ is in $\mathbf{c}\mathcal{M}_\ell$. This proves the lemma. □

Lemma 2.2.2 shows that by defining the integral of a simple process of the form $\xi 1_{]\!]S,T]\!]}$ with respect to $M \in \mathbf{c}\mathcal{M}_\ell$ in a manner corresponding to ordinary integrals, namely by putting $L = \xi(M^T - M^S)$ and interpreting L_t as the integral of $\xi 1_{]\!]S,T]\!]}$ over $[0,t]$ with respect to M, we obtain an element of $\mathbf{c}\mathcal{M}_\ell$ characterised by a simple quadratic covariation structure in relation to other elements of $\mathbf{c}\mathcal{M}_\ell$. We take this characterisation as our defining feature of the stochastic integral with respect to integrators in $\mathbf{c}\mathcal{M}_\ell$. We will later show that this yields an integral which in certain cases may also be interpreted as a particular limit of ordinary Riemann sums. The next theorem shows that under certain circumstances, it is possible to construct an element of $\mathbf{c}\mathcal{M}_\ell$ with the desired covariation structure. This yields the construction of the stochastic integral with respect to a continuous local martingale.

Theorem 2.2.3. *Let $M \in \mathbf{c}\mathcal{M}_\ell$ and let H be progressive. There is $L \in \mathbf{c}\mathcal{M}_\ell$, unique up to evanescence, such that for all $N \in \mathbf{c}\mathcal{M}_\ell$, H is almost surely integrable with respect to $[M,N]$ and $[L,N] = H \cdot [M,N]$, if and only if H^2 is almost surely integrable with respect to $[M]$. In the affirmative case, we define L as the stochastic integral of H with respect to M, and denote it by $H \cdot M$.*

Proof. First assume that H^2 is almost surely integrable with respect to $[M]$. We need to prove that H is integrable with respect to $[M,N]$ for all $N \in \mathbf{c}\mathcal{M}_\ell$, and we need to prove that there exists a process $L \in \mathbf{c}\mathcal{M}_\ell$, unique up to evanescence, such that $[L,N] = H \cdot [M,N]$ for all $N \in \mathbf{c}\mathcal{M}_\ell$.

Fix $N \in \mathbf{c}\mathcal{M}_\ell$. By the Kunita-Watanabe inequality of Theorem 1.4.12, it holds that

$$\int_0^t |H_s|\,|\,\mathrm{d}[M,N]_s| \leq \left(\int_0^t H_s^2 \,\mathrm{d}[M]_s\right)^{\frac{1}{2}} \left(\int_0^t \mathrm{d}[N]_s\right)^{\frac{1}{2}} = \left(\int_0^t H_s^2 \,\mathrm{d}[M]_s\right)^{\frac{1}{2}} [N]_t^{\frac{1}{2}}$$

almost surely for all $t \geq 0$. As the latter is finite almost surely, we find that H is almost surely integrable with respect to $[M,N]$. Next, we consider uniqueness of L. Assume that we have two processes $L, L' \in \mathbf{c}\mathcal{M}_\ell$ such that for all $N \in \mathbf{c}\mathcal{M}_\ell$, it holds that $[L,N] = H \cdot [M,N]$ and $[L',N] = H \cdot [M,N]$. Then $[L - L', N] = 0$ up to indistinguishability for all $N \in \mathbf{c}\mathcal{M}_\ell$. As $L - L' \in \mathbf{c}\mathcal{M}_\ell$, we in particular obtain that $[L - L']$ is evanescent, so by Lemma 1.4.9, $L - L'$ is evanescent and thus L and L' are indistinguishable. This proves uniqueness.

It remains to prove that there exists $L \in \mathbf{c}\mathcal{M}_\ell$ such that $[L,N] = H \cdot [M,N]$ for all $N \in \mathbf{c}\mathcal{M}_\ell$. Assume first that $E(H^2 \cdot [M])_\infty$ is finite. In this case, the Kunita-Watanabe inequality of

Theorem 1.4.12 and the Cauchy-Schwartz inequality yields that for all $N \in \mathbf{c}\mathcal{M}^2$,

$$E\int_0^\infty |H_s||\,\mathrm{d}[M,N]_s| \le E\left(\int_0^\infty H_s^2\,\mathrm{d}[M]_s\right)^{\frac{1}{2}}([N]_\infty)^{\frac{1}{2}} \le \left(E\int_0^\infty H_s^2\,\mathrm{d}[M]_s\right)^{\frac{1}{2}}(E[N]_\infty)^{\frac{1}{2}}$$

which is finite. Therefore, we can define $\varphi : \mathbf{c}\mathcal{M}^2 \to \mathbb{R}$ by putting $\varphi(N) = E\int_0^\infty H_s\,\mathrm{d}[M,N]_s$. Then φ is linear, and by Lemma A.1.7, it holds that $|\varphi(N)| \le (E\int_0^\infty H_s^2\,\mathrm{d}[M]_s)^{1/2}\|N\|_2$, where the seminorm denotes the previously defined seminorm on $\mathbf{c}\mathcal{M}^2$. Thus, φ is Lipschitz, and therefore continuous. Theorem 1.3.5 then yields the existence of a process $L \in \mathbf{c}\mathcal{M}^2$, unique up to indistinguishability, such that $\varphi(N) = EL_\infty N_\infty$ for all $N \in \mathbf{c}\mathcal{M}^2$. By Lemma 1.4.10, it holds for $N \in \mathbf{c}\mathcal{M}^2$ that $[L,N] \in \mathbf{c}\mathcal{V}^i$ and $LN - [L,N] \in \mathbf{c}\mathcal{M}^u$, so we obtain $E[L,N]_\infty = EL_\infty N_\infty = E\int_0^\infty H_s\,\mathrm{d}[M,N]_s$. Fixing $N \in \mathbf{c}\mathcal{M}^2$ and letting T be some stopping time, we have $N^T \in \mathbf{c}\mathcal{M}^2$ as well, and so

$$E[L,N]_T = E[L,N^T]_\infty = E\int_0^\infty H_s\,\mathrm{d}[M,N^T]_s = E\int_0^\infty H_s\,\mathrm{d}[M,N]_s^T = E\int_0^T H_s\,\mathrm{d}[M,N]_s,$$

so $E([L,N]_T - \int_0^T H_s\,\mathrm{d}[M,N]_s) = 0$ for any stopping time T, therefore Lemma 1.2.9 shows that $[L,N] - H\cdot[M,N]$ is in $\mathbf{c}\mathcal{M}^u$. As $[L,N] - H\cdot[M,N]$ has paths of finite variation, Lemma 1.4.5 yields $[L,N] = H\cdot[M,N]$ up to indistinguishability. We have now shown that there is $L \in \mathbf{c}\mathcal{M}^2$ such that $[L,N] = H\cdot[M,N]$ for any $N \in \mathbf{c}\mathcal{M}^2$. In order to extend this to $N \in \mathbf{c}\mathcal{M}_\ell$, let $N \in \mathbf{c}\mathcal{M}_\ell$. Using Lemma 1.4.4, let (T_n) be a localising sequence such that $N^{T_n} \in \mathbf{c}\mathcal{M}^2$. Then, $[L,N]^{T_n} = [L,N^{T_n}] = H\cdot[M,N^{T_n}] = H\cdot[M,N]^{T_n} = (H\cdot[M,N])^{T_n}$ up to indistinguishability by Lemma 1.4.9, and as a consequence, $[L,N] = H\cdot[M,N]$ up to indistinguishability, since T_n tends to infinity almost surely. We have now shown that when H is progressive such that $(H^2\cdot[M])_\infty$ is integrable, there exists $L \in \mathbf{c}\mathcal{M}_\ell$ such that $[L,N] = H\cdot[M,N]$ for any $N \in \mathbf{c}\mathcal{M}_\ell$, and so existence is proven in this case.

Next, consider the case where we merely assume that H^2 is almost surely integrable with respect to $[M]$. As $H^2\cdot[M]$ is in $\mathbf{c}\mathcal{A}$, there is then by Lemma 1.1.11 a localising sequence (T_n) such that $H^2\cdot[M^{T_n}] = (H^2\cdot[M])^{T_n} \in \mathbf{c}\mathcal{A}^i$. By our previous results, there is a unique element $L^n \in \mathbf{c}\mathcal{M}_\ell$ such that for all $N \in \mathbf{c}\mathcal{M}_\ell$, $[L^n,N] = H\cdot[M^{T_n},N] = H\cdot[M,N]^{T_n} = (H\cdot[M,N])^{T_n}$. Then $[(L^{n+1})^{T_n},N] = [L^{n+1},N]^{T_n} = ((H\cdot[M,N])^{T_n})^{T_{n+1}} = (H\cdot[M,N])^{T_n} = [L^n,N]$. By uniqueness, $(L^{n+1})^{T_n} = L^n$. Therefore, the processes (L^n) may be pasted together to a process L by defining $L_t = L_t^n$ whenever $t \le T_n$. In particular, $L^{T_n} = (L^n)^{T_n}$, so $L \in \mathbf{c}\mathcal{M}_\ell$ and we have the relation $[L,N]^{T_n} = [L^{T_n},N] = [L^n,N] = (H\cdot[M,N])^{T_n}$, showing that $[L,N] = H\cdot[M,N]$ up to indistinguishability, as desired. This proves the existence in the case where H^2 is almost surely integrable with respect to $[M]$.

We have now shown that when H^2 is almost surely integrable with respect to $[M]$, then H is almost surely integrable with respect to $[M,N]$ for all $N \in \mathbf{c}\mathcal{M}_\ell$, and there is $L \in \mathbf{c}\mathcal{M}_\ell$, unique up to evanescence, such that $[L,N] = H \cdot [M,N]$ for all $N \in \mathbf{c}\mathcal{M}_\ell$. We also need to show the converse. Therefore, assume that for all $N \in \mathbf{c}\mathcal{M}_\ell$, H is almost surely integrable with respect to $[M,N]$ and there is a process $L \in \mathbf{c}\mathcal{M}_\ell$ such that $[L,N] = H \cdot [M,N]$. In this case, we in particular obtain that $[M,L] = [L,M] = H \cdot [M,M] = H \cdot [M]$. Therefore, our assumptions show that $\int_0^t |H_s| \, \mathrm{d}[M]_s$ and $\int_0^t |H_s| \, \mathrm{d}(H \cdot [M])_s$ are finite. However, by the transformation theorem for integration with respect to measures with densities, we have $\int_0^t H_s \, \mathrm{d}(H \cdot [M])_s = \int_0^t H_s^2 \, \mathrm{d}[M]_s$, so H^2 is almost surely integrable with respect to $[M]$, as desired. □

Theorem 2.2.3 yields the existence of a process $H \cdot M$ for any $M \in \mathbf{c}\mathcal{M}_\ell$ and any progressive H such that H^2 is almost surely integrable with respect to $[M]$. We call this process $H \cdot M$ the stochastic integral with respect to M. Combined with Theorem 2.2.1, we have now proved the existence of integral processes with respect to both elements of $\mathbf{c}\mathcal{V}$ and $\mathbf{c}\mathcal{M}_\ell$. However, the processes which may be integrated depend on the integrator. We would like to identify a common set of integrands which may be integrated against any process in $\mathbf{c}\mathcal{V}$ or $\mathbf{c}\mathcal{M}_\ell$. This will yield a set of integrands which may be integrated against any element of $\mathbf{c}\mathcal{S}$. The following lemma shows how to obtain this.

Lemma 2.2.4. *Assume that H is progressive and that there is a localising sequence (T_n) such that $H^{T_n} 1_{(T_n > 0)}$ is bounded for all n. For any $A \in \mathbf{c}\mathcal{V}$, H is almost surely integrable with respect to A, and for any $M \in \mathbf{c}\mathcal{M}_\ell$, H^2 is almost surely integrable with respect to $[M]$.*

Proof. First consider some $A \in \mathbf{c}\mathcal{V}$, we need to show that H is almost surely integrable with respect to A. Fix $t \geq 0$, and let ω be such that $T_n(\omega) > t$ from a point onwards, this is almost surely the case as T_n almost surely tends to infinity. Fix n so large that $T_n(\omega) > t$. In particular, $T_n(\omega) > 0$, and we find that for $s \leq t$, $H_s(\omega) = (H^{T_n} 1_{(T_n > 0)})_s$, so $H_s(\omega)$ is bounded on $[0,t]$, therefore integrable over $[0,t]$ with respect to $A(\omega)$. Thus, H is almost surely integrable with respect to A, as desired. By the same reasoning, we find that H^2 is almost surely integrable with respect to $[M]$. □

We let $\mathfrak{I}$ denote the set of all H such that H is progressive and there is a localising sequence (T_n) with $H^{T_n} 1_{(T_n > 0)}$ bounded for all n. For $H \in \mathfrak{I}$, we know by Theorem 2.2.1 that for any $A \in \mathbf{c}\mathcal{V}$, there exists a process $H \cdot A \in \mathbf{c}\mathcal{V}$ which almost surely agrees with the pathwise Lebesgue integral of H with respect to A, and by Theorem 2.2.3, we know that for any $M \in \mathbf{c}\mathcal{M}_\ell$, there exists a process $L \in \mathbf{c}\mathcal{M}_\ell$ such that for any $N \in \mathbf{c}\mathcal{M}_\ell$, H is almost surely integrable with respect to $[M,N]$ and $[L,N] = H \cdot [M,N]$. These properties will allow us to define the stochastic integral of any process $H \in \mathfrak{I}$ with respect to any $X \in \mathbf{c}\mathcal{S}$. The following lemma shows that the set $\mathfrak{I}$ contains many useful processes.

Lemma 2.2.5. *Let H be adapted and continuous. Then $H \in \mathfrak{I}$.*

Proof. Put $T_n = \inf\{t \geq 0 \mid |H_t| > n\}$. As H is adapted and has continuous paths, (T_n) is a localising sequence. Pathwisely, if $T_n = 0$, $H^{T_n} 1_{(T_n > 0)}$ is zero, and if $T_n > 0$, H^{T_n} is bounded by n. Thus, $H^{T_n} 1_{(T_n > 0)}$ is bounded by n, we conclude $H \in \mathfrak{I}$. □

We are now ready to introduce the stochastic integral with respect to continuous semimartingales and prove its basic properties.

Theorem 2.2.6 (Existence of the stochastic integral, continuous semimartingales)**.** *Consider $X \in \mathbf{c}\mathcal{S}$ and let $H \in \mathfrak{I}$. It holds that for any decomposition $X = X_0 + M + A$ of X, H^2 is integrable with respect to $[M]$ and H is integrable with respect to A. Furthermore, there is a process $H \cdot X$ in $\mathbf{c}\mathcal{S}$, unique up to indistinguishability, such that for any decomposition $X = X_0 + M + A$, $H \cdot X$ is indistinguishable from the process $H \cdot M + H \cdot A$. We call $H \cdot X$ the stochastic integral of H with respect to X.*

Proof. Fix any decomposition $X = X_0 + M + A$. From Lemma 2.2.4, H is almost surely integrable with respect to A and H^2 is almost surely integrable with respect to $[M]$. Therefore, by Theorem 2.2.1, $H \cdot A$ is well-defined as a pathwise Lebesgue integral, and by Theorem 2.2.3, $H \cdot M$ is well-defined as well. Put $Y = H \cdot M + H \cdot A$, we claim that Y satisfies the criteria of the theorem. To see this, let $X = Z_0 + N + B$ be some other decomposition. We need to prove that Y is indistinguishable from $H \cdot N + H \cdot B$. From what we just proved, $H \cdot N$ and $H \cdot B$ are both well-defined. From Theorem 2.1.2, M and N are indistinguishable and A and B are indistinguishable. In particular, $[M, N']$ and $[N, N']$ are indistinguishable for all $N' \in \mathbf{c}\mathcal{M}_\ell$, leading to that $H \cdot M$ and $H \cdot N$ are indistinguishable. And as A and B are indistinguishable, $H \cdot A$ and $H \cdot B$ are indistinguishable. Therefore, Y is indistinguishable from $H \cdot N + H \cdot B$. This proves existence. Uniqueness follows immediately. □

Lemma 2.2.7. *Let $X, Y \in \mathbf{c}\mathcal{S}$, let $H, K \in \mathfrak{I}$ and let T be a stopping time. The following properties for the stochastic integral hold up to indistinguishability.*

1. *$\mathfrak{I}$ is a linear space, and $H \cdot X$ is a linear mapping in both H and X.*
2. *$H \cdot X$ is a continuous semimartingale with decomposition $H \cdot X = H \cdot M + H \cdot A$.*

3. $H1_{[\![0,T]\!]}$ *is in* $\mathfrak{I}$ *and* $(H \cdot X)^T = H1_{[\![0,T]\!]} \cdot X = H \cdot X^T$.

4. *It holds that* $HK \in \mathfrak{I}$ *and* $K \cdot (H \cdot X) = KH \cdot X$.

5. *We have* $[H \cdot X, Y] = H \cdot [X, Y]$ *and* $[H \cdot X] = H^2 \cdot [X]$.

6. *If* $F \in \mathcal{F}_0$*, it holds that* $1_F H \in \mathfrak{I}$*,* $1_F X \in \mathbf{c}\mathcal{S}$ *and* $(1_F H) \cdot X = 1_F(H \cdot X) = H \cdot (1_F X)$.

7. *If* $H^T = K^T$*, then* $(H \cdot X)^T = (K \cdot X)^T$.

Proof. **Proof of (1).** Let $\alpha, \beta \in \mathbb{R}$. As $H, K \in \mathfrak{I}$, there are localising sequences (T_n) and (S_n) such that $H^{T_n} 1_{(T_n>0)}$ and $K^{S_n} 1_{(S_n>0)}$ are bounded. Using Lemma 1.4.2, $(S_n \wedge T_n)$ is also a localising sequence, and we find that

$$
\begin{aligned}
(\alpha H + \beta K)^{S_n \wedge T_n} 1_{(S_n \wedge T_n > 0)} &= (\alpha H^{S_n \wedge T_n} + \beta K^{S_n \wedge T_n}) 1_{(S_n>0)} 1_{(T_n>0)} \\
&= \alpha (H^{T_n} 1_{(T_n>0)})^{S_n} 1_{(S_n>0)} + \beta (K^{S_n} 1_{(S_n>0)})^{T_n} 1_{(T_n>0)},
\end{aligned}
$$

and since $H^{T_n} 1_{(T_n>0)}$ and $K^{S_n} 1_{(S_n>0)}$ are bounded, this shows that $\alpha H + \beta K$ is in $\mathfrak{I}$. It remains to prove that $H \cdot X$ is linear in both the integrand H and the integrator X. We first fix X with decomposition $X = X_0 + M + A$ and consider the integral as a mapping in H. We commence by showing that $(\alpha H + \beta K) \cdot M = \alpha (H \cdot M) + \beta (K \cdot M)$. By the characterisation in Theorem 2.2.3, we need to show that $[\alpha(H \cdot M) + \beta(K \cdot M), N] = (\alpha H + \beta K) \cdot [M, N]$ for any $N \in \mathbf{c}\mathcal{M}_\ell$. Let $N \in \mathbf{c}\mathcal{M}_\ell$ be given, we then have, again using the characterisation in Theorem 2.2.3,

$$
\begin{aligned}
[\alpha(H \cdot M) + \beta(K \cdot M), N] &= \alpha [H \cdot M, N] + \beta [K \cdot M, N] \\
&= \alpha (H \cdot [M, N]) + \beta (K \cdot [M, N]) \\
&= (\alpha H + \beta K) \cdot [M, N],
\end{aligned}
$$

as desired. As we have $(\alpha H + \beta K) \cdot A = \alpha (H \cdot A) + \beta (H \cdot A)$ when $A \in \mathbf{c}\mathcal{V}$ by the ordinary properties of Lebesgue integrals, this proves that the stochastic integral is linear in the integrand. Next, we prove that it is linear in the integrator. Fix $H \in \mathfrak{I}$, we consider X and Y in $\mathbf{c}\mathcal{S}$ and wish to prove that $H \cdot (\alpha X + \beta Y) = \alpha (H \cdot X) + \beta (H \cdot Y)$. Assume that we have decompositions $X = X_0 + M + A$ and $Y = Y_0 + N + B$. We first prove that $H \cdot (\alpha M + \beta N) = \alpha (H \cdot M) + \beta (H \cdot N)$. Fixing any $N' \in \mathbf{c}\mathcal{M}_\ell$, we have

$$
\begin{aligned}
[\alpha(H \cdot M) + \beta(H \cdot N), N'] &= \alpha [H \cdot M, N'] + \beta [H \cdot N, N'] \\
&= \alpha (H \cdot [M, N']) + \beta (H \cdot [N, N']) \\
&= H \cdot [\alpha M + \beta N, N'],
\end{aligned}
$$

so that $\alpha(H\cdot M)+\beta(H\cdot N)$ is the stochastic integral of H with respect to $\alpha M+\beta N$. Therefore, as $\alpha X+\beta Y$ has continuous martingale part $\alpha M+\beta N$ and continuous finite variation part $\alpha A+\beta B$, we obtain using what was just proven as well as the linearity properties of ordinary Lebesgue integrals,

$$\begin{aligned} H\cdot(\alpha X+\beta Y) &= H\cdot(\alpha M+\beta N)+H\cdot(\alpha A+\beta B) \\ &= \alpha(H\cdot M)+\beta(H\cdot N)+\alpha(H\cdot A)+\beta(H\cdot B) \\ &= \alpha(H\cdot X)+\beta(H\cdot Y), \end{aligned}$$

as desired.

Proof of (2). This follows immediately from the construction of the integral.

Proof of (3). Assume that $H\in\mathfrak{I}$, we first show that $H1_{[\![0,T]\!]}$ is in $\mathfrak{I}$ as well. Note that as $(t\leq T)=(T<t)^c\in\mathcal{F}_t$, the process $t\mapsto 1_{[\![0,T]\!]}(t)$ is adapted and left-continuous, so by 1.1.4, this process is progressive, and therefore, $H1_{[\![0,T]\!]}$ is progressive. Let (T_n) be a localising sequence such that $H^{T_n}1_{(T_n>0)}$ is bounded. Then $(H1_{[\![0,T]\!]})^{T_n}1_{(T_n>0)}=H^{T_n}1_{(T_n>0)}1_{[\![0,T]\!]}$ is bounded as well. We conclude that $H1_{[\![0,T]\!]}\in\mathfrak{I}$, as desired. In order to prove the identities for the stochastic integral, let $X\in\mathbf{c}\mathcal{S}$ with decomposition $X=X_0+M+A$. We then have

$$(H\cdot A)^T_t=\int_0^{T\wedge t}H_s\,\mathrm{d}A_s=\int_0^t(H1_{[\![0,T]\!]})_s\,\mathrm{d}A_s=\int_0^t H_s\,\mathrm{d}A^T_s,$$

so that $(H\cdot A)^T=(H1_{[\![0,T]\!]})\cdot A=H\cdot A^T$. As regards the martingale part, let $N\in\mathbf{c}\mathcal{M}_\ell$, then $[(H\cdot M)^T,N]=[H\cdot M,N]^T=(H\cdot[M,N])^T$. Therefore, $[(H\cdot M)^T,N]=H1_{[\![0,T]\!]}\cdot[M,N]$, proving $(H\cdot M)^T=H1_{[\![0,T]\!]}\cdot M$, and $[(H\cdot M)^T,N]=H\cdot[M^T,N]$, which shows that $(H\cdot M)^T=H\cdot M^T$. Collecting our results for the continuous martingale and continuous finite variation parts, the result follows.

Proof of (4). As $H,K\in\mathfrak{I}$, we know that there exists a localising sequence (T_n) such that $H^{T_n}1_{(T_n>0)}$ and $K^{T_n}1_{(T_n>0)}$ are bounded. As $(HK)^{T_n}1_{(T_n>0)}=H^{T_n}1_{(T_n>0)}K^{T_n}1_{(T_n>0)}$, and HK is progressive, we conclude $HK\in\mathfrak{I}$. As regards the integral identity, assume that X has decomposition $X=X_0+M+A$. By the properties of ordinary Lebesgue integrals, $K\cdot(H\cdot A)=KH\cdot A$. As regards the martingale parts, let $N\in\mathbf{c}\mathcal{M}_\ell$, we then have

$$\begin{aligned} [K\cdot(H\cdot M),N] &= K\cdot[H\cdot M,N] \\ &= K\cdot(H\cdot[M,N]) \\ &= KH\cdot[M,N], \end{aligned}$$

which shows that $K\cdot(H\cdot M)$ satisfies the criterion for being the stochastic integral of KH with respect to M, so $K\cdot(H\cdot M)=KH\cdot M$. Collecting our results and using linearity of the integral in the integrator, we find

$$\begin{aligned} K\cdot(H\cdot X) &= K\cdot(H\cdot M+H\cdot A)\\ &= K\cdot(H\cdot M)+K\cdot(H\cdot A)\\ &= KH\cdot M+KH\cdot A\\ &= KH\cdot X, \end{aligned}$$

as desired.

Proof of (5). Let $X=X_0+M+A$ and $Y=Y_0+N+B$, as the martingale part of $H\cdot X$ is $H\cdot M$, we then find $[H\cdot X,Y]=[H\cdot M,N]=H\cdot[M,N]=H\cdot[X,Y]$. In particular, this yields $[H\cdot X]=[H\cdot X,H\cdot X]=H\cdot[X,H\cdot X]=H\cdot[H\cdot X,X]=H^2\cdot[X]$.

Proof of (6). First note that $1_FH\in\mathfrak{I}$ as 1_F is progressive, and by Lemma 2.1.5, $1_FX\in\mathbf{c}\mathcal{S}$. Let $X=X_0+M+A$. By the properties of ordinary Lebesgue integrals, we know that $(1_FH)\cdot A=1_F(H\cdot A)=H\cdot(1_FA)$ up to indistinguishability. Therefore, it suffices to prove $(1_FH)\cdot M=1_F(H\cdot M)=H\cdot(1_FM)$. By Lemma 1.4.3, all three processes are in $\mathbf{c}\mathcal{M}_\ell$. Therefore, it suffices to prove that their quadratic covariation with any $N\in\mathbf{c}\mathcal{M}_\ell$ are equal. Let $N\in\mathbf{c}\mathcal{M}_\ell$. By Theorem 2.2.3, $[(1_FH)\cdot M,N]=1_FH\cdot[M,N]=1_F(H\cdot[M,N])$, while Lemma 1.4.9 shows that we have $[1_F(H\cdot M),N]=1_F[H\cdot M,N]=1_F(H\cdot[M,N])$ and $[H\cdot 1_FM,N]=H\cdot[1_FM,N]=H\cdot 1_F[M,N]=1_F(H\cdot[M,N])$. Thus, the quadratic covariation with N is equal to $1_F(H\cdot[M,N])$ for all three processes, and so Lemma 1.4.9 shows that $(1_FH)\cdot M=1_F(H\cdot M)=H\cdot(1_FM)$, as desired.

Proof of (7). From what we already have shown, we find

$$\begin{aligned} (H\cdot X)^T &= H1_{[\![0,T]\!]}\cdot X=H^T1_{[\![0,T]\!]}\cdot X\\ &= K^T1_{[\![0,T]\!]}\cdot X=K1_{[\![0,T]\!]}\cdot X=(K\cdot X)^T, \end{aligned}$$

as desired. □

Lemma 2.2.8. *Let $X\in\mathbf{c}\mathcal{S}$, let $S\leq T$ be stopping times and let ξ be bounded and $\mathcal{F}_S$ measurable. If $H=\xi 1_{]\!]S,T]\!]}$, then $H\in\mathfrak{I}$ and $H\cdot X=\xi(X^T-X^S)$. If $H=\xi 1_{[\![S]\!]}$, then $H\in\mathfrak{I}$ and $H\cdot X$ is evanescent. In particular, the integrals of $\xi 1_{]\!]S,T]\!]}$, $\xi 1_{[\![S,T]\!]}$, $\xi 1_{]\!]S,T[\![}$ and $\xi 1_{[\![S,T[\![}$ are indistinguishable.*

Proof. With $H=\xi 1_{]\!]S,T]\!]}$, we find by Lemma 2.2.2 that H is progressive. As it is also bounded, we obtain $H\in\mathfrak{I}$, and Lemma 2.2.2 also yields $H\cdot M=\xi(M^T-M^S)$. Using the properties of ordinary Lebesgue integrals, we furthermore have $H \cdot A = \xi(A^T - A^S)$, and therefore $H \cdot X = \xi(X^T - X^S)$. This proves the first claim.

Next, consider $H = \xi 1_{[\![S]\!]}$. As $H = \xi 1_{[\![S,\infty[\![} - \xi 1_{]\!]S,\infty[\![}$, and both of these processes are adapted and either left-continuous or right-continuous, Lemma 1.1.4 shows that H is progressive. As H is also bounded, $H \in \mathfrak{I}$. For any $N \in \mathbf{c}\mathcal{M}_\ell$, $H \cdot [M,N]$ is evanescent as $[M,N]$ is continuous and therefore accords zero measure to one-point sets. Therefore, $H \cdot M$ is evanescent. As A is continuous, the measures induced by the paths of A also accord zero measure to one-point sets, and so $H \cdot A$ is evanescent, leading us to conclude that $H \cdot X$ is evanescent. By the linearity properties of the integral shown in Lemma 2.2.7, the final claims follow. □

This concludes the construction of the stochatic integral and the proofs of its basic properties. In the following section, we will consider some more advanced properties.

2.3 Itô's formula

In this section, we prove some results for stochastic integrals which are of fundamental importance: the dominated convergence theorem for stochastic integrals, the characterisation of stochastic integrals and the quadratic covariation as particular limits, and Itô's formula, which is the stochastic version of the fundamental theorem of analysis.

Theorem 2.3.1 (Dominated convergence theorem). *Let X be a continuous semimartingale and let $t \geq 0$ be some constant. Assume that (H^n) is a sequence of progressive processes and that H is another progressive process. If it almost surely holds that H^n converges pointwise to H on $[\![0,t]\!]$ and $|H^n|$ and $|H|$ are bounded by $K \in \mathfrak{I}$ on $[\![0,t]\!]$, then H^n and H are in $\mathfrak{I}$ as well, and*

$$\sup_{s \leq t} |(H^n \cdot X)_s - (H \cdot X)_s| \xrightarrow{P} 0.$$

Proof. First note that H^n and H are in $\mathfrak{I}$ as well, as the same localising sequence of stopping times (T_n) which ensures that $K^{T_n} 1_{(T_n > 0)}$ is bounded also works for H^n and H.

As regards the convergence result, we first prove the result in the case where $K \leq c$ for some constant $c > 0$, in which case H^n and H are bounded by c as well. Let $X = X_0 + M + A$ be the decomposition of X. Using Lemma A.1.7, we then have

$$\begin{aligned}
\sup_{s\leq t}|(H^n\cdot X)_s-(H\cdot X)_s| &\leq \sup_{s\leq t}|(H^n\cdot M)_s-(H\cdot M)_s|+\sup_{s\leq t}|(H^n\cdot A)_s-(H\cdot A)_s|\\
&\leq ((H^n-H)\cdot M)_t^*+\sup_{s\leq t}\int_0^t|H_s^n-H_s|\,|\,\mathrm{d}A_s|\\
&= ((H^n-H)\cdot M)_t^*+((H^n-H)\cdot V_A)_t,
\end{aligned}$$

and so it will suffice to show that each of the two latter terms converge in probability to zero. Considering the last of the two terms, we note that as it almost surely holds that H_s^n tends pointwise to H_s for $s\leq t$ and is bounded by a constant, the dominated convergence theorem applied pathwise yields that $((H^n-H)\cdot V_A)_t$ tends almost surely to zero, in particular the convergence holds in probability. As for the martingale part, another application of the dominated convergence theorem yields $\lim_n[(H^n-H)\cdot M]_t=\lim_n\int_0^t(H_s^n-H_s)^2\,\mathrm{d}[M]_s=0$ almost surely, so that in particular, we have convergence in probability. Lemma 1.4.11 then allows us to conclude that $((H^n-H)\cdot M)_t^*\xrightarrow{P}0$ as well, finally allowing us to obtain $\sup_{s\leq t}|(H^n\cdot X)_s-(H\cdot X)_s|\xrightarrow{P}0$, as desired.

Now consider the general case. Let (T_n) be a sequence such that $K^{T_n}1_{(T_n>0)}$ is bounded. We may then use Lemma 2.2.7 to obtain

$$\begin{aligned}
&1_{(T_k>t)}\sup_{s\leq t}|(H^n\cdot X)_s-(H\cdot X)_s|\\
=\ &\sup_{s\leq t}|(H^n\cdot X)_s-(H\cdot X)_s|1_{(T_k>t)}\leq\sup_{s\leq t}|(H^n\cdot X)_s^{T_k}-(H\cdot X)_s^{T_k}|1_{(T_k>0)}\\
=\ &\sup_{s\leq t}|(H^n1_{(T_k>0)}1_{[\![0,T_k]\!]}\cdot X)_s-(H1_{(T_k>0)}1_{[\![0,T_k]\!]}\cdot X)_s|.
\end{aligned}$$

As $H^n1_{(T_k>0)}1_{[\![0,T_k]\!]}$ and $H1_{(T_k>0)}1_{[\![0,T_k]\!]}$ are in $\mathfrak{I}$ and are bounded by the same constant as $K^{T_k}1_{(T_k>0)}$, we find from our previous results that $1_{(T_k>t)}\sup_{s\leq t}|(H^n\cdot X)_s-(H\cdot X)_s|$ converges in probability to zero. As (T_k) tends almost surely to infinity, it holds that $\lim_k P(T_k\leq t)=0$ and so Lemma A.2.1 shows that $\sup_{s\leq t}|(H^n\cdot X)_s-(H\cdot X)_s|$ tends to zero in probability, as was to be proven. □

Next, we prove limit characterisations of the stochastic integral and the quadratic covariation which provide the integral interpretation of $H\cdot X$ and the quadratic covariation interpretation of $[X,Y]$. We first introduce some notation. Fix $t\geq 0$. We say that a finite increasing sequence $(t_0,\ldots,t_K)$ with $0=t_0\leq\cdots\leq t_K=t$ is a partition of $[0,t]$. We refer to $\max_{k\leq K}|t_k-t_{k-1}|$ as the mesh of the partition.

Theorem 2.3.2. *Let $X\in\mathbf{c}\mathcal{S}$ and let H be adapted and continuous. Let $t\geq 0$ and assume* *that $(t^n_k)_{k \le K_n}$, $n \ge 1$, is a sequence of partitions of $[0,t]$ with mesh tending to zero. Then*

$$\sum_{k=1}^{K_n} H_{t^n_{k-1}}(X_{t^n_k} - X_{t^n_{k-1}}) \xrightarrow{P} (H \cdot X)_t.$$

Proof. First note that as H is adapted and continuous, $H \in \mathfrak{I}$ by Lemma 2.2.5, and so the stochastic integral is well-defined.

We define the sequence of processes $H^n = H_t 1_{[\![t]\!]} + \sum_{k=1}^{K_n} H_{t^n_{k-1}} 1_{[\![t^n_{k-1}, t^n_k[\![}$. Since H is continuous and the mesh of the partitions converges to zero, we find that H^n converges pointwise to $H1_{[\![0,t]\!]}$. Also note that as H is continuous and adapted, H^* is also continuous and adapted and so in $\mathfrak{I}$, and both H^n and H are bounded by H^*. By Lemma 2.2.8, we obtain $\sum_{k=1}^{K_n} H_{t^n_{k-1}}(X_{t^n_k} - X_{t^n_{k-1}}) = (H^n \cdot X)_t$, and by the dominated convergence theorem 2.3.1, this converges to $(H1_{[\![0,t]\!]} \cdot X)_t = (H \cdot X)_t$ in probability, as was to be proven. □

Theorem 2.3.3 (Integration-by-parts formula)**.** *Let X and Y be continuous semimartingales. Let $t \ge 0$ and assume that $(t^n_k)_{k \le K_n}$, $n \ge 1$, is a sequence of partitions of $[0,t]$ with mesh tending to zero. Then*

$$\sum_{k=1}^{K_n}(X_{t^n_k} - X_{t^n_{k-1}})(Y_{t^n_k} - Y_{t^n_{k-1}}) \xrightarrow{P} [X,Y]_t,$$

and the identity $X_tY_t = X_0Y_0 + (Y \cdot X)_t + (X \cdot Y)_t + [X,Y]_t$ holds.

Proof. Our first step is to prove the two relations

$$\sum_{k=1}^{K_n}(X_{t^n_k} - X_{t^n_{k-1}})^2 \xrightarrow{P} [X,X]_t$$
$$X_t^2 = X_0^2 + 2(X \cdot X)_t + [X]_t,$$

afterwards we will extend the result to the general case by a polarization argument. We first consider the case of $M \in \mathbf{c}\mathcal{M}_\ell$. Note that

$$M_t^2 = \sum_{k=1}^{K_n}(M^2_{t^n_k} - M^2_{t^n_{k-1}}) = 2\sum_{k=1}^{K_n} M_{t^n_{k-1}}(M_{t^n_k} - M_{t^n_{k-1}}) + \sum_{k=1}^{K_n}(M_{t^n_k} - M_{t^n_{k-1}})^2,$$

and since M is continuous and adapted, $\sum_{k=1}^{K_n} M_{t^n_{k-1}}(M_{t^n_k} - M_{t^n_{k-1}}) \xrightarrow{P} (M \cdot M)_t$ by Theorem 2.3.2, and therefore $\sum_{k=1}^{K_n}(M_{t^n_k} - M_{t^n_{k-1}})^2 \xrightarrow{P} M_t^2 - 2(M \cdot M)_t$. We wish to argue that the process $M^2 - 2(M \cdot M)$ is almost surely increasing. To this end, let $0 \le p \le q$ be two dyadic rationals. There exists $j \geq 1$ and naturals $n_p \leq n_q$ such that $p = n_p 2^{-j}$ and $q = n_q 2^{-j}$. Consider the particular partitions of $[0,p]$ and $[0,q]$ given by putting $p_k^n = k2^{-(n+j)}$ for $k \leq n_p 2^n$ and $q_k^n = k2^{-(n+j)}$ for $k \leq n_q 2^n$, respectively. Using Lemma A.2.2 and the convergence result just proven, we then obtain

$$M_p^2 - 2(M \cdot M)_p = \lim_n \sum_{k=1}^{n_p 2^n} (M_{p_k^n} - M_{p_{k-1}^n})^2 \leq \lim_n \sum_{k=1}^{n_q 2^n} (M_{q_k^n} - M_{q_{k-1}^n})^2 = M_q^2 - 2(M \cdot M)_q,$$

almost surely, where the limits are in probability. As $\mathbb{D}_+$ is countable and dense in $\mathbb{R}_+$, we conclude that $M^2 - 2(M \cdot M)$ is almost surely increasing. By picking a particular modification of $M \cdot M$, we may assume that $M^2 - 2(M \cdot M)$ has sample paths which are all increasing. By definition, $M^2 - [M]$ is in $\mathbf{c}\mathcal{M}_\ell$. As $M \cdot M$ is also in $\mathbf{c}\mathcal{M}_\ell$, we find that $M^2 - 2(M \cdot M) - [M]$ is a process in $\mathbf{c}\mathcal{M}_\ell$ which has paths of finite variation. By Lemma 1.2.12, the process is then evanescent, and we conclude that $M^2 = 2(M \cdot M) + [M]$ up to evanescence. As a consequence, $\sum_{k=1}^{K_n} (M_{t_k^n} - M_{t_{k-1}^n})^2 \xrightarrow{P} [M]_t$. This shows the results in the case of a continuous local martingale. Next, we consider the case $X = X_0 + M + A$, where $M \in \mathbf{c}\mathcal{M}_\ell$ and $A \in \mathbf{c}\mathcal{V}$. We first note that

$$\begin{aligned} \left| \sum_{k=1}^{K_n} (M_{t_k^n} - M_{t_{k-1}^n})(A_{t_k^n} - A_{t_{k-1}^n}) \right| &\leq \max_{1 \leq k \leq K_n} |M_{t_k^n} - M_{t_{k-1}^n}| \sum_{k=1}^{K_n} |A_{t_k^n} - A_{t_{k-1}^n}| \\ &\leq (V_A)_t \max_{1 \leq k \leq K_n} |M_{t_k^n} - M_{t_{k-1}^n}|, \end{aligned}$$

and as M has sample paths which are uniformly continuous on $[0,t]$, the latter tends almost surely to zero, allowing us to conclude $\sum_{k=1}^{K_n} (M_{t_k^n} - M_{t_{k-1}^n})(A_{t_k^n} - A_{t_{k-1}^n}) \xrightarrow{P} 0$. Analogously, we may argue that $\sum_{k=1}^{K_n} (A_{t_k^n} - A_{t_{k-1}^n})^2 \xrightarrow{P} 0$. Combining our results and recalling $[X] = [M]$, we obtain $\sum_{k=1}^{K_n} (X_{t_k^n} - X_{t_{k-1}^n})^2 = \sum_{k=1}^{K_n} (M_{t_k^n} - M_{t_{k-1}^n} + A_{t_k^n} - A_{t_{k-1}^n})^2 \xrightarrow{P} [X]_t$. In order to obtain the integration-by-parts formula from this, we note that

$$X_t^2 = X_0^2 + \sum_{k=1}^{K_n} (X_{t_k^n}^2 - X_{t_{k-1}^n}^2) = X_0^2 + 2 \sum_{k=1}^{K_n} X_{t_{k-1}^n} (X_{t_k^n} - X_{t_{k-1}^n}) + \sum_{k=1}^{K_n} (X_{t_k^n} - X_{t_{k-1}^n})^2,$$

and we now know that the former term converges in probability to $2(X \cdot X)_t$ and the latter term converges in probability to $[X]_t$, so we conclude $X_t^2 = X_0^2 + 2(X \cdot X)_t + [X]_t$. We have now proven both the convergence result for the quadratic variation as well as the integration-by-parts formula. It remains to prove the same results for a pair of continuous semimartingales X and Y. Consider first the convergence to the quadratic covariation. Define two processes $Z = X + Y$ and $W = X - Y$. We then have $Z + W = 2X$ and $Z - W = 2Y$, yielding

$$\begin{aligned} (Z_{t_k^n} - Z_{t_{k-1}^n})^2 - (W_{t_k^n} - W_{t_{k-1}^n})^2 &= (2X_{t_k^n} - 2X_{t_{k-1}^n})(2Y_{t_k^n} - 2Y_{t_{k-1}^n}) \\ &= 4(X_{t_k^n} - X_{t_{k-1}^n})(Y_{t_k^n} - Y_{t_{k-1}^n}), \end{aligned}$$

and we know from our previous results that $\sum_{k=1}^{K_n}(Z_{t_k^n}-Z_{t_{k-1}^n})^2$ converges in probability to $[Z]_t$ and that $\sum_{k=1}^{K_n}(W_{t_k^n}-W_{t_{k-1}^n})^2$ converges in probability to $[W]_t$. By Lemma 1.4.9, we have $[Z]_t-[W]_t=[X+Y]_t-[X-Y]_t=4[X,Y]_t$ almost surely, so collecting our results, we finally conclude $\sum_{k=1}^{K_n}(X_{t_k^n}-X_{t_{k-1}^n})(Y_{t_k^n}-Y_{t_{k-1}^n})\xrightarrow{P}[X,Y]_t$, as desired. Analogously, we find

$$\begin{aligned}
4X_tY_t &= Z_t^2-W_t^2\\
&= Z_0^2-W_0^2+2(Z\cdot Z)_t-2(W\cdot W)+[Z]_t-[W]_t\\
&= 4X_0Y_0+2((X+Y)\cdot(X+Y))_t-2((X-Y)\cdot(X-Y))_t+4[X,Y]_t\\
&= 4X_0Y_0+4(X\cdot Y)_t+4(Y\cdot X)_t+4[X,Y]_t,
\end{aligned}$$

yielding the integration-by-parts formula in the general case. This concludes the proof. □

Lemma 2.3.4. *Let X and Y be continuous semimartingales and let H be adapted and continuous. Let $t\geq 0$ and assume that $(t_k^n)_{k\leq K_n}$, $n\geq 1$, is a sequence of partitions of $[0,t]$ with mesh tending to zero. Then*

$$\sum_{k=1}^{K_n} H_{t_{k-1}^n}(X_{t_k^n}-X_{t_{k-1}^n})(Y_{t_k^n}-Y_{t_{k-1}^n})\xrightarrow{P}(H\cdot[X,Y])_t.$$

Proof. As in the proof of Theorem 2.3.3, by polarization, it will suffice to consider a single semimartingale X and prove $\sum_{k=1}^{K_n} H_{t_{k-1}^n}(X_{t_k^n}-X_{t_{k-1}^n})^2\xrightarrow{P}(H\cdot[X])_t$. Also note that we may assume without loss of generality that X has initial value zero. To prove the result in this case, note that from Theorem 2.3.3, $[X]_t=X_t^2-2(X\cdot X)_t$, so that using Lemma 2.2.7, we find $H\cdot[X]=H\cdot X^2-2H\cdot(X\cdot X)=H\cdot X^2-2HX\cdot X$, where $X^2\in\mathbf{c}\mathcal{S}$ since $X^2=2(X\cdot X)+[X]$. On the other hand,

$$\sum_{k=1}^{K_n} H_{t_{k-1}^n}(X_{t_k^n}-X_{t_{k-1}^n})^2 = \sum_{k=1}^{K_n} H_{t_{k-1}^n}(X_{t_k^n}^2-X_{t_{k-1}^n}^2)-2\sum_{k=1}^{K_n} H_{t_{k-1}^n}X_{t_{k-1}^n}(X_{t_k^n}-X_{t_{k-1}^n}),$$

so that two applications of Theorem 2.3.2 immediately yield the result. □

We are now ready to prove Itô's formula. We denote by $C^2(\mathbb{R}^p)$ the set of mappings $f:\mathbb{R}^p\to\mathbb{R}$ such that all second-order partial derivatives of f exist and are continuous. Furthermore, for any open set U in $\mathbb{R}^p$, we denote by $C^2(U)$ the set of mappings $f:U\to\mathbb{R}$ with the same property. We say that a process X with values in $\mathbb{R}^p$ is a p-dimensional continuous semimartingale if each of its coordinate processes X^i, where $X_t=(X_t^1,\dots,X_t^p)$, is a continuous semimartingale.

Theorem 2.3.5 (Itô's formula)**.** *Let X be a p-dimensional continuous semimartingale and let $f : \mathbb{R}^p \to \mathbb{R}$ be C^2. Then*

$$f(X_t) = f(X_0) + \sum_{i=1}^p \int_0^t \frac{\partial f}{\partial x_i}(X_s)\,\mathrm{d}X_s^i + \frac{1}{2}\sum_{i=1}^p\sum_{j=1}^p \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s)\,\mathrm{d}[X^i, X^j]_s,$$

up to indistinguishability.

Proof. Let $t_k^n = kt2^{-n}$. We may use the relation $f(X_t) - f(X_0) = \sum_{k=1}^{2^n} f(X_{t_k^n}) - f(X_{t_{k-1}^n})$ and Theorem A.1.17 to obtain $f(X_t) = f(X_0) + S_t^n + T_t^n + R_t^n$ where

$$\begin{aligned}
S_t^n &= \sum_{i=1}^p \sum_{k=1}^{2^n} \frac{\partial f}{\partial x_i}(X_{t_{k-1}^n})(X_{t_k^n}^i - X_{t_{k-1}^n}^i) \\
T_t^n &= \frac{1}{2}\sum_{i=1}^p\sum_{j=1}^p\sum_{k=1}^{2^n} \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{t_{k-1}^n})(X_{t_k^n}^i - X_{t_{k-1}^n}^i)(X_{t_k^n}^j - X_{t_{k-1}^n}^j) \\
R_t^n &= \sum_{i=1}^p\sum_{j=1}^p\sum_{k=1}^{2^n} r_2^{ij}(X_{t_{k-1}^n}, X_{t_k^n})(X_{t_k^n}^i - X_{t_{k-1}^n}^i)(X_{t_k^n}^j - X_{t_{k-1}^n}^j),
\end{aligned}$$

and $r_2^{ij}(x,y)$ is the remainder from Theorem A.1.17. By Theorem 2.3.2, S_t^n converges in probability to $\sum_{i=1}^p \int_0^t \frac{\partial f}{\partial x_i}(X_s)\,\mathrm{d}X_s^i$, and by Lemma 2.3.4, T_t^n converges in probability to $\frac{1}{2}\sum_{i=1}^p\sum_{j=1}^p \int_0^t \frac{\partial f}{\partial x_i \partial x_j}(X_s)\,\mathrm{d}[X^i, X^j]_s$. Therefore, it will suffice to show that the remainder term R_t^n converges in probability to zero. Note that while we have no guarantee that r_2^{ij} is measurable, we know that R_t^n is always measurable, since $R_t^n = f(X_t) - f(X_0) - S_t^n - T_t^n$, and so the proposition that R_t^n converges in probability to zero is well-defined. To prove this proposition, first fix $i, j \leq p$. Recalling $|2xy| \leq x^2 + y^2$, we find

$$\begin{aligned}
&\left|\sum_{k=1}^{2^n} r_2^{ij}(X_{t_{k-1}^n}, X_{t_k^n})(X_{t_k^n}^i - X_{t_{k-1}^n}^i)(X_{t_k^n}^j - X_{t_{k-1}^n}^j)\right| \\
\leq\ & \frac{1}{2}\left(\max_{k \leq 2^n} |r_2^{ij}(X_{t_{k-1}^n}, X_{t_k^n})|\right)\left(\sum_{k=1}^{2^n}(X_{t_k^n}^i - X_{t_{k-1}^n}^i)^2 + \sum_{k=1}^{2^n}(X_{t_k^n}^j - X_{t_{k-1}^n}^j)^2\right),
\end{aligned}$$

where the latter factor converges to $[X^i]_t + [X^j]_t$ by Theorem 2.3.3. Now note that by Theorem A.1.17, there is a mapping $\xi : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$ such that $\xi(x,y)$ is always on the line segment between x and y and $r_2^{ij}(X_{t_{k-1}^n}, X_{t_k^n}) = \frac{\partial^2 f}{\partial x_i \partial x_j}(\xi(X_{t_{k-1}^n}, X_{t_k^n})) - \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{t_{k-1}^n})$. In particular, we have

$$\max_{k \leq 2^n} |r_2^{ij}(X_{t_{k-1}^n}, X_{t_k^n})| \leq \max_{k \leq 2^n} \sup_{t \in [0,1]} \left|\frac{\partial^2 f}{\partial x_i \partial x_j}(X_{t_{k-1}^n} + t(X_{t_k^n} - X_{t_{k-1}^n})) - \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{t_{k-1}^n})\right|,$$

and the latter is measurable, since by continuity, the supremum may be reduced to a countable one. Combining our conclusions, we see that in order to show that $R^n_t \xrightarrow{P} 0$, it suffices to show that for any $i, j \leq p$, it holds that

$$\max_{k \leq 2^n} \sup_{t \in [0,1]} \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{t^n_{k-1}} + t(X_{t^n_k} - X_{t^n_{k-1}})) - \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{t^n_{k-1}}) \right| \xrightarrow{P} 0.$$

To do so, fix $i, j \leq p$. Define $T^i_m = \inf\{t \geq 0 | |X^i_t| > m\}$ and $T_m = \min\{T^1_m, \ldots, T^p_m\}$. As (T^i_m) tends to infinity as m tends to infinity, we conclude that (T_m) tends to infinity as well. By Lemma A.2.1, since $\lim_m P(T_m > t) = 1$, it suffices to prove that

$$1_{(T_m > t)} \max_{k \leq 2^n} \sup_{t \in [0,1]} \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{t^n_{k-1}} + t(X_{t^n_k} - X_{t^n_{k-1}})) - \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{t^n_{k-1}}) \right| \xrightarrow{P} 0.$$

Now, on $(T_m > 0)$, each coordinate of the process X^{T_m} is bounded by m. Therefore, defining $Y^m_t = X^{T_m}_t 1_{(T_m > 0)}$, Y^m is bounded by m, and the above is equal to

$$1_{(T_m > t)} \max_{k \leq 2^n} \sup_{t \in [0,1]} \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(Y^m_{t^n_{k-1}} + t(Y^m_{t^n_k} - Y^m_{t^n_{k-1}})) - \frac{\partial^2 f}{\partial x_i \partial x_j}(Y^m_{t^n_{k-1}}) \right|,$$

so we need to show that this converges in probability to zero. Fix $\varepsilon > 0$. As the mapping $\frac{\partial^2 f}{\partial x_i \partial x_j}$ is continuous by our assumptions, it is uniformly continous on the compact set $[-m, m]^p$. Pick $\delta > 0$ parrying ε for this uniform continuity. For each ω, $Y^m(\omega)$ is continuous, and therefore uniformly continous on $[0, t]$. Pick η parrying δ for this uniform continuity. Now pick n so large that $t2^{-n} \leq \eta$. Note that n depends on ω. We then find $1_{(T_m > t)} \max_{k \leq 2^n} \sup_{t \in [0,1]} |\frac{\partial^2 f}{\partial x_i \partial x_j}(Y^m_{t^n_{k-1}} + t(Y^m_{t^n_k} - Y^m_{t^n_{k-1}})) - \frac{\partial^2 f}{\partial x_i \partial x_j}(Y^m_{t^n_{k-1}})| \leq \varepsilon$, proving convergence almost surely, in particular convergence in probability. Combining our conclusions, we obtain $R^n_t \xrightarrow{P} 0$. As limits in probability are almost surely determined, we conclude $f(X_t) = f(X_0) + \sum_{i=1}^p \int_0^t \frac{\partial f}{\partial x_i}(X_s) \, \mathrm{d}X^i_s + \frac{1}{2} \sum_{i=1}^p \sum_{j=1}^p \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s) \, \mathrm{d}[X^i, X^j]_s$ almost surely. As the processes on both sides of this equality are continuous and $t \geq 0$ was arbitrary, Lemma 1.1.5 shows that we have equality up to indistinguishability. This proves the theorem. $\square$

Note that in the case where X has paths of finite variation, it holds that the martingale part of X has paths of finite variation as well, so $[X]$ is evanescent and Theorem 2.3.5 reduces to $f(X_t) = f(X_0) + \int_0^t f'(X_s) \, \mathrm{d}X_s$, which is the classical version of the fundamental theorem of analysis. This highlights the manner in which Itô's formula is an extension of the classical theorem to the case where the integrands do not have paths of finite variation. Another interpretation of Itô's formula is that the space of p-dimensional continuous semimartingales is stable under C^2 transformations, and given a p-dimensional continuous semimartingale X

and a mapping $f \in C^2(\mathbb{R})$, Itô's formula shows how to obtain the decomposition of $f(X_t)$ into its continuous local martingale and continuous finite variation parts.

In practical applications, we will occasionally be considering cases where we wish to apply Itô's formula, but the function f is only defined on some open set. The following corollary shows that Itô's formula also holds in this case.

Corollary 2.3.6. *Let U be an open set in $\mathbb{R}^p$, let X be a p-dimensional continuous semimartingale taking its values in U and let $f : U \to \mathbb{R}$ be C^2. Then*

$$f(X_t) = f(X_0) + \sum_{i=1}^p \int_0^t \frac{\partial f}{\partial x_i}(X_s)\,\mathrm{d}X_s^i + \frac{1}{2}\sum_{i=1}^p\sum_{j=1}^p \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s)\,\mathrm{d}[X^i, X^j]_s,$$

up to indistinguishability.

Proof. Let $\|\cdot\|$ be some norm on $\mathbb{R}^p$ and let $d(x,y) = \|x-y\|$. Define the set U_m by putting $U_m = \{x \in \mathbb{R}^p \mid d(x,U^c) < \frac{1}{m}\}$. Put $F_m = U_m^c$, then $F_m = \{x \in \mathbb{R}^p \mid d(x,U^c) \geq \frac{1}{m}\}$. Our plan is to in some sense localise to F_m and prove the result there using Lemma A.1.18 and Theorem 2.3.5. As $x \mapsto d(x,U^c)$ is continuous, U_m is open and F_m is closed. Define T_m by putting $T_m = \inf\{t \geq 0 \mid X_t \in U_m\}$. Note that $T_m = \inf\{t \geq 0 \mid d(X_t,U^c) < \frac{1}{m}\}$, so as the process $d(X_t,U^c)$ is continuous and adapted, Lemma 1.1.9 shows that T_m is a stopping time. As (U_m) is decreasing, (T_m) is increasing. We wish to argue that (T_m) tends to infinity almost surely and that on $(T_m > 0)$, X^{T_m} takes its values in F_m.

To prove that (T_m) tends to infinity almost surely, first note that by continuity, we always have $d(X_{T_m},U^c) \leq \frac{1}{m}$. Assume that there is ω such that $T_m(\omega)$ has a finite limit $T(\omega)$. We then obtain $(X_T(\omega),U^c) = \lim_m d(X_{T_m}(\omega),U^c) = 0$. As U^c is closed, this implies $X_T(\omega) \in U^c$, a contradiction. Thus, we conclude that (T_m) tends almost surely to infinity. To show that on $(T_m > 0)$, X^{T_m} takes its values in F_m, we merely note that on this set, $X_t \notin U_m$ for $t < T_m$, so $X_t \in F_m$ for $t < T_m$, and by continuity of X and closedness of F_m, $X_{T_m} \in F_m$ as well. Thus, X^{T_m} takes its values in F_m on $(T_m > 0)$.

Now let m be so large that F_m is nonempty, this is possible as $U = \cup_{n=1}^\infty F_m$ and U is nonempty because X takes its values in U. Let y_m be some point in F_m. Define the process Y^m by putting $(Y^m)_t^i = 1_{(T_m>0)}(X^i)_t^{T_m} + y_m^i 1_{(T_m=0)}$. Y^m is then a p-dimensional continuous semimartingale taking its values in F_m. Now, by Lemma A.1.18, there is a C^2 mapping $g_m : \mathbb{R}^p \to \mathbb{R}$ such that g_m and f agree on F_m. By Theorem 2.3.5, Itô's formula holds using

Y^m and g_m, and as g_m and f agree on F_m, we obtain

$$f(Y^m_t) = f(Y^m_0) + \sum_{i=1}^p \int_0^t \frac{\partial f}{\partial x_i}(Y^m_s)\,\mathrm{d}(Y^m)^i_s + \frac{1}{2}\sum_{i=1}^p\sum_{j=1}^p \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(Y^m_s)\,\mathrm{d}[(Y^m)^i,(Y^m)^j]_s,$$

up to indistinguishability. We wish to argue that as m tends to infinity, all terms in the above converge to the corresponding terms with Y^m exchanged by X. Consider the first-order terms. For any $i \le p$, we may use Lemma 2.2.7 to obtain

$$\begin{aligned} 1_{(T_m>t)} \int_0^t \frac{\partial f}{\partial x_i}(Y^m_s)\,\mathrm{d}(Y^m)^i_s &= 1_{(T_m>t)} \int_0^t \frac{\partial f}{\partial x_i}(Y^m_s)\,\mathrm{d}(X^i)^{T_m}_s \\ &= 1_{(T_m>t)} \int_0^t 1_{(T_m>0)}\frac{\partial f}{\partial x_i}(X^{T_m}_s)\,\mathrm{d}(X^i)^{T_m}_s \\ &= 1_{(T_m>t)} \int_0^t \frac{\partial f}{\partial x_i}(X^{T_m}_s)\,\mathrm{d}(X^i)^{T_m}_s \\ &= 1_{(T_m>t)} \int_0^t \frac{\partial f}{\partial x_i}(X^{T_m}_s)1_{[\![0,T_m]\!]}\,\mathrm{d}X^i_s \\ &= 1_{(T_m>t)} \int_0^t \frac{\partial f}{\partial x_i}(X_s)\,\mathrm{d}X^i_s, \end{aligned}$$

and with an application of Lemma 1.4.9, the analogous statement is obtained for the second-order terms. Also, $1_{(T_m>t)}f(Y^m_t) = 1_{(T_m>t)}f(X_t)$ and $1_{(T_m>t)}f(Y^m_0) = 1_{(T_m>t)}f(X_0)$. All in all, we conclude that Itô's formula holds almost surely at time $t \ge 0$ on $(T_m > t)$, and letting m tend to infinity, we obtain that the formula holds at any time $t \ge 0$. By Lemma 1.1.5, the result holds up to indistinguishability and the proof is concluded. □

2.4 Conclusion

We have now come to the end of our treatment of stochastic integration for continuous semimartingales. We end the chapter with some comments on the literature treating the subject as well as some remarks on extensions of the theory.

The presentation of the theory in this monograph is for the most part inspired by the books Karatzas & Shreve (1991), He et al. (1992), Rogers & Williams (2000a), Rogers & Williams (2000b), Kallenberg (2002) and Protter (2005). The proof of the existence of the quadratic variation process using Mazur's lemma is inspired by Beiglböck et al. (2010).

The results on stochastic calculus with respect to continuous semimartingales given here is a small part of the bulk. Immediate further results include local time processes, martingale inequalities such as the Burkholder-Davis-Gundy inequalities, Girsanov's change of measure results, et cetera. Such topics are covered for example in the books Karatzas & Shreve (1991) and Rogers & Williams (2000b). A major application of stochastic calculus is to stochastic differential equations, in particular analysis of the solutions of stochastic differential equations of the form

$$\mathrm{d}X_t = a(X_t)\,\mathrm{d}t + b(X_t)\,\mathrm{d}W_t,$$

where W is an $\mathcal{F}_t$ Brownian motion and a process $X \in \mathbf{c}\mathcal{S}$ is sought which solves the above equation. Important results of the theory consider criteria for existence and uniqueness and relations to martingale problems, as well as criteria for stationarity and ergodicity, and problems of estimation. Main parts of the theory are given in Rogers & Williams (2000b) and Karatzas & Shreve (1991).

Another extension of the theory is to integration with respect to semimartingales with jumps, defined as processes X with decompositions $X = X_0 + M + A$, where M is a right-continous local martingale with left limits, and A is a right-continuous process of finite variation with left limits. The theory of martingales with jumps is more convoluted than the theory of continuous martingales, and the extension of the main results of stochastic calculus from the continuous case to the discontinuous case often involves substantial difficulties. The general theory is treated in the books Dellacherie & Meyer (1978), He et al. (1992), Rogers & Williams (2000b) and Protter (2005).

2.5 Exercises

Exercise 2.1. Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion. Show that $W \in \mathbf{c}\mathcal{S}$.

Exercise 2.2. Let X be continuous, adapted process. Define $F_c(X)$ as the set of ω such that $X_t(\omega)$ is convergent as $t \to \infty$ to a finite limit. Show that $F_c(X) \in \mathcal{F}$.

Exercise 2.3. Let $M \in \mathbf{c}\mathcal{M}_\ell$. Show that $F_c(M) = F_c([M])$ almost surely in the sense that $P(F_c(M)\Delta F_c([M])) = 0$, where $F\Delta G = (F \setminus G) \cup (G \setminus F)$ for $F, G \in \mathcal{F}$.

Exercise 2.4. Let $X \in \mathbf{c}\mathcal{S}$ with $X = M + A$, where $M \in \mathbf{c}\mathcal{M}_\ell$ and $A \in \mathbf{c}\mathcal{A}$. Prove that $F_c(X) = (\sup_t X_t < \infty)$ almost surely in the sense that $P(F_c(X)\Delta(\sup_t X_t < \infty)) = 0$.

Exercise 2.5. Let $X \in \mathbf{c}\mathcal{S}$ with $X = M + A$, where $M \in \mathbf{c}\mathcal{M}_\ell$ and $A \in \mathbf{c}\mathcal{A}$. Show that $F_c(X) = F_c(M) \cap F_c(A)$ almost surely in the sense that $P(F_c(X)\Delta(F_c(M) \cap F_c(A))) = 0$.

Exercise 2.6. Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion and let $H \in \mathfrak{I}$. Show that $H \cdot W$ is in $\mathbf{c}\mathcal{M}^2$ if and only if $E\int_0^\infty H_s^2 \,\mathrm{d}s$ is finite. Show that if it holds that for any $t \geq 0$, $E\int_0^t H_s^2 \,\mathrm{d}s$ is finite, then $H \cdot W$ is in $\mathbf{c}\mathcal{M}$ and $E(H \cdot W)_t^2 = E\int_0^t H_s^2 \,\mathrm{d}s$.

Exercise 2.7. Let $M \in \mathbf{c}\mathcal{M}_\ell$. Show that the mapping μ_M from $\mathcal{B}_+ \otimes \mathcal{F}$ to $[0, \infty]$ defined by putting, for any $A \in \mathcal{B}_+ \otimes \mathcal{F}$, $\mu_M(A) = \int\int 1_A(t, \omega) \,\mathrm{d}[M](\omega)_t \,\mathrm{d}P(\omega)$, is a well-defined nonnegative measure on $\mathcal{B}_+ \otimes \mathcal{F}$. Show that μ_M is always σ-finite, and if $M \in \mathbf{c}\mathcal{M}^2$, μ_M is bounded. Denote by $\mathcal{L}^2(M)$ the $\mathcal{L}^2$ space of $(\mathbb{R}_+ \times \Omega, \Sigma^p, \mu_M)$. Show that for $H \in \mathfrak{I}$, $H \cdot M$ is in $\mathbf{c}\mathcal{M}^2$ if and only if $H \in \mathcal{L}^2(\mu_M)$, and in the affirmative, $\|H \cdot M\|_2 = \|H\|_M$, where $\|\cdot\|_M$ denotes the $\mathcal{L}^2$ norm of the space $\mathcal{L}^2(M)$.

Exercise 2.8. Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion and let H be bounded, adapted and continuous. Show that for any fixed $t \geq 0$, $(W_{t+h} - W_t)^{-1}\int_t^{t+h} H_s \,\mathrm{d}W_s$ converges in probability to H_t, where we define $\int_t^{t+h} H_s \,\mathrm{d}W_s = (H \cdot W)_{t+h} - (H \cdot W)_t$.

Exercise 2.9. Let X be a continuous process. Let $t \geq 0$ and let $t_k^n = kt2^{-n}$. Let $p > 0$ and assume that $\sum_{k=1}^{2^n} |X_{t_k^n} - X_{t_{k-1}^n}|^p$ is convergent in probability. Show that $\sum_{k=1}^{2^n} |X_{t_k^n} - X_{t_{k-1}^n}|^q$ converges to zero in probability for $q > p$.

Exercise 2.10. Let $0 < H < 1$ and X be a continuous adapted process such that X has finite-dimensional distributions which are normally distributed with mean zero and such that for any s and t with $s,t \geq 0$, $EX_sX_t = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$. Such a process is called a fractional Brownian motion with Hurst parameter H. Show that if $H = \frac{1}{2}$, then X has the distribution of a Brownian motion. Show that if $H \neq \frac{1}{2}$, then X is not in $\mathbf{c}\mathcal{S}$.

Exercise 2.11. Let W be a p-dimensional $\mathcal{F}_t$ Brownian motion. Let $f : \mathbb{R}^p \to \mathbb{R}$ be C^2. Show that $f(W_t)$ is a contiuous local martingale if $\sum_{i=1}^p \frac{\partial^2 f}{\partial x_i^2}(x) = 0$ for all $x \in \mathbb{R}^p$.

Exercise 2.12. Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be C^2. Show that $f(t, W_t)$ is a continuous local martingale if $\frac{\partial f}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x) = 0$ for $(t,x) \in \mathbb{R}^2$. Show that in the affirmative, it holds that $f(t,W_t) = f(0,0) + \int_0^t \frac{\partial f}{\partial x}(s,W_s)\,\mathrm{d}W_s$.

Exercise 2.13. Let $M \in \mathbf{c}\mathcal{M}_\ell$ and define the process $\mathcal{E}(M)$ by $\mathcal{E}(M)_t = \exp(M_t - \frac{1}{2}[M]_t)$. Show that $\mathcal{E}(M)$ is a continuous local martingale with initial value one. Show that $\mathcal{E}(M)$ is the unique solution in Y in $\mathbf{c}\mathcal{S}$ to the stochastic differential equation $Y_t = 1 + \int_0^t Y_s\,\mathrm{d}M_s$.

Exercise 2.14. Let $M, N \in \mathbf{c}\mathcal{M}_\ell$. Show that $\mathcal{E}(M)$ and $\mathcal{E}(N)$ are indistinguishable if and only if M and N are indistinguishable.

Exercise 2.15. Fix $t \geq 0$ and define $Z_t^n = \prod_{k=1}^{2^n}(M_{t_k} - M_{t_{k-1}})$, where $t_k^n = kt2^{-n}$. Show that Z_t^n converges in probability to $\mathcal{E}(M)_t$.

Exercise 2.16. Let $M \in \mathbf{c}\mathcal{M}_\ell$. Show that $\mathcal{E}(M)$ is a nonnegative supermartingale. Show that $E\mathcal{E}(M)_t \leq 1$ for all $t \geq 0$. Show that $\mathcal{E}(M)$ is almost surely convergent and that the limit $\mathcal{E}(M)_\infty$ satisfies that $E\mathcal{E}(M)_\infty \leq 1$.

Exercise 2.17. Show that $\mathcal{E}(M)$ is a uniformly integrable martingale if and only if it holds that $E\mathcal{E}(M)_\infty = 1$, and show that $\mathcal{E}(M)$ is a martingale if and only if $E\mathcal{E}(M)_t = 1$ for all $t \geq 0$.

Exercise 2.18. Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion and let $f : \mathbb{R}_+ \to \mathbb{R}$ be continuous. Show that $f \in \mathfrak{I}$ in the sense that the process $(t,\omega) \mapsto f(t)$ is in $\mathfrak{I}$. Fix $t \geq 0$ and find the distribution of $\int_0^t f(s)\,\mathrm{d}W_s$.

Exercise 2.19. Let $X, Y \in \mathbf{c}\mathcal{S}$ and $f, g \in C^2(\mathbb{R})$. With $f(X)_t = f(X_t)$ and $g(Y)_t = g(Y_t)$, show that $[f(X), g(Y)]_t = \int_0^t f'(X_s)g'(Y_s)\,\mathrm{d}[X,Y]_s$ for all $t \geq 0$ up to indistinguishability. Use this to identify the quadratic variation of W^p when W is an $\mathcal{F}_t$ Brownian motion and $p \geq 1$.

Exercise 2.20. Let W be a p-dimensional $\mathcal{F}_t$ Brownian motion. Find the quadratic covariation process of $W_t^i W_t^j$ for $i, j \leq p$.

Exercise 2.21. Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion. Define X as the process given by $X_t = \int_0^t \operatorname{sgn}(W_s)\,\mathrm{d}W_s$. Show that $X_t W_t$ and $X_t W_t^2$ are both integrable with $EX_t W_t = 0$ and $EX_t W_t^2 = 2^{\frac{5}{2}} t^{\frac{3}{2}} (3\sqrt{\pi})^{-1}$.

Exercise 2.22. Let X be in $\mathbf{c}\mathcal{S}$ with initial value zero and let $A \in \mathbf{c}\mathcal{V}$. Define M^α by putting $M_t^\alpha = \exp(\alpha X_t - \frac{\alpha^2}{2} A_t)$. Show that if $M^\alpha \in \mathbf{c}\mathcal{M}_\ell$ for all $\alpha \in \mathbb{R}$, then X is in $\mathbf{c}\mathcal{M}_\ell$ and $[X] = A$.

Exercise 2.23. Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion. Let $f : \mathbb{R} \to \mathbb{R}$ be C^2 with $f(0) = \alpha$ and assume that f satisfies that $|f''(x)| \leq C + \exp(\beta|x|)$ for some $C, \beta > 0$. Show that $Ef(B_t) = \alpha + \frac{1}{2} \int_0^t Ef''(B_s)\,\mathrm{d}s$. Use this to prove that for all $p \in \mathbb{N}$, $EB_t^{2p-1} = 0$ and $EB_t^{2p} = t^p \prod_{i=1}^p (2i-1)$.

Exercise 2.24. Assume that $M \in \mathbf{c}\mathcal{M}^b$ and put $t_k^n = kt2^{-n}$. Show that the sequence $\sum_{k=1}^{2^n} (M_{t_k} - M_{t_{k-1}})^2$ converges in $\mathcal{L}^1$ to $[M]_t$.

Exercise 2.25. Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion. Define X as the process given by $X_t = \int_0^t \sin W_s + \cos W_s\,\mathrm{d}W_s$. Argue that the stochastic integral is well-defined. Find the mean and variance of X_t for $t \geq 0$.

Exercise 2.26. Let $X_t = \int_0^t \sin W_s\,\mathrm{d}W_s$, where W is a one-dimensional $\mathcal{F}_t$ Brownian motion. Compute the covariance between X_s and X_t for $0 \leq s \leq t$.

Exercise 2.27. With W a one-dimensional $\mathcal{F}_t$ Brownian motion, show that $\exp(\frac{1}{2}t)\sin W_t$ and $(W_t + t)\exp(-W_t - \frac{1}{2}t)$ are in $\mathbf{c}\mathcal{M}_\ell$.

Exercise 2.28. Let W be a one-dimensional $\mathcal{F}_t$ Brownian motion. Without using direct moment calculations, show that the processes $W_t^3 - 3tW_t$ and $W_t^4 - 6tW_t^2 + 3t^2$ are in $\mathbf{c}\mathcal{M}$.

Appendix A

Appendices

In these appendices, we review the general analysis, measure theory and probability theory which are used in the main text, but whose subject matter is either taken to be more or less well-known or taken to be sufficiently different from our main interests to merit separation.

A.1 Analysis and measure theory

We begin by considering some results on signed measures and mappings of finite variation. Let $(E, \mathcal{E})$ be a measurable space. A signed measure on $(E, \mathcal{E})$ is a mapping $\mu : \mathcal{E} \to \mathbb{R}$ such that $\mu(\emptyset) = 0$ and such that whenever (A_n) is a sequence of disjoint sets in $\mathcal{E}$, $\sum_{n=1}^\infty |\mu(A_n)|$ is convergent and $\mu(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \mu(A_n)$.

Theorem A.1.1. *Let μ be a signed measure on $(E, \mathcal{E})$. There exists a bounded nonnegative measure $|\mu|$ on $(E, \mathcal{E})$ such that $|\mu|(A) = \sup \sum_{n=1}^\infty |\mu(A_n)|$, where the supremum is taken over all mutually disjoint sequences (A_n) in $\mathcal{E}$ with $A = \cup_{n=1}^\infty A_n$. $|\mu|$ is called the total variation measure of μ. In particular, $|\mu(A)| \leq |\mu|(A) \leq |\mu|(E)$, so every signed measure is bounded.*

Proof. See Theorem 6.2 and Theorem 6.4 of Rudin (1987). $\square$

Theorem A.1.2 (Jordan-Hahn decomposition)**.** *Let μ be a signed measure on $(E, \mathcal{E})$. There is a unique pair of positive bounded singular measures μ^+ and μ^- such that $\mu = \mu^+ - \mu^-$, given by $\mu^+ = \frac{1}{2}(|\mu|+\mu)$ and $\mu^- = \frac{1}{2}(|\mu|-\mu)$. This decomposition also satisfies $|\mu| = \mu^+ + \mu^-$. We call this the Jordan-Hahn decomposition of μ.*

Proof. By Section 6.6 of Rudin (1987) and Theorem 6.14 of Rudin (1987), the explicit construction of μ^+ and μ^- satisfies the requirements of the theorem. For uniqueness, assume that $\mu = \nu^+ - \nu^-$, where ν^+ and ν^- is another pair of singular positive bounded measures. Assume that ν^+ is concentrated on F^+ and ν^- is concentrated on F^-, while μ^+ is concentrated on E^+ and μ^- is concentrated on E^-, where F^+ and F^- are disjoint and E^+ and E^- are disjoint. Then, for any $A \in \mathcal{E}$, $\mu^+(A) = \mu^+(A \cap E^+) = \mu(A \cap E^+) \leq \nu^+(A \cap E^+) \leq \nu^+(A)$ and $\nu^+(A) = \nu^+(A \cap F^+) = \mu(A \cap F^+) \leq \mu^+(A \cap F^+) \leq \mu^+(A)$, so ν^+ and μ^+ are equal and therefore ν^- and μ^- are equal as well. □

Lemma A.1.3. *Let μ be a signed measure on $(E, \mathcal{E})$. Let $\mathbb{D}$ be an algebra generating $\mathcal{E}$. Then $|\mu|(E) = \sup \sum_{k=1}^n |\mu(A_k)|$, where the supremum is taken over finite disjoint partitions (A_k) of E, and each element A_k is in $\mathbb{D}$.*

Proof. We first show that $|\mu|(E) = \sup \sum_{k=1}^n |\mu(A_k)|$, where the sum is taken over finite disjoint partitions (A_k) of E, and each element A_k is in $\mathcal{E}$. To this end, let $\varepsilon > 0$. There is a countable disjoint measurable partition (A_n) of E such that $|\mu|(E) \leq \varepsilon + \sum_{n=1}^\infty |\mu(A_n)|$. Since $|\mu|$ is a bounded positive measure, the sum $\sum_{n=1}^\infty |\mu|(A_n)$ is convergent, and therefore, there is k such that $|\mu(\cup_{n=k}^\infty A_n)| = |\sum_{n=k}^\infty \mu(A_n)| \leq \sum_{n=k}^\infty |\mu(A_n)| \leq \sum_{n=k}^\infty |\mu|(A_n) \leq \varepsilon$. As all the numbers in the chain of inequalities are nonnegative, we find in particular that $||\mu(\cup_{n=k}^\infty A_n)| - \sum_{n=k}^\infty |\mu(A_n)|| \leq \varepsilon$ and thus

$$|\mu|(E) \leq \varepsilon + \sum_{n=1}^\infty |\mu(A_n)| = \varepsilon + \sum_{n=1}^{k-1} |\mu(A_n)| + \sum_{n=k}^\infty |\mu(A_n)| \leq 2\varepsilon + |\mu(\cup_{n=k}^\infty A_n)| + \sum_{n=1}^{k-1} |\mu(A_n)|,$$

and since the family of sets $A_1, \ldots, A_{k-1}, \cup_{n=k}^\infty A_n$ is a finite disjoint partition of E with each element in $\mathcal{E}$, and $\varepsilon > 0$ was arbitrary, we conclude that $|\mu|(E) = \sup \sum_{k=1}^n |\mu(A_k)|$, where the supremum is over finite disjoint measurable partitions of E.

Next, we show that it suffices to consider partitions with each element in $\mathbb{D}$. Let $\varepsilon > 0$, we need to locate a finite disjoint partition (A_n) of E with elements from $\mathbb{D}$ such that $|\mu|(E) \leq \varepsilon + \sum_{n=1}^k \mu(A_k)$. From what we have just shown, there is a finite disjoint partition (A_n) of E with each A_n in $\mathcal{E}$ such that $|\mu|(E) \leq \varepsilon + \sum_{n=1}^k |\mu(A_n)|$. For any $n \leq k$, we may use Theorem 1.3.11 of Ash (2000) to obtain some $B_n \in \mathbb{D}$ with $|\mu|(A_n \Delta B_n) \leq \frac{1}{k}\varepsilon 2^{-k}$, where the symmetric difference $A_n \Delta B_n$ is defined by $A_n \Delta B_n = (A_n \cap B_n^c) \cup (A_n^c \cap B_n)$.

Now let $\mathbb{P}_k$ denote the set of all subsets of $\{1,\ldots,k\}$, and define the set C_α for any $\alpha \in \mathbb{P}_k$ by putting $C_\alpha = \{x \in E \mid \ \forall\ n \leq k : x \in B_n \text{ if } n \in \alpha \text{ and } x \in B_n^c \text{ if } n \notin \alpha\}$. C_α is the intersection of the B_n's with $n \in \alpha$ and the B_n^c's with $n \notin \alpha$. In particular, the family $(C_\alpha)_{\alpha \in \mathbb{P}_k}$ consists of mutually disjoint sets. As for each $x \in E$ and each $n \leq k$, we either have $x \in B_n$ or $x \in B_n^c$, the family $(C_\alpha)_{\alpha \in \mathbb{P}_k}$ is a finite disjoint partition of E, and as $\mathbb{D}$ is an algebra, $C_\alpha \in \mathbb{D}$ for all $\alpha \in \mathbb{P}_k$. We claim that $|\mu|(E) \leq 3\varepsilon + \sum_{\alpha \in \mathbb{P}_k} |\mu(C_\alpha)|$.

To prove this, we first note that for any $n \leq k$, we have

$$\begin{aligned}
||\mu(A_n)| - |\mu(B_n)|| &\leq |\mu(A_n) - \mu(B_n)| \\
&= |\mu(A_n \cap B_n) + \mu(A_n \cap B_n^c) - (\mu(A_n \cap B_n) + \mu(A_n^c \cap B_n))| \\
&= |\mu(A_n \cap B_n^c) - \mu(A_n^c \cap B_n)| \leq |\mu(A_n \cap B_n^c)| + |\mu(A_n^c \cap B_n)| \\
&\leq |\mu|(A_n \cap B_n^c) + |\mu|(A_n^c \cap B_n) = |\mu|(A_n \Delta B_n),
\end{aligned}$$

which is less than $\frac{1}{k}\varepsilon 2^{-k}$. Therefore, we obtain

$$|\mu|(E) \leq \varepsilon + \sum_{n=1}^k |\mu(A_n)| \leq \varepsilon + \sum_{n=1}^k |\mu(B_n)| + \sum_{n=1}^k ||\mu(A_n)| - |\mu(B_n)|| \leq 2\varepsilon + \sum_{n=1}^k |\mu(B_n)|.$$

Note that $B_n = \cup_{\alpha \in \mathbb{P}_k : n \in \alpha} C_\alpha$, with each pair of sets in the union being mutually disjoint. We will argue that $|\mu|(B_n) \leq |\mu|(C_{\{n\}}) + \frac{1}{k}\varepsilon$. To see this, consider some $\alpha \in \mathbb{P}_k$ with more than one element, assume for definiteness that $n, m \in \alpha$ with $n \neq m$. As the A_n's are disjoint, we then find

$$\begin{aligned}
|\mu|(C_\alpha) &\leq |\mu|(B_n \cap B_m) = |\mu|(B_n \cap A_n \cap B_m) + |\mu|(B_n \cap A_n^c \cap B_m) \\
&\leq |\mu|(A_n \cap B_m) + |\mu|(B_n \cap A_n^c) = |\mu|(A_n \cap A_m^c \cap B_m) + |\mu|(B_n \cap A_n^c) \\
&\leq |\mu|(A_m^c \cap B_m) + |\mu|(B_n \cap A_n^c) \leq |\mu|(A_m \Delta B_m) + |\mu|(A_n \Delta B_n),
\end{aligned}$$

which is less than $\frac{2}{k}\varepsilon 2^{-k}$. We now note, using $C_{\{n\}} \subseteq B_n$, that

$$\begin{aligned}
|\mu(B_n)| &\leq |\mu(C_{\{n\}})| + |\mu(B_n) - \mu(C_{\{n\}})| \\
&= |\mu(C_{\{n\}})| + |\mu(B_n \setminus C_{\{n\}})| \\
&\leq |\mu(C_{\{n\}})| + |\mu|(B_n \setminus C_{\{n\}}).
\end{aligned}$$

However, $|\mu|(B_n \setminus C_{\{n\}}) = |\mu|(B_n) - |\mu|(C_{\{n\}}) = \sum_{\alpha \in \mathbb{P}_k : n \in \alpha, \alpha \neq \{n\}} |\mu|(C_\alpha)$ and as there are less than 2^{k-1} elements in the sum, with each element according to what was already proven has a value of less than $\frac{2}{k}\varepsilon 2^{-k}$, we find $|\mu|(B_n) \leq |\mu|(C_{\{n\}}) + \frac{1}{k}\varepsilon$. We may now conclude $|\mu|(E) \leq 2\varepsilon + \sum_{n=1}^k |\mu(B_n)| \leq 3\varepsilon + \sum_{n=1}^k |\mu(C_{\{n\}})| \leq 3\varepsilon + \sum_{\alpha \in \mathbb{P}_k} |\mu(C_\alpha)|$. As ε was arbitrary, we conclude that $|\mu(E)| = \sup \sum_{n=1}^k |\mu(A_n)|$, where the supremum is taken over finite disjoint partitions (A_n) of E, and each element A_n is in $\mathbb{D}$. $\square$

We next introduce finite variation mappings and consider their connection to pairs of positive singular measures. These types of mappings will be important in our consideration of continuous-time stochastic processes of finite variation, of which the quadratic covariation will be a primary example. Consider a mapping $f : \mathbb{R}_+ \to \mathbb{R}$. We define the variation of f on $[0,t]$ by $V_f(0) = 0$ and $V_f(t) = \sup \sum_{k=1}^n |f(t_k) - f(t_{k-1})|$, where the supremum is over partitions $0 = t_0 < \cdots < t_n = t$. We say that f is of finite variation on $[0,t]$ if $V_f(t)$ is finite. We say that f is of finite variation if $V_f(t)$ is finite for $t \geq 0$. We say that f is of bounded variation if $\sup_t V_f(t)$ is finite. Finally, by $\mathbf{cFV}$, we denote the continuous mappings $f : \mathbb{R}_+ \to \mathbb{R}$ of finite variation, and by $\mathbf{cFV}_0$, we denote the elements of $\mathbf{cFV}$ which are zero at zero. In general, we will focus our attention on $\mathbf{cFV}_0$ and not $\mathbf{cFV}$, as pinning our mapping to zero at zero makes for a better concordance with measure theory. Note that any monotone function has finite variation, and for any increasing function $f : \mathbb{R}_+ \to \mathbb{R}$ with initial value zero, $V_f(t) = f(t)$.

Lemma A.1.4. *Let $f \in \mathbf{cFV}_0$. Then V_f is continuous.*

Proof. See Carothers (2000), Theorem 13.9. □

Theorem A.1.5. *Let $f \in \mathbf{cFV}_0$. There is a unique decomposition $f = f^+ - f^-$ such that f^+ and f^- are increasing functions in $\mathbf{cFV}_0$ with the property that there exists two unique positive singular measures μ_f^+ and μ_f^- with zero point mass at zero such that for any $0 \leq a \leq b$, $\mu_f^+(a,b] = f^+(b) - f^+(a)$ and $\mu_f^-(a,b] = f^-(b) - f^-(a)$. The decomposition is given by $f^+ = \frac{1}{2}(V_f + f)$ and $f^- = \frac{1}{2}(V_f - f)$. In particular, the measures μ_f^+ and μ_f^- are finite on bounded intervals, and $(\mu_f^+ + \mu_f^-)(a,b] = V_f(b) - V_f(a)$.*

Proof. We first show that the explicit construction of f^+ and f^- satisfies the properties required. We note that f^+ and f^- are increasing and zero at zero, and so, as monotone functions are of finite variation, we conclude that f^+ and f^- are in $\mathbf{cFV}_0$, continuity being a consequence of Lemma A.1.4. By Theorem 1.4.4 of Ash (2000), there exists unique nonnegative measures μ_f^+ and μ_f^- with zero point mass at zero such that for any $0 \leq a \leq b$, $\mu_f^+(a,b] = f^+(b) - f^+(a)$ and $\mu_f^-(a,b] = f^-(b) - f^-(a)$. Then $(\mu_f^+ + \mu_f^-)(a,b] = V_f(b) - V_f(a)$ as well. It remains to prove that μ_f^+ and μ_f^- are singular, and to this end, it suffices to prove that the measures are singular on $[0,t]$ for any $t \geq 0$.

To do so, fix $t \geq 0$. Put $\mu_f^t = \mu_f^+ - \mu_f^-$ on $\mathcal{B}_t$, then μ_f^t is a signed measure on $\mathcal{B}_t$, and for any $0 \leq a \leq b \leq t$, $\mu_f^t(a,b] = f(b) - f(a)$. We consider the total variation of μ_f^t. Fix $0 \leq a \leq b \leq t$ and let $\mathbb{D}$ be the set of finite unions of intervals of the form $(c,d]$ with $a \leq c \leq d \leq b$, $\mathbb{D}$ is an algebra generating the Borel-σ-algebra on $(a,b]$. Lemma A.1.3 shows that we have $|\mu_f^t|(a,b] = \sup \sum_{n=1}^k |\mu_f^t(A_n)|$, where (A_n) is a finite disjoint partition of $(a,b]$ with elements from $\mathbb{D}$. In particular, we obtain $|\mu_f^t|(a,b] \le V_f(b) - V_f(a)$, and as we trivially have $V_f(b) - V_f(a) \le |\mu_f^t|(a,b]$, we have equality. Thus, $|\mu_f^t|(a,b] = V_f(b) - V_f(a)$. Let $(\mu_f^t)^+$ and $(\mu_f^t)^-$ be the Jordan-Hahn decomposition of Theorem A.1.2, we then obtain

$$(\mu_f^t)^+(a,b] = \tfrac{1}{2}(|\mu_f^t|(a,b] + \mu_f^t(a,b]) = \tfrac{1}{2}(V_f(b) - V_f(a) + f(b) - f(a)) = \mu_f^+(a,b],$$

and so we find that $(\mu_f^t)^+$ and μ_f^+ agree on $\mathcal{B}_t$. Analogously, $(\mu_f^t)^-$ and μ_f^- agree on $\mathcal{B}_t$ as well. As the components of the Jordan-Hahn decomposition are singular, we conclude that μ_f^+ and μ_f^- are singular on $[0,t]$, and so μ_f^+ and μ_f^- are singular measures.

It remains to prove uniqueness. Assume that $f = g^+ - g^-$ is another decomposition with the same properties. Let ν_f^+ and ν_f^- be the two corresponding singular nonnegative measures. As earlier, we may then define $\nu_f^t = \nu_f^+ - \nu_f^-$ on $\mathcal{B}_t$. Then ν_f^t and μ_f^t are equal, and so in particular, we have the Jordan-Hahn decompositions $\mu_f^t = \mu_f^+ - \mu_f^-$ and $\mu_f^t = \nu_f^t = \nu_f^+ - \nu_f^-$ on $\mathcal{B}_t$. By uniqueness of the decomposition, we conclude $\mu_f^+ = \nu_f^+$ and $\mu_f^- = \nu_f^-$, and so $f^+ = g^+$ and $f^- = g^-$, proving uniqueness. □

Theorem A.1.5 shows that finite variation mappings correspond to pairs of positive singular measures. As stated in the theorem, for any $f \in \mathbf{cFV}_0$, we denote by f^+ and f^- the positive and negative parts of f, given by $f^+ = \frac{1}{2}(V_f + f)$ and $f^- = \frac{1}{2}(V_f - f)$. Furthermore, we denote by μ_f^+ and μ_f^- the two corrresponding positive singular measures, and we put $|\mu_f| = \mu_f^+ + \mu_f^-$ and call $|\mu_f|$ the total variation measure of f. By Theorem A.1.5, $|\mu_f|$ is the measure induced by the increasing function V_f using Theorem 1.4.4 of Ash (2000). As μ_f^+ and μ_f^- has finite mass on bounded intervals, so does $|\mu_f|$, in particular we have $|\mu_f|([0,t]) = V_f(t)$ according to Theorem A.1.5. Also note that if f is increasing, $V_f = f$ and so μ^- is zero.

These results lead to a concept of integration with respect to a function of finite variation. Let $f \in \mathbf{cFV}_0$ and let $h : \mathbb{R}_+ \to \mathbb{R}$ be some measurable function. We say that h is integrable with respect to f if $\int_0^t |h(s)| \,\mathrm{d}|\mu_f|_s$ is finite for all $t \ge 0$, and in the affirmative, we put $\int_0^t h(s)\,\mathrm{d}f(s) = \int_0^t h(s)\,\mathrm{d}(\mu_f^+)_s - \int_0^t h(s)\,\mathrm{d}(\mu_f^-)_s$ and call $\int_0^t h(s)\,\mathrm{d}f(s)$ the integral of h with respect to f over $[0,t]$. Furthermore, we denote by $\int_0^t h(s)|\,\mathrm{d}f_s|$ the integral $\int_0^t h(s)\,\mathrm{d}|\mu_f|_s$. Next, we consider some further properties of finite variation mappings and their integrals.

Lemma A.1.6. *Let $f \in \mathbf{cFV}_0$. Then $|f(t)| \le V_f(t)$ for all $t \ge 0$.*

Proof. As $\{0, t\}$ is a partition of $[0, t]$, it holds that

$$|f(t)| = |f(t) - f(0)| \leq \sup \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| = V_f(t),$$

as required. □

Lemma A.1.7. *Let $f \in \mathbf{cFV}_0$ and let h be integrable with respect to f. It then holds that* $|\int_0^t h(s)\,\mathrm{d}f_s| \leq \int_0^t |h(s)||\,\mathrm{d}f_s|$.

Proof. We find

$$\begin{aligned}
\left|\int_0^t h(s)\,\mathrm{d}f_s\right| &= \left|\int_0^t h(s)\,\mathrm{d}\mu_f^+ - \int_0^t h(s)\,\mathrm{d}\mu_f^-\right| \leq \left|\int_0^t h(s)\,\mathrm{d}\mu_f^+\right| + \left|\int_0^t h(s)\,\mathrm{d}\mu_f^-\right| \\
&\leq \int_0^t |h(s)|\,\mathrm{d}\mu_f^+ + \int_0^t |h(s)|\,\mathrm{d}\mu_f^- = \int_0^t |h(s)|\,\mathrm{d}|\mu_f|,
\end{aligned}$$

and the latter is what we denote by $\int_0^t |h(s)||\,\mathrm{d}f_s|$. □

Lemma A.1.8 (Integration by parts)**.** *Let $f, g \in \mathbf{cFV}$, then for any $t \geq 0$,*

$$f(t)g(t) = f(0)g(0) + \int_0^t f(s)\,\mathrm{d}g_s + \int_0^t g(s)\,\mathrm{d}f_s.$$

Proof. See Section IV.18 of Rogers & Williams (2000b). □

Lemma A.1.9. *Let $f \in \mathbf{cFV}_0$. Then $V_f(t) = \sup \sum_{k=1}^n |f(t_k) - f(t_{k-1})|$, with the supremum taken over partitions in $\mathbb{Q}_+ \cup \{t\}$.*

Proof. It suffices to show that for any $\varepsilon > 0$ and any partition $(t_0, \ldots, t_n)$ of $[0, t]$, there exists another partition $(q_0, \ldots, q_n)$ such that $|\sum_{k=1}^n |f(t_k) - f(t_{k-1})| - \sum_{k=1}^n |f(q_k) - f(q_{k-1})| < \varepsilon$. To this end, choose δ parrying $\frac{\varepsilon}{2n}$ for the continuity of f in $t_0, \ldots, t_n$, and let, for $k \leq n$, q_k be some rational with $|q_k - t_k| \leq \delta$. By picking q_k close enough to t_k, we may ensure that $(q_0, \ldots, q_n)$ is in fact a partition of $[0, t]$. Then $|(f(t_k) - f(t_{k-1})) - (f(q_k) - f(q_{k-1}))| \leq \frac{\varepsilon}{n}$, and since $|\cdot|$ is a contraction, this implies $||f(t_k) - f(t_{k-1})| - |f(q_k) - f(q_{k-1})|| \leq \frac{\varepsilon}{n}$, finally yielding

$$\begin{aligned}
&\left|\sum_{k=1}^n |f(t_k) - f(t_{k-1})| - \sum_{k=1}^n |f(q_k) - f(q_{k-1})|\right| \\
\leq\ &\sum_{k=1}^n ||f(t_k) - f(t_{k-1})| - |f(q_k) - f(q_{k-1})|| \leq \varepsilon.
\end{aligned}$$

This proves the result. □

This concludes our results on finite variation mappings and signed measures. Next, we consider two results which will aid us in the proof of the Kunita-Watanabe inequality.

Lemma A.1.10. *Let $\alpha, \gamma \geq 0$ and $\beta \in \mathbb{R}$. It then holds that $|\beta| \leq \sqrt{\alpha}\sqrt{\gamma}$ if and only if $\lambda^2\alpha + 2\lambda\beta + \gamma \geq 0$ for all $\lambda \in \mathbb{Q}$.*

Proof. First note that by continuity, the requirement that $\lambda^2\alpha + 2\lambda\beta + \gamma \geq 0$ for all $\lambda \in \mathbb{Q}$ is equivalent to the same requirement for all $\lambda \in \mathbb{R}$.

Consider first the case $\alpha = 0$. If $|\beta| \leq \sqrt{\alpha}\sqrt{\gamma}$, clearly $\beta = 0$, and the criterion is trivially satisfied. Conversely, assume that the criterion holds, which in this case is equivalent to $2\lambda\beta + \gamma \geq 0$ for all $\lambda \in \mathbb{Q}$. Letting λ tend to infinity or minus infinity depending on the sign of β, the requirement that γ be nonnegative forces $\beta = 0$, so that $\beta \leq \sqrt{\alpha}\sqrt{\gamma}$. This proves the result in the case $\alpha = 0$. Next, consider the case $\alpha \neq 0$, so that $\alpha > 0$. The mapping $\lambda^2\alpha + 2\lambda\beta + \gamma \geq 0$ takes its minimum at $-\frac{\beta}{\alpha}$, and the minimum value is

$$\inf_{\lambda \in \mathbb{R}} \lambda^2\alpha + 2\lambda\beta + \gamma = \left(-\frac{\beta}{\alpha}\right)^2 \alpha - \frac{2\beta^2}{\alpha} + \gamma = \frac{1}{\alpha}(\alpha\gamma - \beta^2),$$

which is nonnegative if and only if $|\beta| \leq \sqrt{\alpha}\sqrt{\gamma}$. This proves the result. $\square$

Lemma A.1.11. *Let $f, g, h : \mathbb{R}_+ \to \mathbb{R}$ be in $\mathbf{cFV}_0$, with f and g increasing. If it holds for all $0 \leq s \leq t$ that $|h(t) - h(s)| \leq \sqrt{f(t) - f(s)}\sqrt{g(t) - g(s)}$, then for any measurable $x, y : \mathbb{R}_+ \to \mathbb{R}$, we have*

$$\int_0^\infty |x(t)y(t)|\,|\,\mathrm{d}h(t)| \leq \left(\int_0^\infty x(t)^2 \,\mathrm{d}f(t)\right)^{\frac{1}{2}} \left(\int_0^\infty y(t)^2 \,\mathrm{d}g(t)\right)^{\frac{1}{2}}.$$

Proof. Let μ_f, μ_g and μ_h be the measures corresponding to the finite variation mappings f, g and h. Clearly, the measures μ_f, μ_g and μ_h are all absolutely continuous with respect to $\nu = \mu_f + \mu_g + |\mu_h|$. Then, by the Radon-Nikodym Theorem, there exists densities φ_f, φ_g and φ_h of the three measures with respect to ν, and it therefore suffices to prove

$$\left(\int_0^\infty |x(t)y(t)||\varphi_h(t)| \,\mathrm{d}\nu(t)\right)^2 \leq \left(\int_0^\infty x(t)^2\varphi_f(t) \,\mathrm{d}\nu(t)\right) \left(\int_0^\infty y(t)^2\varphi_g(t) \,\mathrm{d}\nu(t)\right).$$

To this end, we wish to argue that $|\varphi_h(t)| \leq \sqrt{\varphi_f(t)}\sqrt{\varphi_g(t)}$ almost everywhere with respect to ν. By Lemma A.1.10, this is equivalent to proving that almost everywhere in t with respect to ν, it holds that for all $\lambda \in \mathbb{Q}$ that $\lambda^2\varphi_f(t) + 2\lambda\varphi_h(t) + \varphi_g(t) \geq 0$. As a countable intersection of null sets is again a null set, it suffices to prove that for any $\lambda \in \mathbb{Q}$, it holds that $\lambda^2\varphi_f(t) + 2\lambda\varphi_h(t) + \varphi_g(t) \geq 0$ almost everywhere with respect to ν. However, for any $0 \leq s \leq t$, we have

$$\int_s^t \lambda^2\varphi_f(u) + 2\lambda\varphi_h(u) + \varphi_g(u)\,\mathrm{d}\nu(u) = \lambda^2\mu_f(s,t] + 2\lambda\mu_h(s,t] + \mu_g(s,t],$$

and as $|\mu_h(s,t]| \leq \sqrt{\mu_f(s,t]}\sqrt{\mu_g(s,t]}$ by assumption, the above is nonnegative by Lemma A.1.10. By a monotone class argument, we obtain that $\int_A \lambda^2\varphi_f(t)+2\lambda\varphi_h(t)+\varphi_g(t)\,\mathrm{d}\nu(t) \geq 0$ for any $A \in \mathcal{B}_+$, in particular $\lambda^2\varphi_f(t) + 2\lambda\varphi_h(t) + \varphi_g(t) \geq 0$ almost everywhere with respect to ν. Thus, we finally conclude $|\varphi_h(t)| \leq \sqrt{\varphi_f(t)}\sqrt{\varphi_g(t)}$. The Cauchy-Schwartz inequality then immediately yields

$$\begin{aligned}
\int_0^\infty |x(t)y(t)||\varphi_h(t)|\,\mathrm{d}\nu(t) &\leq \int_0^\infty |x(t)\sqrt{\varphi_f(t)}||y(t)\sqrt{\varphi_g(t)}|\,\mathrm{d}\nu(t)\\
&\leq \left(\int_0^\infty x(t)^2\varphi_f(t)\,\mathrm{d}\nu(t)\right)^{\frac{1}{2}}\left(\int_0^\infty y(t)^2\varphi_g(t)\,\mathrm{d}\nu(t)\right)^{\frac{1}{2}}\\
&= \left(\int_0^\infty x(t)^2\,\mathrm{d}f(t)\right)^{\frac{1}{2}}\left(\int_0^\infty y(t)^2\,\mathrm{d}g(t)\right)^{\frac{1}{2}},
\end{aligned}$$

as desired. □

Finally, we end the section with a few assorted results from measure theory and analysis.

Theorem A.1.12. *Let P be a probability measure on $(\Omega, \mathcal{F})$ and let (ν_ω) be a family of uniformly bounded nonnegative measures on $(E, \mathcal{E})$, in the sense that there is $c > 0$ such that $\nu_\omega(E) \leq c$ for all $\omega \in \Omega$. Assume that $\omega \mapsto \nu_\omega(A)$ is $\mathcal{F}$ measurable for all $A \in \mathcal{E}$. There exists a unique nonnegative measure λ on $\mathcal{F} \otimes \mathcal{E}$, called the integration of (ν_ω) with respect to P, uniquely characterized by the requirement that for $F \in \mathcal{F}$ and $A \in \mathcal{E}$, it holds that $\lambda(F \times A) = \int_F \nu_\omega(A)\,\mathrm{d}P(\omega)$.*

Proof. This follows from Theorem 2.6.2 of Ash (2000). □

Theorem A.1.13 (Tonelli and Fubini theorems for integration measures)**.** *Let P be a probability measure on $(\Omega, \mathcal{F})$ and let (ν_ω) be a family of uniformly bounded nonnegative measures on $(E, \mathcal{E})$ such that $\omega \mapsto \nu_\omega(A)$ is $\mathcal{F}$ measurable for all $A \in \mathcal{E}$. Let λ be the integration of (ν_ω) with respect to P. Let $f : \Omega \times E \to \mathbb{R}$ be $\mathcal{F} \otimes \mathcal{E}$ measurable. Then, the following holds.*

1. *If f is nonnegative, $\omega \mapsto \int f(\omega, x)\,\mathrm{d}\nu_\omega(x)$ is $\mathcal{F}$ measurable and we have the equality $\int f(\omega, x)\,\mathrm{d}\lambda(\omega, x) = \int\int f(\omega, x)\,\mathrm{d}\nu_\omega(x)\,\mathrm{d}P(\omega)$.*

2. *If f is integrable with respect to λ, $\int f(\omega,x)\,\mathrm{d}\nu_\omega(x)$ exists and is finite P almost surely, and as a function of ω defines an $\mathcal{F}$ measurable function if it is taken to be zero on the null set where $\int f(\omega,x)\,\mathrm{d}\nu_\omega(x)$ does not exist. Furthermore, it holds that $\int f(\omega,x)\,\mathrm{d}\lambda(\omega,x) = \int\int f(\omega,x)\,\mathrm{d}\nu_\omega(x)\,\mathrm{d}P(\omega)$.*

Proof. See Theorem 2.6.4 of Ash (2000). □

Lemma A.1.14. *Let X be some integrable variable. Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. If $E1_F X \geq 0$ for all $F \in \mathcal{G}$, then $E(X|\mathcal{G}) \geq 0$ almost surely.*

Proof. Pick $n \in \mathbb{N}$ and define $F = (E(X|\mathcal{G}) \leq -\frac{1}{n})$. As $E(X|\mathcal{G})$ is $\mathcal{G}$ measurable, we have $F \in \mathcal{G}$ and therefore obtain $E1_F X = E1_F E(X|\mathcal{G}) \leq -\frac{1}{n}P(F)$. Therefore, $P(F) = 0$. By the continuity properties of probability measures, we conclude $P(E(X|\mathcal{G}) < 0) = 1$, so that $E(X|\mathcal{G}) \geq 0$ almost surely. □

Lemma A.1.15. *Let $X \geq 0$. It holds that X has mean zero if and only if X is almost surely zero.*

Proof. Clearly, X has mean zero if X is almost surely zero. Assume instead that X has mean zero. For any $n \in \mathbb{N}$, we have $EX \geq EX1_{(X \geq \frac{1}{n})} \geq \frac{1}{n}P(X \geq \frac{1}{n})$. Thus, we conclude $P(X \geq \frac{1}{n}) = 0$ for all n, and therefore $P(X > 0) = 0$, so that X is almost surely zero. □

Lemma A.1.16. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be some continuous mapping. Let $U(f,a,b)$ denote the number of upcrossings from a to b of f, meaning that*

$$U(f,a,b) = \sup\{n \mid \exists\, 0 \leq s_1 < t_1 < \cdots s_n < t_n : f(s_k) < a, f(t_k) > b, k \leq n\}.$$

The mapping f_t has a limit in $[-\infty,\infty]$ as t tends to infinity if $U(f,a,b)$ is finite for all $a, b \in \mathbb{Q}$ with $a < b$.

Proof. Assume that $U(f,a,b)$ is finite for all $a, b \in \mathbb{Q}$ with $a < b$. Assume, expecting a contradiction, that $f(t)$ does not converge to any limit in $[-\infty,\infty]$ as t tends to infinity. Then $\liminf_t f(t) < \limsup_t f(t)$, and in particular there exists $a, b \in \mathbb{Q}$ with $a < b$ such that $\liminf_t f(t) < a < b < \limsup_t f(t)$.

Now consider $U(f,a,b)$, we wish to derive a contradiction with our assumption that $U(f,a,b)$ is finite. If $U(f,a,b)$ is zero, either $f(t) \geq a$ for all $t \geq 0$, or $f(t) < a$ for some t and $f(t) \leq b$ from a point onwards. In this first case, $\liminf_t f(t) \geq a$, and in the second case,

$\limsup_t f(t) \leq b$, both leading to contradictions. Therefore, $U(f,a,b)$ must be nonzero. As we have assumed that $U(f,a,b)$ is finite, we obtain that either $f(t) \geq a$ from a point onwards, or $f(t) \leq b$ from a point onwards. In the first case, $\liminf_t f(t) \geq a$ and in the second case, $\limsup_t f(t) \leq b$. Again, we obtain a contradiction, and so conclude that $f(t)$ must exist as a limit in $[-\infty,\infty]$. $\square$

For the next results, we use the notation that for any open set U in $\mathbb{R}^p$, $C^2(U)$ denotes the set of mappings $f : U \to \mathbb{R}$ such that all second-order partial derivatives of f exists and are continuous. Furthermore, $C^\infty(U)$ denotes the set of $f : U \to \mathbb{R}$ such that all partial derivatives of any order of f exists, and $C_c^\infty(U)$ denotes the set of elements f in $C^\infty(U)$ which have compact support in the sense that $\{x \in U | f(x) \neq 0\}$ is contained in a compact set.

Theorem A.1.17. *Let $f \in C^2(\mathbb{R}^p)$, and let $x, y \in \mathbb{R}^p$. It then holds that*

$$f(y) = f(x) + \sum_{i=1}^p \frac{\partial f}{\partial x_i}(x)(y_i - x_i) + \frac{1}{2}\sum_{i=1}^p\sum_{j=1}^p \frac{\partial^2 f}{\partial x_i \partial x_j}(x)(y_i - x_i)(y_j - x_j) + R_2(x,y),$$

where $R_2(x,y) = \sum_{i=1}^p \sum_{j=1}^p r_2^{ij}(y,x)(y_i - x_i)(y_j - x_j)$, and

$$r_2^{ij}(y,x) = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\xi(x,y)) - \frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right),$$

where $\xi(x,y)$ is some element on the line segment between x and y.

Proof. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(t) = f(x + t(y - x))$. Note that $g(1) = f(y)$ and $g(0) = f(x)$. We will prove the theorem by applying the one-dimensional Taylor formula, see Apostol (1964) Theorem 7.6, to g. Clearly, $g \in C^2(\mathbb{R})$, and we obtain $g(1) = g(0) + g'(0) + \frac{1}{2}g''(s)$, where $0 \leq s \leq 1$. Applying the chain rule, we find $g'(t) = \sum_{i=1}^p \frac{\partial f}{\partial x_i}(x + t(y - x))(y_i - x_i)$ and $g''(t) = \sum_{i=1}^p \sum_{j=1}^p \frac{\partial^2 f}{\partial x_i \partial x_j}(x + t(y - x))(y_i - x_i)(y_j - x_j)$. Substituting and writing $\xi = x + s(y - x)$, we may conclude

$$f(y) = f(x) + \sum_{i=1}^p \frac{\partial f}{\partial x_i}(x)(y_i - x_i) + \frac{1}{2}\sum_{i=1}^p\sum_{j=1}^p \frac{\partial^2 f}{\partial x_i \partial x_j}(\xi)(y_i - x_i)(y_j - x_j).$$

In particular, we find $R_2(x,y) = \sum_{i=1}^p \sum_{j=1}^p r_2^{ij}(y,x)(y_i - x_i)(y_j - x_j)$, where $r_2^{ij} : \mathbb{R}^2 \to \mathbb{R}$ is defined by putting

$$r_2^{ij}(y,x) = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\xi) - \frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right),$$

where ξ of course depends on x and y, as it is on the line segment between the two. This proves the result. $\square$

Lemma A.1.18. *Let U be an open set in $\mathbb{R}^p$ and let $f \in C^2(U)$. Let $\varepsilon > 0$. With $\|\cdot\|$ denoting some norm on $\mathbb{R}^p$ and $d(x,y) = \|x-y\|$, put $F = \{x \in \mathbb{R}^p \mid d(x,U^c) \geq \varepsilon\}$. There exists $g \in C^2(\mathbb{R}^p)$ such that f and g agree on F.*

Proof. Let $G = \{x \in \mathbb{R}^p \mid d(x,U^c) \geq \frac{\varepsilon}{2}\}$ and $H = \{x \in \mathbb{R}^p \mid d(x,U^c) \geq \frac{\varepsilon}{4}\}$. We first prove that there exists a mapping $\chi \in C^\infty(\mathbb{R}^p)$ such that χ is one on F and zero on H^c. From Lemma 2.1 of Grubb (2008) and Section 0.B of Zimmer (1990), there exists some mapping $\psi \in C_c^\infty(\mathbb{R}^p)$ such that $\int_{\mathbb{R}^p} \psi(x)\,\mathrm{d}x = 1$ and ψ is zero outside of the open Euclidean ball B centered at the origin with radius $\frac{\varepsilon}{4}$. Define $\chi : \mathbb{R}^p \to \mathbb{R}$ by $\chi(x) = \int_{\mathbb{R}^p} 1_G(y)\psi(x-y)\,\mathrm{d}y$, this is well-defined as ψ has compact support, and compact sets have finite Lebesgue measure. We claim that χ satisfies the requirements. Applying the methods of the proof of Proposition B.3 of Zimmer (1990), we find that $\chi \in C^\infty(\mathbb{R}^p)$. Note that by the translation invariance of Lebesgue measure, we have

$$\chi(x) = \int_{\mathbb{R}^p} 1_G(x-y)\psi(y)\,\mathrm{d}y = \int_B 1_G(x-y)\psi(y)\,\mathrm{d}y.$$

Now, given some $x \in F$, we find that for any $y \in B$, $d(x,U^c) \leq d(x-y,U^c) + \|y\|$ and so $d(x-y,U^c) \geq d(x,U^c) - \|y\| \geq \varepsilon - \frac{\varepsilon}{4} \geq \frac{\varepsilon}{2}$. Thus, $x-y \in G$ and so $\chi(x) = \int_B \psi(y)\,\mathrm{d}y = 1$. Conversely, if $x \in H^c$, it holds that $d(x-y,U^c) \leq d(x,U^c) + \|y\| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$, so $x - y \notin G$, and $\chi(x) = 0$. Thus, χ is in $C^\infty(\mathbb{R}^p)$ and $\chi(x) = 1$ when $x \in F$ and $\chi(x) = 0$ when $x \in H^c$. We now define $g : \mathbb{R}^p \to \mathbb{R}$ by putting $g(x) = f(x)\chi(x)$ when $x \in U$ and $g(x) = 0$ otherwise. We claim that g satisfies the requirements of the lemma.

To see this, first note that when $x \in F$, $g(x) = f(x)\chi(x) = f(x)$, so g and f agree on F. Therefore, we merely need to check that g is C^2. To see this, note that on U, g is the product of an C^2 mapping and an C^∞ mapping, so g is C^2 on U. Conversely, as χ is zero on H^c, we find that g is in particular C^2 on H^c. As $H \subseteq U$, $U^c \subseteq H^c$ and so $\mathbb{R}^p = U \cup H^c$. Therefore, we conclude that g is in $C^2(\mathbb{R}^p)$, as desired. □

Lemma A.1.19 (Dynkin's lemma)**.** *Let E be some set, and let $\mathbb{E}$ be a family of subsets of E which is stable under intersections. Let $\mathbb{D}$ be another family of subsets of E such that $E \in \mathbb{D}$, if $A, B \in \mathbb{D}$ with $A \subseteq B$ then $B \setminus A \in \mathbb{D}$ and if (A_n) is an increasing sequence in $\mathbb{D}$, then $\cup_{n=1}^\infty A_n \in \mathbb{D}$. Such a family is called a Dynkin class. If $\mathbb{E} \subseteq \mathbb{D}$, then $\sigma(\mathbb{E}) \subseteq \mathbb{D}$.*

Proof. See Theorem 2.1.3 of Karatzas & Shreve (1991). □

A.2 Convergence results and uniform integrability

In this section, we recall some basic results on convergence of random variables. Let $(\Omega, \mathcal{F}, P)$ be a probability triple. Let X_n be a sequence of random variables and let X be another random variable. By $\mathcal{L}^p$, $p \geq 1$, we denote the variables X where $E|X|^p$ is finite. If $X_n(\omega)$ converges to $X(\omega)$ for all ω except on a null set, we say that X_n converges almost surely to X and write $X_n \xrightarrow{\text{a.s.}} X$. If it holds for all $\varepsilon > 0$ that $\lim_n P(|X_n - X| > \varepsilon) = 0$, we say that X_n converges in probability to X under P and write $X_n \xrightarrow{P} X$. If $\lim_n E|X_n - X|^p = 0$, we say that X_n converges to X in $\mathcal{L}^p$ and write $X_n \xrightarrow{\mathcal{L}^p} X$. Convergence in $\mathcal{L}^p$ and almost sure convergence both imply convergence in probability. Convergence in probability implies convergence almost surely along a subsequence.

The following lemmas will be useful at various points in the main text.

Lemma A.2.1. *Let X_n be a sequence of random variables, let X be another random variable and let $(F_n) \subseteq \mathcal{F}$. Assume that $X_n 1_{F_k} \xrightarrow{P} X 1_{F_k}$ for all $k \geq 1$ and that $\lim_k P(F_k^c) = 0$. Then $X_n \xrightarrow{P} X$ as well.*

Proof. For any $\varepsilon > 0$, we find

$$\begin{aligned} P(|X_n - X| > \varepsilon) &= P((|X_n - X| > \varepsilon) \cap F_k) + P((|X_n - X| > \varepsilon) \cap F_k^c) \\ &\leq P(|X_n 1_{F_k} - X 1_{F_k}| > \varepsilon) + P(F_k^c), \end{aligned}$$

and may therefore conclude $\limsup_n P(|X_n - X| > \varepsilon) \leq P(F_k^c)$. Letting k tend to infinity, we obtain $X_n \xrightarrow{P} X$. □

Lemma A.2.2. *Let (X_n) and (Y_n) be two sequences of variables convergent in probability to X and Y, respectively. If $X_n \leq Y_n$ almost surely for all n, then $X \leq Y$ almost surely.*

Proof. Picking nested subsequences, we find that for some subsequence, X_{n_k} tends almost surely to X and Y_{n_k} tends almost surely to Y. From the properties of ordinary convergence, we obtain $X \leq Y$ almost surely. □

Next, we consider the concept of uniform integrability, its basic properties and its relation to convergence of random variables. Let $(X_i)_{i \in I}$ be a family of random variables. We say that X_i is uniformly integrable if it holds that

$$\lim_{\lambda \to \infty} \sup_{i \in I} E|X_i| 1_{(|X_i| > \lambda)} = 0.$$

Note that as $\sup_{i\in I} E|X_i|1_{(|X_i|>\lambda)}$ is decreasing in λ, the limit always exists in $[0,\infty]$. We will review some basic results about uniform integrability. We refer the results mainly for discrete sequences of variables, but many results extend to sequences indexed by $\mathbb{R}_+$ as well.

Lemma A.2.3. *Let $(X_i)_{i\in I}$ be some family of variables. (X_i) is uniformly integrable if and only if it holds that (X_i) is bounded in $\mathcal{L}^1$, and for every $\varepsilon > 0$, it holds that there is $\delta > 0$ such that whenever $F \in \mathcal{F}$ with $P(F) \leq \delta$, we have $E1_F|X_i| \leq \varepsilon$ for all $i \in I$.*

Proof. First assume that $(X_i)_{i\in I}$ is uniformly integrable. Clearly, we then have

$$\begin{aligned}\sup_{i\in I} E|X_i| &\leq \sup_{i\in I} E|X_i|1_{(|X_i|>\lambda)} + \sup_{i\in I} E|X_i|1_{(|X_i|\leq\lambda)} \\ &\leq \lambda + \sup_{i\in I} E|X_i|1_{(|X_i|>\lambda)},\end{aligned}$$

and as the latter term converges to zero, it is in particular finite from a point onwards, and so $\sup_{i\in I} E|X_i|$ is finite, proving that (X_i) is bounded in $\mathcal{L}^1$. Now fix $\varepsilon > 0$. For any $\lambda > 0$, we have $E1_F|X_i| = E1_F|X_i|1_{(|X_i|>\lambda)} + E1_F|X_i|1_{(|X_i|\leq\lambda)} \leq \sup_{i\in I} E|X_i|1_{(|X_i|>\lambda)} + \lambda P(F)$. Therefore, picking λ so large that $\sup_{i\in I} E|X_i|1_{(|X_i|>\lambda)} \leq \frac{\varepsilon}{2}$ and putting $\delta = \frac{\varepsilon}{2\lambda}$, we obtain $E1_F|X_i| \leq \varepsilon$ for all $i \in I$, as desired.

In order to obtain the converse, assume that $(X_i)_{i\in I}$ is bounded in $\mathcal{L}^1$ and that for all $\varepsilon > 0$, there is $\delta > 0$ such that whenever $F \in \mathcal{F}$ with $P(F) \leq \delta$, we have $E1_F|X_i| \leq \varepsilon$ for all $i \in I$. We need to prove that $(X_i)_{i\in I}$ is uniformly integrable. Fix $\varepsilon > 0$, we wish to prove that there is $\lambda > 0$ such that $\sup_{i\in I} E|X_i|1_{(|X_i|>\lambda)} \leq \varepsilon$. To this end, let $\delta > 0$ be such that whenever $P(F) \leq \delta$, we have $E1_F|X_i| \leq \varepsilon$ for all $i \in I$. Note that by Markov's inequality, $P(|X_i| > \lambda) \leq \frac{1}{\lambda}E|X_i| \leq \frac{1}{\lambda}\sup_{i\in I} E|X_i|$, which is finite as $(X_i)_{i\in I}$ is bounded in $\mathcal{L}^1$. Therefore, there is $\lambda > 0$ such that $P(|X_i| > \lambda) \leq \delta$ for all i. For this λ, we then have $E|X_i|1_{(|X_i|>\lambda)} \leq \varepsilon$ for all $i \in I$, in particular $\sup_{i\in I} E|X_i|1_{(|X_i|>\lambda)} \leq \varepsilon$ for this λ and all larger λ as well, proving $\lim_{\lambda\to\infty} \sup_{i\in I} E|X_i|1_{(|X_i|>\lambda)} = 0$ and thus proving uniform integrability. □

Lemma A.2.4. *The property of being uniformly integrable satisfies the following properties.*

1. *If $(X_i)_{i\in I}$ is a finite family of integrable variables, then (X_i) is uniformly integrable.*
2. *If $(X_i)_{i\in I}$ and $(Y_j)_{j\in J}$ are uniformly integrable, then the union is uniformly integrable.*
3. *If $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ are uniformly integrable, so is $(\alpha X_i + \beta Y_i)_{i\in I}$ for $\alpha, \beta \in \mathbb{R}$.*
4. *If $(X_i)_{i\in I}$ is uniformly integrable and $J \subseteq I$, then $(X_j)_{j\in J}$ is uniformly integrable.*

5. *If $(X_i)_{i\in I}$ is bounded in $\mathcal{L}^p$ for some $p>1$, $(X_i)_{i\in I}$ is uniformly integrable.*
6. *If $(X_i)_{i\in I}$ is uniformly integrable and $|Y_i|\le|X_i|$, then $(Y_i)_{i\in I}$ is uniformly integrable.*

Proof. **Proof of (1).** Assume that $(X_i)_{i\in I}$ is a finite family of integrable variables. The dominated convergence theorem then yields

$$\lim_{\lambda\to\infty}\sup_{i\in I}E|X_i|1_{(|X_i|>\lambda)}\le\lim_{\lambda\to\infty}\sum_{i\in I}E|X_i|1_{(|X_i|>\lambda)}=\sum_{i\in I}E\lim_{\lambda\to\infty}|X_i|1_{(|X_i|>\lambda)},$$

which is zero. Therefore, $(X_i)_{\in I}$ is uniformly integrable.

Proof of (2). As the maximum function is continuous, we find

$$\begin{aligned}&\lim_{\lambda\to\infty}\max\{\sup_{i\in I}E|X_i|1_{(|X_i|>\lambda)},\sup_{j\in J}E|Y_j|1_{(|Y_j|>\lambda)}\}\\=\ &\max\{\lim_{\lambda\to\infty}\sup_{i\in I}E|X_i|1_{(|X_i|>\lambda)},\lim_{\lambda\to\infty}\sup_{j\in J}E|Y_j|1_{(|Y_j|>\lambda)}\},\end{aligned}$$

which is zero when the two families $(X_i)_{i\in I}$ and $(Y_j)_{j\in J}$ are uniformly integrable, and the result follows.

Proof of (3). Assume that $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ are uniformly integrable. If α and β are both zero, the result is trivial, so we assume that this is not the case. Let $\varepsilon>0$. Using Lemma A.2.3, pick $\delta>0$ such that whenever $P(F)\le\delta$, we have the inequalities $E1_F|X_i|\le\varepsilon(|\alpha|+|\beta|)^{-1}$ and $E1_F|Y_i|\le\varepsilon(|\alpha|+|\beta|)^{-1}$ for any $i\in I$. Then

$$\begin{aligned}E1_F|\alpha X_i+\beta Y_i|&\le|\alpha|E1_F|X_i|+|\beta|E1_F|Y_i|\\&\le|\alpha|\varepsilon(|\alpha|+|\beta|)^{-1}+|\beta|\varepsilon(|\alpha|+|\beta|)^{-1}\le\varepsilon,\end{aligned}$$

so that by Lemma A.2.3, the result holds.

Proof of (4). As $J\subseteq I$, we have $\sup_{j\in J}E|X_j|1_{(|X_j|>\lambda)}\le\sup_{i\in I}E|X_i|1_{(|X_i|>\lambda)}$, and the result follows.

Proof of (5). Assume that $(X_i)_{i\in I}$ is bounded in $\mathcal{L}^p$ for some $p>1$. We have

$$\lim_{\lambda\to\infty}\sup_{i\in I}E|X_i|1_{(|X_i|>\lambda)}\le\lim_{\lambda\to\infty}\lambda^{1-p}\sup_{i\in I}E|X_i|^p1_{(|X_i|>\lambda)}\le\sup_{i\in I}E|X_i|^p\lim_{\lambda\to\infty}\lambda^{1-p},$$

which is zero, as $p-1>0$, so $(X_i)_{i\in I}$ is uniformly integrable.

Proof of (6). In the case where $(X_i)_{\in I}$ is uniformly integrable and $(Y_i)_{i\in I}$ is such that $|Y_i|\le|X_i|$, we get $E|Y_i|1_{(|Y_i|>\lambda)}\le E|X_i|1_{(|X_i|>\lambda)}$ for all i, and it follows immediately that $(Y_i)_{i\in I}$ is uniformly integrable. □

Lemma A.2.5. *Let* (X_n) *be a sequence of random variables indexed by* $\mathbb{N}$*, and let* X *be another variable.* X_n *converges in* $\mathcal{L}^1$ *to* X *if and only if* (X_n) *is uniformly integrable and converges in probability to* X*. If* (X_t) *is a sequence of random variables indexed by* $\mathbb{R}_+$*,* X_t *converges to* X *in* $\mathcal{L}^1$ *if* (X_t) *is uniformly integrable and converges in probability to* X*.*

Proof. Consider first the discrete-time case. Assume that X_n converges to X in $\mathcal{L}^1$, we need to prove that (X_n) is uniformly integrable. We use the criterion from Lemma A.2.3. As (X_n) is convergent in $\mathcal{L}^1$, (X_n) is bounded in $\mathcal{L}^1$, and X_n converges to X in probability. Fix $\varepsilon > 0$ and let m be such that whenever $n \geq m$, $E|X_n - X| \leq \frac{\varepsilon}{3}$. As the finite-variable family $\{X_1, \ldots, X_m, X\}$ is uniformly integrable by Lemma A.2.4, using Lemma A.2.3 we may obtain $\delta > 0$ such that whenever $P(F) \leq \delta$, $E1_F|X| \leq \frac{\varepsilon}{3}$ and $E1_F|X_n| \leq \frac{\varepsilon}{3}$ for $n \leq m$. We then obtain that for all such $F \in \mathcal{F}$,

$$\begin{aligned}
\sup_n E1_F|X_n| &\leq \sup_{n\leq m} E1_F|X_n| + \sup_{n\geq m} E1_F|X_n| \\
&\leq \frac{\varepsilon}{3} + E1_F|X| + \sup_{n\geq m} E1_F|X_n - X| \\
&\leq \frac{2\varepsilon}{3} + \sup_{n\geq m} E|X_n - X| \leq \varepsilon,
\end{aligned}$$

so (X_n) is uniformly integrable.

Consider the converse statement, where we assume that (X_n) is uniformly integrable and converges to X in probability. As (X_n) is uniformly integrable, (X_n) is bounded in $\mathcal{L}^1$. Using that there is a subsequence (X_{n_k}) converging to X almost surely, we obtain by Fatou's lemma that $E|X| = E\lim_k |X_{n_k}| \leq \liminf_k E|X_{n_k}| \leq \sup_n E|X_n|$, so X is integrable. By Lemma A.2.4, $(X_n - X)$ is uniformly integrable. Let $\varepsilon > 0$. Using Lemma A.2.3, we pick $\delta > 0$ such that whenever $P(F) \leq \delta$, we have $E1_F|X_n - X| \leq \varepsilon$. As $X_n \xrightarrow{P} X$, there is m such that whenever $n \geq m$, we have $P(|X_n - X| \leq \varepsilon) \leq \delta$. For such n, we then find $E|X_n - X| = E1_{(|X_n - X| \leq \varepsilon)}|X_n - X| + E1_{(|X_n - X| > \varepsilon)}|X_n - X| \leq 2\varepsilon$, proving that X_n tends to X in $\mathcal{L}^1$.

As for the case of a family $(X_t)_{t\geq 0}$ indexed by $\mathbb{R}_+$, we see that the proof that X_t is convergent in $\mathcal{L}^1$ to X if X_t is convergent in probability to X and is uniformly integrable may be copied more or less verbatim from the discrete-time case. □

Note that in Lemma A.2.5, one cannot obtain a double implication in the statements regarding sequences indexed by $\mathbb{R}_+$. As a counterexample, simply put $X_t = \sum_{n=1}^\infty n 1_{(t=1-\frac{1}{n})}$. Then X_t converges to zero in $\mathcal{L}^1$, but $(X_t)_{t\geq 0}$ is not uniformly integrable as it is not even bounded in $\mathcal{L}^1$.

Lemma A.2.6. *Let X be any integrable random variable on probability space $(\Omega, \mathcal{F}, P)$. Let I be the set of all sub-σ-algebras of $\mathcal{F}$. Then, $(E(X|\mathcal{G}))_{\mathcal{G}\in I}$ is uniformly integrable.*

Proof. Using Jensen's inequality and the fact that $(E(|X||\mathcal{G}) > \lambda) \in \mathcal{G}$, we have

$$\sup_{\mathcal{G}\in I} E|E(X|\mathcal{G})|1_{(|E(X|\mathcal{G})|>\lambda)} \leq \sup_{\mathcal{G}\in I} EE(|X||\mathcal{G})1_{(E(|X||\mathcal{G})>\lambda)} = \sup_{\mathcal{G}\in I} E|X|1_{(E(|X||\mathcal{G})>\lambda)}.$$

Fix $\varepsilon > 0$, we show that for λ large enough, the above is smaller than ε. To this end, note that for any sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$, we have $P(E(|X||\mathcal{G}) > \lambda) \leq \frac{1}{\lambda}EE(|X||\mathcal{G}) = \frac{1}{\lambda}E|X|$ by Markov's inequality. Applying Lemma A.2.3 with the family $\{X\}$, we know that there is $\delta > 0$ such that whenever $P(F) \leq \delta$, $E1_F|X| \leq \varepsilon$. Therefore, picking λ so large that $\frac{1}{\lambda}E|X| \leq \delta$, we obtain $P(E(|X||\mathcal{G}) > \lambda) \leq \delta$ and so $\sup_{\mathcal{G}\in I} E|X|1_{(E(|X||\mathcal{G})>\lambda)} \leq \varepsilon$. This concludes the proof. □

Lemma A.2.7 (Mazur's lemma)**.** *Let (X_n) be sequence of variables bounded in $\mathcal{L}^2$. There exists a sequence (Y_n) such that each Y_n is a convex combination of a finite set of elements in $\{X_n, X_{n+1}, \ldots\}$ and (Y_n) is convergent in $\mathcal{L}^2$.*

Proof. Let α_n be the infimum of EZ^2, where Z ranges through all finite convex combinations of elements in $\{X_n, X_{n+1}, \ldots\}$, and define $\alpha = \sup_n \alpha_n$. If $Z = \sum_{k=n}^{K_n} \lambda_k X_k$ for some convex weights $\lambda_n, \ldots, \lambda_{K_n}$, we obtain $\sqrt{EZ^2} \leq \sum_{k=n}^{K_n} \lambda_k \sqrt{EX_k^2} \leq \sup_n \sqrt{EX_n^2}$, in particular we have $\alpha_n \leq \sup_n EX_n^2$ and so $\alpha \leq \sup_n EX_n^2$ as well, proving that α is finite. For each n, there is a variable Y_n which is a finite convex combination of elements in $\{X_n, X_{n+1}, \ldots\}$ such that $E(Y_n)^2 \leq \alpha_n + \frac{1}{n}$. Let n be so large that $\alpha_n \geq \alpha - \frac{1}{n}$, and let $m \geq n$, we then obtain

$$\begin{aligned}
E(Y_n - Y_m)^2 &= 2EY_n^2 + 2EY_m^2 - E(Y_n + Y_m)^2 \\
&= 2EY_n^2 + 2EY_m^2 - 4E(\tfrac{1}{2}(Y_n + Y_m))^2 \\
&\leq 2(\alpha_n + \tfrac{1}{n}) + 2(\alpha_m + \tfrac{1}{m}) - 4\alpha_n \\
&= 2(\tfrac{1}{n} + \tfrac{1}{m}) + 2(\alpha_m - \alpha_n).
\end{aligned}$$

As (α_n) is convergent, it is Cauchy. Therefore, the above shows that (Y_n) is Cauchy in $\mathcal{L}^2$, therefore convergent, proving the lemma. □

A.3 Discrete-time martingales

In this section, we review the basic results from discrete-time martingale theory. Assume given a probability field $(\Omega, \mathcal{F}, P)$. If $(\mathcal{F}_n)$ is a sequence of σ-algebras indexed by $\mathbb{N}$ which are increasing in the sense that $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$, we say that $(\mathcal{F}_n)$ is a filtration. We then refer to $(\Omega, \mathcal{F}, (\mathcal{F}_n), P)$ as a filtered probability space. In the remainder of this section, we will assume given a filtered probability space of this kind.

A discrete-time stochastic process is a sequence $X = (X_n)$ of random variables defined on $(\Omega, \mathcal{F})$. If X_n is $\mathcal{F}_n$ measurable, we say that the process X is adapted. If X is adapted and $E(X_n|\mathcal{F}_k) = X_k$ whenever $n \geq k$, we say that X is a martingale. If instead $E(X_n|\mathcal{F}_k) \leq X_k$, we say that X is a supermartingale and if $E(X_n|\mathcal{F}_k) \geq X_k$, we say that X is a submartingale. Any martingale is also a submartingale and a supermartingale. Furthermore, if X is a supermartingale, then $-X$ is a submartingale and vice versa.

A stopping time is a random variable $T : \Omega \to \mathbb{N} \cup \{\infty\}$ such that $(T \leq n) \in \mathcal{F}_n$ for any $n \in \mathbb{N}$. We say that T is finite if T maps into $\mathbb{N}$. We say that T is bounded if T maps into a bounded subset of $\mathbb{N}$. If X is a stochastic process and T is a stopping time, we denote by X^T the process $X^T_n = X_{T \wedge n}$ and call X^T the process stopped at T. Furthermore, we define the stopping time σ-algebra $\mathcal{F}_T$ by putting $\mathcal{F}_T = \{A \in \mathcal{F} | A \cap (T \leq n) \in \mathcal{F}_n \text{ for all } n \in \mathbb{N}_0\}$. $\mathcal{F}_T$ is a σ-algebra, and if T is constant, the stopping time σ-algebra is the same as the filtration σ-algebra.

Lemma A.3.1 (Doob's upcrossing lemma). *Let Z be a supermartingale which is bounded in $\mathcal{L}^1$. Define $U(Z, a, b) = \sup\{n \mid \exists\, 1 \leq s_1 < t_1 < \cdots s_n < t_n : Z_{s_k} < a, Z_{t_k} > b, k \leq n\}$ for any $a, b \in \mathbb{R}$ with $a < b$. We refer to $U(Z, a, b)$ as the number of upcrossings from a to b by Z. Then*

$$EU(Z, a, b) \leq \frac{|a| + \sup_n E|Z_n|}{b - a}$$

Proof. See Corollary II.48.4 of Rogers & Williams (2000a). □

Theorem A.3.2 (Doob's supermartingale convergence theorem). *Let Z be a supermartingale. If Z is bounded in $\mathcal{L}^1$, Z is almost surely convergent. If Z is uniformly integrable, Z is also convergent in $\mathcal{L}^1$, and the limit Z_∞ satisfies that for all n, $E(Z_\infty|\mathcal{F}_n) \leq Z_n$ almost surely.*

Proof. That Z converges almost surely follows from Theorem II.49.1 of Rogers & Williams

(2000a). The results for the case where Z is uniformly integrable follows from Theorem II.50.1 of Rogers & Williams (2000a). $\square$

Theorem A.3.3 (Uniformly integrable martingale convergence theorem). *Let M be a discrete-time martingale. The following are equivalent:*

1. *M is uniformly integrable.*
2. *M is convergent almost surely and in $\mathcal{L}^1$.*
3. *There is some integrable variable ξ such that $M_n = E(\xi|\mathcal{F}_n)$ for $n \geq 1$.*

In the affirmative, with M_∞ denoting the limit of M_n almost surely and in $\mathcal{L}^1$, we have for all $n \geq 1$ that $M_n = E(M_\infty|\mathcal{F}_n)$ almost surely, and $M_\infty = E(\xi|\mathcal{F}_\infty)$, where $\mathcal{F}_\infty = \sigma(\cup_{n=1}^\infty \mathcal{F}_n)$.

Proof. From Theorem II.50.1 in Rogers & Williams (2000a), it follows that if (1) holds, then (2) and (3) holds as well. From Theorem II.50.3 of Rogers & Williams (2000a), we find that if (3) holds, then (1) and (2) holds. Finally, (2) implies (1) by Lemma A.2.5.

In the affirmative case, Theorem II.50.3 of Rogers & Williams (2000a) shows that we have $M_\infty = E(\xi|\mathcal{F}_\infty)$, and so in particular, $M_n = E(\xi|\mathcal{F}_n) = E(E(\xi|\mathcal{F}_\infty)|\mathcal{F}_n) = E(M_\infty|\mathcal{F}_n)$ almost surely. $\square$

Lemma A.3.4 (Doob's $\mathcal{L}^2$ inequality). *Let M be a martingale such that $\sup_{n\geq 1} EM_n^2$ is finite. Then M is convergent almost surely and in $\mathcal{L}^2$ to a square-integrable variable M_∞, and $EM_\infty^{*2} \leq 4EM_\infty^2$, where $M_\infty^* = \sup_{n\geq 0}|M_n|$ and $M_\infty^{*2} = (M_\infty^*)^2$.*

Proof. This is Theorem II.52.6 of Rogers & Williams (2000a). $\square$

Lemma A.3.5 (Optional sampling theorem). *Let Z be a discrete-time supermartingale, and let $S \leq T$ be two stopping times. If Z is uniformly integrable, then Z is almost surely convergent, Z_S and Z_T are integrable, and $E(Z_T|\mathcal{F}_S) \leq Z_S$.*

Proof. That Z is almost surely convergent follows from Theorem II.49.1 of Rogers & Williams (2000a). That Z_S and Z_T are integrable and that $E(Z_T|\mathcal{F}_S) \leq Z_S$ then follows from Theorem II.59.1 of Rogers & Williams (2000a). $\square$

Finally, we consider backwards martingales. Let $(\mathcal{F}_n)_{n\geq 1}$ be a decreasing sequence of σ-algebras and let (Z_n) be some process. If Z_n is $\mathcal{F}_n$ measurable and integrable, and it holds that $X_n = E(X_k|\mathcal{F}_n)$ for $n \geq k$, we say that (Z_n) is a backwards martingale. If instead $Z_n \leq E(Z_k|\mathcal{F}_n)$, we say that (Z_n) is a backwards supermartingale, and if $Z_n \geq E(Z_k|\mathcal{F}_n)$, we say that (Z_n) is a backwards submartingale.

Note that for both ordinary supermartingales and backwards supermartingales, the definition is essentially the same. Z is a supermartingale when, for $n \geq k$, $E(Z_n|\mathcal{F}_k) \leq Z_k$, while Z is a backwards supermartingale when, for $n \geq k$, $Z_n \leq E(Z_k|\mathcal{F}_n)$. Furthermore, clearly, if Z is a backwards supermartingale, then $-Z$ is a backwards submartingale and vice versa.

Theorem A.3.6 (Backwards supermartingale convergence theorem)**.** *Let $(\mathcal{F}_n)$ be a decreasing sequence of σ-algebras, and let (Z_n) be a backwards supermartingale. If $\sup_{n\geq 1} EZ_n$ is finite, then Z is uniformly integrable and convergent almost surely and in $\mathcal{L}^1$. Furthermore, the limit satisfies $Z_\infty \geq E(Z_n|\mathcal{F}_\infty)$, where $\mathcal{F}_\infty$ is the σ-algebra $\cap_{n=1}^\infty \mathcal{F}_n$.*

Proof. See Theorem II.51.1 of Rogers & Williams (2000a). □

A.4 Brownian motion and the usual conditions

Let $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ be a filtered probability space. Recall that the usual conditions for a filtered probability space are the conditions that the filtration is right-continuous in the sense that $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$, and that for $t \geq 0$, $\mathcal{F}_t$ contains all null sets in $\mathcal{F}$. Our development of the stochastic integral is made under the assumption that the usual conditions hold. This assumption is made for convenience. However, making this assumption, we need to make sure that these assumptions hold in the situations where we would like to apply the stochastic integral, in particular in the case of Brownian motion. This is the subject matter of this section. We are going to show that any filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ has a minimal extension satisfying the usual conditions, and we are going to show that any probability field endowed with a Brownian motion satisfying certain regularity properties with respect to the filtration can be extended to a probability field satisfying the usual conditions with a Brownian motion satisfying the same regularity properties in relation to the filtration.

Our first aim is to understand how to augment a filtered probability space so that it fulfills the usual conditions. In the following, $\mathcal{N}$ denotes the null sets of $\mathcal{F}$. Before we can construct the desired augmentation of a filtered probability space, we need a few lemmas.

Lemma A.4.1. *Letting* $\mathcal{G} = \sigma(\mathcal{F}, \mathcal{N})$*, it holds that* $\mathcal{G} = \{F \cup N \mid F \in \mathcal{F}, N \in \mathcal{N}\}$*, and* P *can be uniquely extended from* $\mathcal{F}$ *to a probability measure* P' *on* $\mathcal{G}$ *by defining* P' *as* $P'(F \cup N) = P(F)$*. The space* $(\Omega, \mathcal{G}, P')$ *is called the completion of* $(\Omega, \mathcal{F}, P)$*.*

Proof. We first prove the equality for $\mathcal{G}$. Define $\mathcal{H} = \{F \cup N \mid F \in \mathcal{F}, N \in \mathcal{N}\}$. It is clear that $\mathcal{H} \subseteq \mathcal{G}$. We need to prove the opposite inclusion. To do so, we prove directly that $\mathcal{H}$ is a σ-algebra containing $\mathcal{F}$ and $\mathcal{N}$, as the inclusion follows from this. It is clear that $\Omega \in \mathcal{H}$. If $H \in \mathcal{H}$ with $H = F \cup N$, we obtain, with $B \in \mathcal{F}$ such that $P(B) = 0$ and $N \subseteq B$,

$$\begin{aligned} H^c &= (F \cup N)^c = (B^c \cap (F \cup N)^c) \cup (B \cap (F \cup N)^c) \\ &= (B \cup F \cup N)^c \cup (B \cap (F \cup N)^c) = (B \cup F)^c \cup (B \cap (F \cup N)^c), \end{aligned}$$

so since $(B \cup F)^c \in \mathcal{F}$ and $B \cap (F \cup N)^c \in \mathcal{N}$, we find $H^c \in \mathcal{H}$. If $(H_n) \subseteq \mathcal{H}$ with $H_n = F_n \cup N_n$, we find $\cup_{n=1}^\infty H_n = (\cup_{n=1}^\infty F_n) \cup (\cup_{n=1}^\infty N_n)$, showing $\cup_{n=1}^\infty H_n \in \mathcal{H}$. We have now proven that $\mathcal{H}$ is a σ-algebra. Since it contains $\mathcal{F}$ and $\mathcal{N}$, we conclude $\mathcal{G} \subseteq \mathcal{H}$. It remains to show that P can be uniquely extended from $\mathcal{F}$ to $\mathcal{G}$. We begin by proving that the proposed extension is well-defined. Let $G \in \mathcal{G}$ with two decompositions $G = F_1 \cup N_1$ and $G = F_2 \cup N_2$. We then have $G^c = F_2^c \cap N_2^c$. Thus, if $\omega \in F_2^c$ and $\omega \in G$, then $\omega \in N_2$, showing $F_2^c \subseteq G^c \cup N_2$ and therefore $F_1 \cap F_2^c \subseteq F_1 \cap (G^c \cup N_2) = F_1 \cap N_2 \subseteq N_2$, and analogously $F_2 \cap F_1^c \subseteq N_1$. We can therefore conclude $P(F_1 \cap F_2^c) = P(F_2 \cap F_1^c) = 0$ and as a consequence,

$$P(F_1) = P(F_1 \cap F_2) + P(F_1 \cap F_2^c) = P(F_2 \cap F_1) + P(F_2 \cap F_1^c) = P(F_2).$$

This means that putting $P'(G) = P(F)$ for $G = F \cup N$ is a definition independent of the representation of G, therefore well-defined. It remains to prove that P' is in fact a probability measure. As P' is nonnegative and $P'(\Omega) = P(\Omega) = 1$, it suffices to show P' is σ-additive. To this end, let (H_n) be a sequence of disjoint sets in $\mathcal{G}$, and assume that $H_n = F_n \cup N_n$. Then $F_n \subseteq H_n$ for all n, so the sequence (F_n) consists of disjoint sets as well. Therefore, we obtain $P'(\cup_{n=1}^\infty H_n) = P(\cup_{n=1}^\infty F_n) = \sum_{n=1}^\infty P(F_n) = \sum_{n=1}^\infty P'(H_n)$, as desired. This completes the proof. $\square$

In the following, we let $(\Omega, \mathcal{G}, P)$ be the completion of $(\Omega, \mathcal{F}, P)$. Note in particular that we use the same symbol for the measure on $\mathcal{F}$ and its extension to the completion $\mathcal{G}$.

Lemma A.4.2. *Let* $\mathcal{H}$ *be a sub-*σ*-algebra of* $\mathcal{G}$*. Then*

$$\sigma(\mathcal{H}, \mathcal{N}) = \{G \in \mathcal{G} \mid G \Delta H \in \mathcal{N} \text{ for some } H \in \mathcal{H}\},$$

where Δ *is the symmetric difference,* $G \Delta H = (G \setminus H) \cup (H \setminus G)$*.*

Proof. Define $\mathcal{K} = \{G \in \mathcal{G} \mid G\Delta H \in \mathcal{N}$ for some $H \in \mathcal{H}\}$. We begin by arguing that the inclusion $\mathcal{K} \subseteq \sigma(\mathcal{H},\mathcal{N})$ holds. Let $G \in \mathcal{K}$ and let $H \in \mathcal{H}$ be such that $G\Delta H \in \mathcal{N}$. We then find $G = (H \cap G) \cup (G \setminus H) = (H \cap (H \cap G^c)^c) \cup (G \setminus H) = (H \cap (H \setminus G)^c) \cup (G \setminus H)$. Since $P(G\Delta H) = 0$, $H \setminus G$ and $G \setminus H$ are both null sets, and we conclude that $G \in \sigma(\mathcal{H},\mathcal{N})$. Thus $\mathcal{K} \subseteq \sigma(\mathcal{H},\mathcal{N})$. To show the other inclusion, we will show that $\mathcal{K}$ is a σ-algebra containing $\mathcal{H}$ and $\mathcal{N}$. If $G \in \mathcal{H}$, we have $G\Delta G = \emptyset \in \mathcal{N}$ and therefore $G \in \mathcal{K}$. If $N \in \mathcal{N}$, we have $N\Delta\emptyset = N \in \mathcal{N}$, so $N \in \mathcal{K}$. We have now shown $\mathcal{H},\mathcal{N} \subseteq \mathcal{K}$.

It remains to show that $\mathcal{K}$ is a σ-algebra. Clearly, $\Omega \in \mathcal{K}$. Assume $G \in \mathcal{K}$ and $H \in \mathcal{H}$ with $G\Delta H \in \mathcal{N}$. We then obtain that $G^c\Delta H^c = (G^c \setminus H^c) \cup (H^c \setminus G^c) = (H \setminus G) \cup (G \setminus H) \in \mathcal{N}$, so $G^c \in \mathcal{K}$ as well. Now assume that $(G_n) \subseteq \mathcal{K}$, and let $H_n \in \mathcal{H}$ be such that $G_n\Delta H_n \in \mathcal{N}$. Then

$$\begin{aligned}(\cup_{n=1}^\infty G_n)\Delta(\cup_{n=1}^\infty H_n) &= (\cup_{n=1}^\infty G_n \setminus \cup_{n=1}^\infty H_n) \cup (\cup_{n=1}^\infty H_n \setminus \cup_{n=1}^\infty G_n)\\ &\subseteq (\cup_{n=1}^\infty G_n \setminus H_n) \cup (\cup_{n=1}^\infty H_n \setminus G_n)\\ &= \cup_{n=1}^\infty G_n\Delta H_n,\end{aligned}$$

so $\cup_{n=1}^\infty G_n \in \mathcal{K}$. We can now conclude that $\mathcal{K}$ is a σ-algebra. Since it contains $\mathcal{H}$ and $\mathcal{N}$, $\sigma(\mathcal{H},\mathcal{N}) \subseteq \mathcal{K}$. □

Theorem A.4.3. *With $\mathcal{F}_{t+} = \cap_{s>t}\mathcal{F}_s$, we have $\cap_{s>t}\sigma(\mathcal{F}_s,\mathcal{N}) = \sigma(\mathcal{F}_{t+},\mathcal{N})$, and the filtration $\mathcal{G}_t = \sigma(\mathcal{F}_{t+},\mathcal{N})$ is the smallest filtration such that $\mathcal{F}_t \subseteq \mathcal{G}_t$ and such that $(\Omega,\mathcal{G},(\mathcal{G}_t),P)$ satisfies the usual conditions, where $\mathcal{G} = \sigma(\cup_{t\geq 0}\mathcal{G}_t)$. We call $(\mathcal{G}_t)$ the usual augmentation of $(\mathcal{F}_t)$. We call $(\Omega,\mathcal{G},P,\mathcal{G}_t)$ the usual augmentation of the filtered probability space $(\Omega,\mathcal{F},P,\mathcal{F}_t)$.*

Proof. We need to prove three things. First, we need to prove the equality stated in the lemma. Second, we need to prove that $(\Omega,\mathcal{F},(\mathcal{G}_t),P)$ satisfies the usual conditions. And third, we need to prove that $\mathcal{G}_t$ is the smallest filtration containing $\mathcal{F}_t$ with this property.

We first prove $\cap_{s>t}\sigma(\mathcal{F}_s,\mathcal{N}) = \sigma(\mathcal{F}_{t+},\mathcal{N})$. Since $\mathcal{F}_{t+} \subseteq \mathcal{F}_s$ for any $s > t$, it is clear that $\sigma(\mathcal{F}_{t+},\mathcal{N}) \subseteq \cap_{s>t}\sigma(\mathcal{F}_s,\mathcal{N})$. Now consider $F \in \cap_{s>t}\sigma(\mathcal{F}_s,\mathcal{N})$. By Lemma A.4.2, for any $s > t$ there is $F_s \in \mathcal{F}_s$ such that $F\Delta F_s \in \mathcal{N}$. Put $G_n = \cup_{k\geq n}F_{t+\frac{1}{k}}$. Much like in the proof of Lemma A.4.2, we obtain $F\Delta G_n \subseteq \cup_{k=n}^\infty F\Delta F_{t+\frac{1}{k}} \in \mathcal{N}$. Put $G = \cap_{n=1}^\infty G_n$. Since G_n is decreasing and $G_n \in \mathcal{F}_{t+\frac{1}{n}}$, $G \in \mathcal{F}_{t+}$. We find

$$\begin{aligned}F\Delta G &= (F \setminus \cap_{n=1}^\infty G_n) \cup (\cap_{n=1}^\infty G_n \setminus F) = (\cup_{n=1}^\infty F \setminus G_n) \cup (\cap_{n=1}^\infty (G_n \setminus F))\\ &\subseteq (\cup_{n=1}^\infty F \setminus G_n) \cup (\cup_{n=1}^\infty (G_n \setminus F)) = \cup_{n=1}^\infty F\Delta G_n \in \mathcal{N},\end{aligned}$$

showing $F \in \sigma(\mathcal{F}_{t+}, \mathcal{N})$ and thereby the inclusion $\cap_{s>t}\sigma(\mathcal{F}_s, \mathcal{N}) \subseteq \sigma(\mathcal{F}_{t+}, \mathcal{N})$.

Next, we argue that $(\Omega, \mathcal{F}, (\mathcal{G}_t), P)$ satisfies the usual conditions. We find that $\mathcal{G}_t$ contains all null sets for all $t \geq 0$. To prove right-continuity of the filtration, we merely note

$$\cap_{s>t}\mathcal{G}_s = \cap_{s>t} \cap_{u>s} \sigma(\mathcal{F}_u, \mathcal{N}) = \cap_{u>t}\sigma(\mathcal{F}_u, \mathcal{N}) = \mathcal{G}_t.$$

Finally, we prove that $\mathcal{G}_t$ is the smallest filtration satisfying the usual conditions such that $\mathcal{F}_t \subseteq \mathcal{G}_t$. To do so, assume that $\mathcal{H}_t$ is another filtration satisfying the usual conditions with $\mathcal{F}_t \subseteq \mathcal{H}_t$. We need to prove $\mathcal{G}_t \subseteq \mathcal{H}_t$. To do so, merely note that since $\mathcal{H}_t$ satisfies the usual conditions, $\mathcal{F}_{t+} \subseteq \mathcal{H}_{t+} = \mathcal{H}_t$, and $\mathcal{N} \subseteq \mathcal{H}_t$. Thus, $\mathcal{G}_t = \sigma(\mathcal{F}_{t+}, \mathcal{N}) \subseteq \mathcal{H}_t$. □

Theorem A.4.3 shows that for any filtered probability space, there exists a minimal augmentation satisfying the usual conditions, and the theorem also shows how to construct this augmentation. Next, we consider a p-dimensional Brownian motion W on our basic, still uncompleted, probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$, understood as a continuous process W with initial value zero such that the increments are independent and normally distributed, with $W_t - W_s$ having the normal distribution with mean zero and variance $(t-s)I_p$, where I_p denotes the identity matrix of order p. We will define a criterion to ensure that the Brownian motion interacts properly with the filtration, and we will show that when augmenting the filtration induced by the Brownian motion, we still obtain a proper interaction between the Brownian motion and the filtration.

Definition A.4.4. *A p-dimensional $\mathcal{F}_t$ Brownian motion is a continuous process W adapted to $\mathcal{F}_t$ such that for any t, the distribution of $s \mapsto W_{t+s} - W_t$ is a p-dimensional Brownian motion independent of $\mathcal{F}_t$.*

The independence in Definition A.4.4 means the following. Fix $t \geq 0$ and let X be the process defined by $X_s = W_{t+s} - W_t$. X is then a random variable with values in $C(\mathbb{R}_+, \mathbb{R}^p)$, the space of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^p$, endowed with the σ-algebra $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^p)$ induced by the coordinate mappings. The independence of X and $\mathcal{F}_t$ means that for any $A \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^p)$ and any $B \in \mathcal{F}_t$, $P((X \in A) \cap B) = P(X \in A)P(B)$.

In what follows, we will assume that the filtration $\mathcal{F}_t$ is the one induced by the Brownian motion W. W is then trivially an $\mathcal{F}_t$ Brownian motion. Letting $(\Omega, \mathcal{G}, P, \mathcal{G}_t)$ be the usual augmentation of $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ as given in Theorem A.4.3, we want to show that W is a $\mathcal{G}_t$ Brownian motion.

Lemma A.4.5. *W is also a Brownian motion with respect to the filtration $\mathcal{F}_{t+}$.*

Proof. It is clear by the continuity of W that W is adapted to $\mathcal{F}_{t+}$. We need to show that for any $t \geq 0$, $s \mapsto W_{t+s} - W_t$ is independent of $\mathcal{F}_{t+}$. Let $t \geq 0$ be given and define $X_s = W_{t+s} - W_t$. We need to show that for $A \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^p)$ and $B \in \mathcal{F}_{t+}$, it holds that $P((X \in A) \cap B) = P(X \in A)P(B)$. To do so, it suffices by Lemma A.1.19 to prove that for any $0 \leq t_1 \leq \cdots \leq t_n$, we have that

$$P((X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n) \cap B) = P(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n)P(B),$$

where $A_1, \ldots, A_n$ are open sets in $\mathbb{R}^p$. To this end, let $0 \leq t_1 \leq \cdots \leq t_n$ and be given let $A_1, \ldots, A_n$ be open sets in $\mathbb{R}^p$, and let $B \in \mathcal{F}_{t+}$. Using continuity of W and the fact that $A_1, \ldots, A_n$ are open,

$$\begin{aligned}
P((X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n) \cap B) &= E1_B \prod_{k=1}^n 1_{A_k}(X_{t_k}) = E1_B \prod_{k=1}^n 1_{A_k}(W_{t+t_k} - W_t) \\
&= E1_B \prod_{k=1}^n \lim_{n\to\infty} 1_{A_k}(W_{t+t_k+\frac{1}{n}} - W_{t+\frac{1}{n}}) \\
&= \lim_{n\to\infty} E1_B \prod_{k=1}^n 1_{A_k}(W_{t+t_k+\frac{1}{n}} - W_{t+\frac{1}{n}}).
\end{aligned}$$

Next, note that for any $\varepsilon > 0$ and $s \geq 0$, $W_{t+s+\varepsilon} - W_{t+\varepsilon}$ is independent of $\mathcal{F}_{t+\varepsilon}$. Therefore, $W_{t+s+\varepsilon} - W_{t+\varepsilon}$ is in particular independent of $\mathcal{F}_{t+}$, and we obtain

$$\begin{aligned}
\lim_{n\to\infty} E1_B \prod_{k=1}^n 1_{A_k}(W_{t+t_k+\frac{1}{n}} - W_{t+\frac{1}{n}}) &= \lim_{n\to\infty} P(B) E \prod_{k=1}^n 1_{A_k}(W_{t+t_k+\frac{1}{n}} - W_{t+\frac{1}{n}}) \\
&= P(B) E \prod_{k=1}^n 1_{A_k}(W_{t+t_k} - W_t) \\
&= P(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n)P(B)
\end{aligned}$$

This proves the claim. □

Theorem A.4.6. *W is a $\mathcal{G}_t$ Brownian motion.*

Proof. It is clear that W is adapted to $\mathcal{G}_t$. We therefore merely need to show that for $t \geq 0$, $s \mapsto W_{t+s} - W_t$ is independent of $\mathcal{G}_t$. Let $t \geq 0$ and define X by putting $X_s = W_{t+s} - W_t$. With $\mathcal{N}$ the null sets of $\mathcal{F}$, we have by Theorem A.4.3 that $\mathcal{G}_t = \sigma(\mathcal{F}_{t+}, \mathcal{N})$. Thus, since both $\mathcal{F}_{t+}$ and the complements of $\mathcal{N}$ contain Ω, the sets of the form $C \cap D$, where $C \in \mathcal{F}_{t+}$ and $D^c \in \mathcal{N}$, form a generating system for $\mathcal{G}_t$, stable under intersections. It will suffice to show $P((X \in A) \cap (C \cap D)) = P(X \in A)P(C \cap D)$ for any $A \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$. Lemma A.4.5 yields $P((X \in A) \cap (C \cap D)) = P((X \in A) \cap C) = P(X \in A)P(C) = P(X \in A)P(C \cap D)$, and the desired conclusion follows. $\square$

A.5 Exercises

Exercise A.1. Let (X_n) be a sequence of integrable variables. Define the integrability index c of (X_n) by putting $c = \inf_{\varepsilon>0} \sup_{F\in\mathcal{F}:P(F)\leq\varepsilon} \sup_{n\geq 1} E1_F|X_n|$. Show that (X_n) is uniformly integrable if and only if (X_n) is bounded in $\mathcal{L}^1$ and $c = 0$. Show that if $X_n \xrightarrow{P} X$, where X is integrable, we have $\limsup_n E|X_n - X| \leq c$.

Exercise A.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with a sequence $(X_n)_{n\geq 1}$ of independent and identically distributed variable with mean ξ. Use the backwards Martingale Convergence Theorem and Kolmogorov's zero-one law to show that $\frac{1}{n}\sum_{k=1}^n X_k$ converges almost surely and in $\mathcal{L}^1$ to ξ.

Exercise A.3. Let $(X_i)_{i\in I}$ be a uniformly integrable family, and let $\mathbb{A}$ denote the closure of this family in $\mathcal{L}^1$, that is, the set of all integrable variables X which are limits in $\mathcal{L}^1$ of sequences (X_{i_n}) in $(X_i)_{i\in I}$. Then $\mathbb{A}$ is uniformly integrable as well.

Exercise A.4. Let (X_n) be a uniformly integrable sequence of variables. Prove that exists a sequence (Y_n) such that each Y_n is a convex combination of a finite set of elements in $\{X_n, X_{n+1}, \ldots\}$ and (Y_n) is convergent in $\mathcal{L}^1$.

Exercise A.5. Let $(\Omega, \mathcal{F}, (\mathcal{F}_n), P)$ be a filtered probability space, and let $(M_n)_{n\geq 0}$ be a martingale, zero at zero, with $\sup_{n\geq 0} EM_n^2$ finite. Define $[M]_n = \sum_{k=1}^n (M_k - M_{k-1})^2$ for $n \geq 1$ and $[M]_0 = 0$. Also define $\langle M\rangle_n = \sum_{k=1}^n E((M_k - M_{k-1})^2|\mathcal{F}_{k-1})$ for $n \geq 1$ and $\langle M\rangle_0 = 0$. Show that $[M]$ and $\langle M\rangle$ are increasing and adapted, and show that both $M^2 - [M]$ and $M^2 - \langle M\rangle$ are uniformly integrable martingales, zero at zero.

Exercise A.6. Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with a sequence $(X_n)_{n\geq 1}$ of independent and identically distributed variable such that for some p with $0 < p < 1$, it holds that $P(X_n = 1) = p$ and $P(X_n = -1) = 1 - p$. Put $Z_n = \sum_{k=1}^n X_k$ for $n \geq 1$. Let $a \in \mathbb{N}$ and define $T_a = \inf\{n \geq 1 \mid Z_n = a\}$ and $T_{-a} = \inf\{n \geq 1 \mid Z_n = -a\}$. Put $c = \frac{1-p}{p}$. Show that when $p = \frac{1}{2}$, $P(T_a > T_{a-}) = \frac{1}{2}$, and when $p \neq \frac{1}{2}$, $P(T_a > T_{a-}) = \frac{1-c^a}{1-c^{2a}}$.

Exercise A.7. Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with a sequence $(X_t)_{t\geq 0}$ of integrable variables, and let X be some other variable. Assume that the mapping $t \mapsto X_t$ from $\mathbb{R}_+$ to $\mathcal{L}^1$ is continuous. Show that $(X_t)_{t\geq 0}$ converges in $\mathcal{L}^1$ to X if and only if $(X_t)_{t\geq 0}$ is uniformly integrable and converges in probability to X.

Exercise A.8. Let (X_n) be a bounded sequence of variables. Show that if X_n converges in probability, it converges in $\mathcal{L}^p$ for any $p \geq 1$.

Exercise A.9. Let (X_n) be a uniformly integrable sequence of variables. Show that $\limsup_n EX_n \leq E \limsup_n X_n$.

Appendix B

Hints and solutions for exercises

B.1 Hints and solutions for Chapter 1

Hints for exercise 1.1. For the inclusion $\mathcal{F}_{S \vee T} \subseteq \sigma(\mathcal{F}_S, \mathcal{F}_T)$, use that for any F, it holds that $F = (F \cap (S \leq T)) \cup (F \cap (T \leq S))$. ∘

Hints for exercise 1.2. Use Lemma 1.1.10 to conclude that T is a stopping time. To show that $X_T = a$ whenever $T < \infty$, Show that whenever T is finite, there is a sequence (u_n) depending on ω such that $T \leq u_n \leq T + \frac{1}{n}$ and such that $X_{u_n} = a$. Use this to obtain the desired result. ∘

Hints for exercise 1.5. Use Lemma 1.1.7, to show $\mathcal{F}_T \subseteq \cap_{n=1}^\infty \mathcal{F}_{T_n}$. In order to obtain the other inclusion, let $F \in \cap_{n=1}^\infty \mathcal{F}_{T_n}$. Show that $F \cap (T \leq t) = \cap_{n=1}^\infty \cup_{m=1}^\infty \cap_{k=m}^\infty F \cap (T_k \leq t + \frac{1}{n})$. Use this and the right-continuity of the filtration to prove that $F \cap (T \leq t) \in \mathcal{F}_t$, and conclude $\cap_{n=1}^\infty \mathcal{F}_{T_n} \subseteq \mathcal{F}_T$ from this. ∘

Hints for exercise 1.7. First show that for any $t \geq 0$, Z_{t_n} converges to Z_t whenever (t_n) is a sequence in $\mathbb{R}_+$ which either increases or decreases to t. Use this to obtain that Z is continuous in $\mathcal{L}^1$ at t using a proof by contradiction. ∘

Hints for exercise 1.9. In order to obtain that $M \in \mathbf{c}\mathcal{M}^u$ when $(M_T)_{T \in \mathcal{C}}$ is uniformly integrable, use the continuity of M, Lemma 1.2.9 and Lemma A.2.5. ∘

Hints for exercise 1.12. To show that M is not uniformly integrable, assume that M is in fact uniformly integrable. Prove that $\frac{1}{t}M_t$ then converges to zero in $\mathcal{L}^1$ and obtain a contradiction from this. ◦

Hints for exercise 1.13. In order to obtain the final two convergences in probability, apply the relation $W_t^2 = 2\sum_{k=1}^{2^n} W_{t_{k-1}^n}(W_{t_k^n} - W_{t_{k-1}^n}) + \sum_{k=1}^{2^n}(W_{t_k^n} - W_{t_{k-1}^n})^2$. ◦

Hints for exercise 1.14. In order to obtain that T is a stopping time, write T as the minimum of two variables which are stopping times according to Lemma 1.1.10. In order to find that the distribution of W_T, use the law of the iterated logarithm to show that $P(T < \infty) = 1$ and conclude that the distribution of W_T is concentrated on $\{a, b\}$. Then use the optional sampling theorem on the uniformly integrable martingale W^T to obtain the distribution of W_T. ◦

Hints for exercise 1.15. In order to prove that M^α is a martingale, recall that for any $0 \leq s \leq t$, $W_t - W_s$ is independent of $\mathcal{F}_s$ and has a normal distribution with mean zero and variance $t - s$. In order to obtain the result on the Laplace transform of the stopping time T, first reduce to the case of $a \geq 0$. Note that by the properties of Brownian Motion, T is almost surely finite. Show that $(M^\alpha)^T$ is uniformly integrable and use the optional sampling theorem in order to obtain the result. ◦

Hints for exercise 1.16. For the first process, use $W_t^3 = (W_t - W_s)^3 + 3W_t^2 W_s - 3W_t W_s^2 + W_s^3$ and calculate conditional expectations using the properties of the $\mathcal{F}_t$ Brownian motion. Apply a similar method to obtain the martingale property of the second process. ◦

Hints for exercise 1.17. To show the equality for $P(T < \infty)$, consider the martingale M^α defined in Exercise 1.15 by $M_t^\alpha = \exp(\alpha W_t - \frac{1}{2}\alpha^2 t)$ for $\alpha \in \mathbb{R}$. Show that the equality $E1_{(T<\infty)}M_T^{2b} = \exp(2ab)P(T < \infty)$ holds. Recalling that $\lim_{t\to\infty}\frac{W_t}{t} = 0$, use the optional sampling theorem and the dominated convergence theorem to show that $E1_{(T<\infty)}M_T^{2b} = 1$. Use this to obtain the result. ◦

Hints for exercise 1.18. First show that we always have $0 < T \leq 1$ and $W_T^2 = a(1-T)$. From Theorem 1.2.1 and Exercise 1.16, it is known that the processes $W_t^2 - t$ and $W_t^4 - 6tW_t^2 + 3t^2$ are in $\mathbf{c}\mathcal{M}$. Use these facts to obtain expressions for ET and ET^2. ◦

Solution to exercise 1.1. By Lemma 1.1.7, $\mathcal{F}_S \subseteq \mathcal{F}_{S\vee T}$ and $\mathcal{F}_T \subseteq \mathcal{F}_{S\vee T}$, showing that

$\sigma(\mathcal{F}_S, \mathcal{F}_T) \subseteq \mathcal{F}_{S\vee T}$. We need to prove the other implication. Let $F \in \mathcal{F}_{S\vee T}$. We find

$$\begin{aligned} F \cap (S \leq T) \cap (T \leq t) &= F \cap (S \leq T) \cap (S \vee T \leq t) \\ &= (F \cap (S \vee T \leq t)) \cap ((S \leq T) \cap (S \vee T \leq t)). \end{aligned}$$

Here, $F \cap (S \vee T \leq t) \in \mathcal{F}_t$ as $F \in \mathcal{F}_{S\vee T}$, and $(S \leq T) \cap (S \vee T \leq t) \in \mathcal{F}_t$ as we have $(S \leq T) \in \mathcal{F}_{S\wedge T} \subseteq \mathcal{F}_{S\vee T}$ by Lemma 1.1.13. We conclude $F \cap (S \leq T) \in \mathcal{F}_T$. Analogously, $F \cap (T \leq S) \in \mathcal{F}_S$. From this, we obtain $F \in \sigma(\mathcal{F}_S, \mathcal{F}_T)$, as desired. □

Solution to exercise 1.2. As one-point sets are closed, we know from Lemma 1.1.10 that T is a stopping time. When $T < \infty$, it holds that $\{t \geq 0 | X_t = a\}$ is nonempty, and for any n, $T + \frac{1}{n}$ is not a lower bound for the set $\{t \geq 0 \mid X_t = a\}$. Therefore, there is $u_n < T + \frac{1}{n}$ such that $u \in \{t \geq 0 \mid X_t = a\}$. Furthermore, as T is a lower bound for $\{t \geq 0 \mid X_t = a\}$, $u_n \geq T$. Thus, by continuity, $X_T = \lim_n X_{u_n} = a$. □

Solution to exercise 1.3. For any ω such that $T(\omega) > 0$, $X(\omega)$ has a discontinuity at $T(\omega)$, so X is almost surely not continuous. Now let $t \geq 0$ and consider a sequence (t_n) tending to t. Let A_n be the interval with endpoints t_n and t. As t_n tends to t, the Lebesgue measure of A_n tends to zero. For any $\varepsilon > 0$ with $\varepsilon < 1$, we then find

$$\lim_n P(|X_{t_n} - X_t| > \varepsilon) = \lim_n P(T \text{ is between } t_n \text{ and } t) = \lim_n \int_0^\infty 1_{A_n} \exp(-t)\,\mathrm{d}t,$$

and the latter is zero by the dominated convergence theorem. Thus, X_{t_n} converges in probability to X_t. □

Solution to exercise 1.4. Fix $t \geq 0$ and let $Z_t^\varepsilon = \sup\{X_s \mid s \in \mathbb{R}_+, |t-s| \leq \varepsilon\}$. Note that as X is continuous, we also have $Z_t^\varepsilon = \sup\{X_s | s \in \mathbb{Q}_+, |t-s| \leq \varepsilon\}$, and therefore Z_t^ε is $\mathcal{F}_{t+\varepsilon}$ measurable. Also note that we always have $X_t \leq Z_t^\varepsilon$. Therefore, $X_t = Z_t^\varepsilon$ if and only if $X_s \leq X_t$ for all $s \geq 0$ with $|t-s| \leq \varepsilon$. In particular, if $0 < \varepsilon' < \varepsilon$, $X_t = Z_t^\varepsilon$ implies $X_t = Z_t^{\varepsilon'}$. Therefore, fixing $\delta > 0$, we obtain

$$F = \cup_{\varepsilon > 0}(X_t = Z_t^\varepsilon) = \cup_{\varepsilon \in \mathbb{Q}_+, 0 < \varepsilon \leq \delta}(X_t = Z_t^\varepsilon) \in \mathcal{F}_{t+\delta}.$$

Consequently, $F \in \cap_{s>t}\mathcal{F}_s = \mathcal{F}_t$, as desired. □

Solution to exercise 1.5. First note that by Lemma 1.1.8, T is a stopping time. In particular, $\mathcal{F}_T$ is well-defined. Using Lemma 1.1.7, the relation $T \leq T_n$ yields $\mathcal{F}_T \subseteq \mathcal{F}_{T_n}$, so that $\mathcal{F}_T \subseteq \cap_{n=1}^\infty \mathcal{F}_{T_n}$. Conversely, let $F \in \cap_{n=1}^\infty \mathcal{F}_{T_n}$, we want to show $F \in \mathcal{F}_T$, and to this end, we have to show $F \cap (T \leq t) \in \mathcal{F}_t$ for any $t \geq 0$. Fixing $t \geq 0$, the convergence of T_n to T yields $F \cap (T \leq t) = \cap_{n=1}^\infty \cup_{m=1}^\infty \cap_{k=m}^\infty F \cap (T_k \leq t + \frac{1}{n})$. Now, as $F \in \cap_{n=1}^\infty \mathcal{F}_{T_n}$, we have $F \in \mathcal{F}_{T_n}$

for all n. Therefore, $F \cap (T_k \leq t + \frac{1}{n}) \in \mathcal{F}_{t+\frac{1}{n}}$, and so $\cup_{m=1}^\infty \cap_{k=m}^\infty F \cap (T_k \leq t + \frac{1}{n}) \in \mathcal{F}_{t+\frac{1}{n}}$. As this sequence of sets is decreasing in n, we find that $\cap_{n=1}^\infty \cup_{m=1}^\infty \cap_{k=m}^\infty F \cap (T_k \leq t + \frac{1}{n})$ is in $\cap_{n=1}^\infty \mathcal{F}_{t+\frac{1}{n}}$. By right-continuity of the filtration, $\cap_{n=1}^\infty \mathcal{F}_{t+\frac{1}{n}} = \mathcal{F}_t$, and so we finally conclude $F \cap (T \leq t) \in \mathcal{F}_t$, proving $F \in \mathcal{F}_T$. We have now shown $\cap_{n=1}^\infty \mathcal{F}_{T_n} \subseteq \mathcal{F}_T$, and so $\mathcal{F}_T = \cap_{n=1}^\infty \mathcal{F}_{T_n}$. This concludes the proof. □

Solution to exercise 1.6. Let $0 \leq s \leq t$. As $\mathcal{G}_s \subseteq \mathcal{F}_s$ and M_s is $\mathcal{G}_s$ measurable, we find $E(M_t|\mathcal{G}_s) = E(E(M_t|\mathcal{F}_s)|\mathcal{G}_s) = E(M_s|\mathcal{G}_s) = M_s$. This proves the result. □

Solution to exercise 1.7. First assume that (t_n) is an increasing sequence in $\mathbb{R}_+$ with finite limit $t \geq 0$. By Lemma 1.2.2 and Lemma A.2.4, (Z_{t_n}) is then a uniformly integrable discrete-time supermartingale. Therefore, by Theorem A.3.2, Z_{t_n} is convergent almost surely and in $\mathcal{L}^1$. As Z is continuous, Z_{t_n} is convergent almost surely to Z_t. Therefore, we conclude that Z_{t_n} converges almost surely to Z_t.

Next, assume that (t_n) is a decreasing sequence in $\mathbb{R}_+$ with finite limit $t \geq 0$. For $n \geq k$, we then have $t_k \geq t_n$ and so $Z_{t_n} \geq E(Z_{t_k}|\mathcal{F}_{t_n})$ by the supermartingale property. Therefore, (Z_{t_n}) is a backwards submartingale. By Lemma A.2.4, it is uniformly integrable, and therefore, by Theorem A.3.6, it is convergent almost surely and in $\mathcal{L}^1$. By continuity of Z, we find that Z_{t_n} converges almost surely to Z_t, and therefore, we may conclude that Z_{t_n} converges to Z_t in $\mathcal{L}^1$.

We are now ready to prove continuity in $\mathcal{L}^1$ of $t \mapsto Z_t$. Fix $t \geq 0$. Assume that Z is not continuous in $\mathcal{L}^1$ at t. Then, there exists $\varepsilon > 0$ such that for any $\delta > 0$, there is $s \geq 0$ with $|s - t| \leq \delta$ such that $E|Z_t - Z_s| \geq \varepsilon$. Therefore, we obtain a sequence (s_n) such that s_n converges to t and $E|Z_{s_n} - Z_t| \geq \varepsilon$. As any sequence in $\mathbb{R}$ has a monotone subsequence, we obtain a sequence (t_n) such that t_n either converges downwards or upwards to t, and such that $E|Z_{t_n} - Z_t| \geq \varepsilon$. This is in contradiction with our earlier conclusions, and we conclude that Z must be continuous in $\mathcal{L}^1$ at t. This proves the result. □

Solution to exercise 1.8. Define $\alpha(t) = \sup_{F \in \mathcal{F}_\infty} \inf_{G \in \mathcal{F}_t} P(G \Delta F)$. We will prove the result by a kind of division into positive and negative parts. First, let $F = (M_t - M_\infty \geq 0)$, we then have $F \in \mathcal{F}_\infty$. Fix $\varepsilon > 0$ and let G be an element of $\mathcal{F}_t$ such that $P(G \Delta F) \leq \alpha(t) + \varepsilon$.

Using that $F^c = (M_t - M_\infty < 0)$, we obtain

$$\begin{aligned} E1_F|M_t - M_\infty| &= E1_{F\cap G}(M_t - M_\infty) + E1_{F\cap G^c}|M_t - M_\infty| \\ &= E1_G(M_t - M_\infty) - E1_{G\cap F^c}(M_t - M_\infty) + E1_{F\cap G^c}|M_t - M_\infty| \\ &= E1_G(M_t - M_\infty) + E1_{G\cap F^c}|M_t - M_\infty| + E1_{F\cap G^c}|M_t - M_\infty| \\ &\leq E1_G(M_t - M_\infty) + 2cP(G\Delta F) \leq E1_G(M_t - M_\infty) + 2c(\alpha(t) + \varepsilon). \end{aligned}$$

Now, by Theorem 1.2.5, we have $M_t = E(M_\infty|\mathcal{F}_t)$ almost surely. As $G \in \mathcal{F}_t$, we therefore find $E1_G(M_t - M_\infty) = E1_G M_t - E1_G M_\infty = 0$. All in all, we obtain $E1_F|M_t - M_\infty| \leq 2c(\alpha(t)+\varepsilon)$. As $\varepsilon > 0$ was arbitrary, this allows us to conclude that $E1_F|M_t - M_\infty| \leq 2c\alpha(t)$. Making the same calculations with the set $(M_t - M_\infty \leq 0)$ and adding up the results, we find $E|M_t - M_\infty| \leq 4c\alpha(t)$, as desired. □

Solution to exercise 1.9. If $M \in \mathbf{c}\mathcal{M}^u$, we have by Theorem 1.2.7 that $M_T = E(M_\infty|\mathcal{F}_T)$ for all $T \in \mathcal{C}$. Therefore, Lemma A.2.6 shows that $(M_T)_{T\in\mathcal{C}}$ is uniformly integrable. Conversely, assume that $(M_T)_{T\in\mathcal{C}}$ is uniformly integrable, we will use Lemma 1.2.9 to show that $M \in \mathbf{c}\mathcal{M}$. Let S be any bounded stopping time, and let (T_n) be a localising sequence with the property that $M^{T_n} \in \mathbf{c}\mathcal{M}^u$. As S is bounded, $M_{T_n\wedge S}$ converges almost surely to M_S. As it holds that $(M_{T_n\wedge S}) \subseteq (M_T)_{T\in\mathcal{C}}$, $(M_{T_n\wedge S})$ is uniformly integrable, and so Lemma A.2.5 shows that $M_{T_n\wedge S}$ converges in $\mathcal{L}^1$ to M_S. In particular, $EM_S = \lim_n EM_{T_n\wedge S} = \lim_n EM_S^{T_n} = 0$ by Theorem 1.2.7. Lemma 1.2.9 then shows that $M \in \mathbf{c}\mathcal{M}$. As $(M_t)_{t\geq 0} \subseteq (M_T)_{T\in\mathcal{C}}$, M is uniformly integrable, and so $M \in \mathbf{c}\mathcal{M}^u$ as well. □

Solution to exercise 1.10. By Lemma 1.4.9, we have

$$[M^T - M^S] = [M^T] - 2[M^T, M^S] + [M^S] = [M]^T - 2[M]^S + [M^S] = [M]^T - [M]^S.$$

Therefore, if $[M]_S = [M]_T$ almost surely, we obtain that $[M^T - M^S]_\infty$ is almost surely zero. As $[M^T - M^S]$ is increasing, this implies that $[M^T - M^S]$ is evanescent, and so by Lemma 1.4.9, $M^T = M^S$ almost surely. □

Solution to exercise 1.11. Since $X^{S\wedge T} = (X^S)^T$, we find by Lemma 1.2.8 that $X^{S\wedge T}$ is in $\mathbf{c}\mathcal{M}^u$. As $X_{S\wedge t} + X_{T\wedge t} = X_{(S\wedge T)\wedge t} + X_{(S\vee T)\wedge t}$, we find $X^{S\vee T} = X^S + X^T - X^{S\wedge T}$, so $X^{S\vee T}$ is in $\mathbf{c}\mathcal{M}^u$ as well, since it is a linear combination of elements in $\mathbf{c}\mathcal{M}^u$. □

Solution to exercise 1.12. Recall from Theorem 1.2.1 that M is a martingale. In order to show that M is not uniformly integrable, assume that M is in fact uniformly integrable, we seek a contradiction. We know by Theorem 1.2.5 that there is M_∞ such that M_t converges almost surely and in $\mathcal{L}^1$ to M_∞. However, we then obtain

$$\lim_{t\to\infty} E|\tfrac{1}{t}M_t| = \lim_{t\to\infty} \tfrac{1}{t}E|M_t| \leq \lim_{t\to\infty} \tfrac{1}{t}E|M_t - M_\infty| + \tfrac{1}{t}E|M_\infty| = 0,$$

so $\frac{1}{t}M_t$ converges in $\mathcal{L}^1$ to zero. In particular $\frac{1}{t}M_t$ converges in distribution to the Dirac measure at zero, which is in contradiction to the fact that as $\frac{1}{t}M_t = (\frac{1}{\sqrt{t}}W_t)^2 - 1$, $\frac{1}{t}M_t$ has the law of a standard normal distribution transformed by the transformation $x \mapsto x^2 - 1$ for any $t \geq 0$. Therefore, M_t cannot be convergent in $\mathcal{L}^1$, and then, again by Theorem 1.2.5, M cannot be uniformly integrable. □

Solution to exercise 1.13. Put $U_n = \sum_{k=1}^{2^n}(W_{t_k^n} - W_{t_{k-1}^n})^2$, we need to show that $U_n \xrightarrow{P} t$. By the properties of the Brownian motion W, the variables $W_{t_k^n} - W_{t_{k-1}^n}$ are independent and normally distributed with mean zero and variance $t2^{-n}$. Defining the variables Z_k^n by putting $Z_k^n = (\frac{1}{\sqrt{t2^{-n}}}(W_{t_k^n} - W_{t_{k-1}^n}))^2$, we find that Z_k^n, $k = 1, \ldots, 2^n$ are independent and distributed as an χ^2 distribution with one degree of freedom, and we have $U_n = t2^{-n}\sum_{k=1}^{2^n} Z_k^n$. As $EZ_k^n = 1$ and $VZ_k^n = 2$, we then obtain $EU_n = t$ and $VU_n = (t2^{-n})^2\sum_{k=1}^{2^n} 2 = 2t^2 2^{-n}$. In particular, Chebychev's inequality yields for any $\varepsilon > 0$ that

$$\lim_{n\to\infty} P(|U_n - t| > \varepsilon) \leq \tfrac{1}{\varepsilon^2}\lim_{n\to\infty} E(U_n - t)^2 = \tfrac{1}{\varepsilon^2}\lim_{n\to\infty} VU_n = \tfrac{1}{\varepsilon^2}2t^2\lim_{n\to\infty} 2^{-n} = 0,$$

which proves that $U_n \xrightarrow{P} t$, as desired. The two convergences in probability then follow from the equalities

$$\begin{aligned} W_t^2 &= 2\sum_{k=1}^{2^n} W_{t_{k-1}^n}(W_{t_k^n} - W_{t_{k-1}^n}) + \sum_{k=1}^{2^n}(W_{t_k^n} - W_{t_{k-1}^n})^2 \\ &= 2\sum_{k=1}^{2^n} W_{t_k^n}(W_{t_k^n} - W_{t_{k-1}^n}) - \sum_{k=1}^{2^n}(W_{t_k^n} - W_{t_{k-1}^n})^2, \end{aligned}$$

by rearrangement and letting n tend to infinity. □

Solution to exercise 1.14. As $T = \min\{\inf\{t \geq 0 \mid W_t = -a\}, \inf\{t \geq 0 \mid W_t = b\}\}$, T is a stopping time by Lemma 1.1.7 and Lemma 1.1.10. As for the distribution of W_T, first note that by the law of the iterated logarithm, $\limsup_{t\to\infty} |W_t|(2t\log\log t)^{-1/2} = 1$ almost surely, so that $|W_t|$ almost surely is larger than any fixed number at some timepoint. Therefore, $P(T < \infty) = 1$. Next, note that since W has initial value zero and both a and b are nonzero, it follows that if T is finite and $T = \inf\{t \geq 0 \mid W_t = -a\}$, we have $W_T = -a$, and if T is finite and $T = \inf\{t \geq 0 \mid W_t = b\}$, we have $W_T = b$. Therefore, W_T is almost surely either $-a$ or b. Furthermore, note that whenever $0 \leq t < T$, $W_t > -a$ and $W_t < b$. Therefore, W^T is a bounded martingale, in particular uniformly integrable, and so $EW_T = 0$ by Theorem 1.2.7. This leads us to conclude

$$\begin{aligned} 0 &= EW_T = bP(W_T = b) - aP(W_T = -a) \\ &= bP(W_T = b) - a(1 - P(W_T = b)) = (a+b)P(W_T = b) - a, \end{aligned}$$

so that $P(W_T = b) = \frac{a}{a+b}$ and $P(W_T = a) = \frac{b}{a+b}$.

Consider now the case where $a \neq b$, we have to show that $[-W^T]$ and $[W^T]$ are the same, while $-W^T$ and W^T have different distributions. From what we already have shown, W_T is concentrated on $\{-a, b\}$ while $-W_T$ is concentrated on $\{a, -b\}$, and in both cases, both points have nonzero probability. As these two sets are not the same when $a \neq b$, $-W_T$ and W_T do not have the same distribution. As these two variables are the limits of $-W^T_t$ and W^T_t as t tends to infinity, it follows that $-W^T$ and W^T cannot have the same distribution. However, by Lemma 1.4.9, we have $[-W^T] = [-W^T, -W^T] = [W^T, W^T] = [W^T]$, so the quadratic variation processes are equal. □

Solution to exercise 1.15. Fix α and let $0 \leq s \leq t$. We then find

$$\begin{aligned} E(M^\alpha_t|\mathcal{F}_s) &= M^\alpha_s E(\exp(\alpha(W_t - W_s) - \tfrac{1}{2}\alpha^2(t-s))|\mathcal{F}_s) \\ &= M^\alpha_s E(\exp(\alpha(W_t - W_s) - \tfrac{1}{2}\alpha^2(t-s))) = M^\alpha_s, \end{aligned}$$

using that $W_t - W_s$ is independent of $\mathcal{F}_s$ and normally distributed, and for any variable X which is standard normally distributed, $E\exp(tX) = \exp(\frac{1}{2}t^2)$ for any $t \in \mathbb{R}$. Thus, M^α is a martingale.

Next, we consider the result on the stopping time T. By symmetry, it suffices to consider the case where $a \geq 0$. By the law of the iterated logarithm, T is almost surely finite. Therefore, W^T is bounded from above by a and $W_T = a$. Fixing some $\alpha \geq 0$, we then find $(M^\alpha)^T_t = \exp(\alpha W^T_t - \frac{1}{2}\alpha^2(T \wedge t)) \leq \exp(\alpha a)$. Therefore, by Lemma A.2.4 $(M^\alpha)^T$ is uniformly integrable. Now, the almost sure limit of $(M^\alpha)^T$ is $\exp(\alpha a - \frac{1}{2}\alpha^2 T)$. By Theorem 1.2.7, we therefore find $1 = E(M^\alpha)^T_\infty = E\exp(\alpha a - \frac{1}{2}\alpha^2 T) = \exp(\alpha a)E\exp(-\frac{1}{2}\alpha^2 T)$. This shows $E\exp(-\frac{1}{2}\alpha^2 T) = \exp(-\alpha a)$. Thus, if we now fix $\beta \geq 0$, we find

$$E\exp(-\beta T) = E\exp(-\tfrac{1}{2}(\sqrt{2\beta})^2 T) = \exp(-\sqrt{2\beta}a),$$

as desired. □

Solution to exercise 1.16. Fix $0 \leq s \leq t$. As $(W_t - W_s)^3 = W_t^3 - 3W_t^2 W_s + 3W_t W_s^2 - W_s^3$, we obtain

$$\begin{aligned} E(W_t^3 - 3tW_t|\mathcal{F}_s) &= E((W_t - W_s)^3|\mathcal{F}_s) + E(3W_t^2 W_s - 3W_t W_s^2 + W_s^3|\mathcal{F}_s) - 3tW_s \\ &= E(W_t - W_s)^3 + 3W_s E(W_t^2|\mathcal{F}_s) - 3W_s^2 E(W_t|\mathcal{F}_s) + W_s^3 - 3tW_s \\ &= 3W_s E(W_t^2 - t|\mathcal{F}_s) - 2W_s^3 = 3W_s(W_s^2 - s) - 2W_s^3 = W_s^3 - sW_s. \end{aligned}$$

As regards the second process, we have $(W_t - W_s)^4 = W_t^4 - 4W_t^3 W_s + 6W_t^2 W_s^2 - 4W_t W_s^3 + W_s^4$, and so

$$\begin{aligned}
E(W_t^4|\mathcal{F}_s) &= E((W_t-W_s)^4|\mathcal{F}_s)+E(4W_t^3W_s-6W_t^2W_s^2+4W_tW_s^3-W_s^4|\mathcal{F}_s)\\
&= 3(t-s)^2+4W_sE(W_t^3|\mathcal{F}_s)-6W_s^2E(W_t^2|\mathcal{F}_s)+4W_s^3E(W_t|\mathcal{F}_s)-W_s^4\\
&= 3(t-s)^2+4W_sE(W_t^3-3tW_t|\mathcal{F}_s)+12tW_s^2-6W_s^2E(W_t^2|\mathcal{F}_s)+3W_s^4\\
&= 3(t-s)^2+4W_s(W_s^3-3sW_s)+12tW_s^2-6W_s^2E(W_t^2|\mathcal{F}_s)+3W_s^4\\
&= 3(t-s)^2+12(t-s)W_s^2-6W_s^2E(W_t^2-t|\mathcal{F}_s)-6tW_s^2+7W_s^4\\
&= 3(t-s)^2+12(t-s)W_s^2-6W_s^2(W_s^2-s)-6tW_s^2+7W_s^4\\
&= 3(t-s)^2+6(t-s)W_s^2+W_s^4.
\end{aligned}$$

Therefore, we find

$$\begin{aligned}
E(W_t^4-6tW_t^2+3t^2|\mathcal{F}_s) &= 3(t-s)^2+6(t-s)W_s^2+W_s^4-6tE(W_t^2|\mathcal{F}_s)+3t^2\\
&= 3(t-s)^2+6(t-s)W_s^2+W_s^4-6tE(W_t^2-t|\mathcal{F}_s)-3t^2\\
&= 3(t-s)^2+6(t-s)W_s^2+W_s^4-6t(W_s^2-s)-3t^2\\
&= W_s^4-6sW_s^2+3(t-s)^2+6st-3t^2\\
&= W_s^4-6sW_s^2+3s^2,
\end{aligned}$$

as desired. □

Solution to exercise 1.17. We have $T=\inf\{t\geq 0 \mid W_t-a-bt\geq 0\}$, where the process W_t-a-bt is continuous and adapted. Therefore, Lemma 1.1.10 shows that T is a stopping time. Now consider $a>0$ and $b>0$. Note that W_t-a-bt has initial value $-a\neq 0$, so we always have $T>0$. In particular, again using continuity, we have $W_T=a+bT$ whenever T is finite. Now, by Exercise 1.15, we know that for any $\alpha\in\mathbb{R}$, the process M^α defined by $M_t^\alpha=\exp(\alpha W_t-\frac{1}{2}\alpha^2 t)$ is a martingale. We then find

$$\begin{aligned}
E1_{(T<\infty)}M_T^\alpha &= E1_{(T<\infty)}\exp(\alpha W_T-\tfrac{1}{2}\alpha^2 T)\\
&= E1_{(T<\infty)}\exp(\alpha(a+bT)-\tfrac{1}{2}\alpha^2 T)\\
&= \exp(\alpha a)E1_{(T<\infty)}\exp(T(\alpha b-\tfrac{1}{2}\alpha^2)).
\end{aligned}$$

Now note that the equation $\alpha b=\frac{1}{2}\alpha^2$ has the unique nonzero solution $\alpha=2b$. Therefore, we obtain $E1_{(T<\infty)}M_T^{2b}=\exp(2ab)P(T<\infty)$. In order to show the desired equality, it therefore suffices to prove $E1_{(T<\infty)}M_T^{2b}=1$. To this end, note that by the law of the iterated logarithm, using that $b\neq 0$, $\lim_{t\to\infty}2bW_t-\frac{1}{2}(2b)^2t=\lim_{t\to\infty}t(2b\frac{W_t}{t}-2b^2)=-\infty$, so that $\lim_{t\to\infty}M_t^{2b}=0$. Therefore, $1_{(T<\infty)}M_T^{2b}=M_T^{2b}$ and it suffices to prove $EM_T^{2b}=1$.

To this end, note that

$$\begin{aligned}M^{2b}_{T\wedge t} &= \exp\left(2bW_{T\wedge t} - \frac{1}{2}(2b)^2(T\wedge t)\right)\\ &\leq \exp\left(2b(a+b(T\wedge t)) - \frac{1}{2}(2b)^2(T\wedge t)\right) = \exp(2ba).\end{aligned}$$

Therefore, $(M^{2b})^T$ is bounded by the constant $\exp(2ba)$, in particular it is uniformly integrable. Thus, Theorem 1.2.7 shows that $EM^{2b}_T = E(M^{2b})^T_\infty = 1$. From this, we finally conclude $P(T<\infty) = \exp(-2ab)$. □

Solution to exercise 1.18. Since $W_t^2 - a(1-t)$ is a continuous adapted process, Lemma 1.1.10 shows that T is a stopping time. Now let $a>0$. As $W_t^2 - a(1-t)$ has initial value $-a \neq 0$, we always have $T>0$. Furthermore, as $a(1-t)<0$ when $t>1$, we must have $T\leq 1$, so T is a bounded stopping time. In particular, T has moments of all orders, and by continuity, $W_T^2 = a(1-T)$.

In order to find the mean of T, recall from Theorem 1.2.1 that $W_t^2 - t$ is a martingale. As T is a bounded stopping time, we find $0 = E(W_T^2 - T) = E(a(1-T)-T) = a-(1+a)ET$ by Theorem 1.2.7, so $ET = \frac{a}{1+a}$. Next, we consider the second moment. Recall by Exercise 1.16 that $W_t^4 - 6tW_t^2 + 3t^2$ is in $\mathbf{c}\mathcal{M}$. Again using Theorem 1.2.7, we obtain

$$\begin{aligned}0 &= E(W_T^4 - 6TW_T^2 + 3T^2)\\ &= E(a^2(1-T)^2 - 6Ta(1-T) + 3T^2)\\ &= E(a^2(1-2T+T^2) + 6a(T^2-T) + 3T^2)\\ &= (a^2+6a+3)ET^2 - (2a^2+6a)ET + a^2.\end{aligned}$$

Recalling that $ET = \frac{a}{1+a}$, we find

$$(2a^2+6a)ET - a^2 = \frac{a(2a^2+6a)}{1+a} - \frac{(1+a)a^2}{1+a} = \frac{a^3+5a^2}{1+a},$$

from which we conclude

$$ET^2 = \frac{a^3+5a^2}{(1+a)(a^2+6a+3)} = \frac{a^3+5a^2}{a^3+7a^2+9a+3},$$

concluding the solution to the exercise. □

Solution to exercise 1.19. By Theorem 1.4.8, we know that $W \in \mathbf{c}\mathcal{M}_\ell$ with $[W]_t = t$. In particular, we find that when $p > \frac{1}{2}$, we have, with W^n denoting the process stopped at the deterministic timepoint n,

$$\lim_{n\to\infty}[n^{-p}W^n]_\infty = \lim_{n\to\infty}[n^{-p}W^n, n^{-p}W^n]_\infty = \lim_{n\to\infty} n^{-2p}[W^n]_\infty = \lim_{n\to\infty} n^{1-2p} = 0,$$

since $1 - 2p < 0$. In particular, $[n^{-p}W^n]_\infty \xrightarrow{P} 0$. Therefore, Lemma 1.4.11 shows that $(n^{-p}W^n)^*_\infty \xrightarrow{P} 0$, which is equivalent to stating that $\frac{1}{n^p}\sup_{0\le s\le n}|W_s| \xrightarrow{P} 0$. $\square$

B.2 Hints and solutions for Chapter 2

Hints for exercise 2.3. Define $T_n = \inf\{t \ge 0 \mid [M]_t > n\}$. Show that M^{T_n} is almost surely convergent for all n. Use that $F_c([M]) \subseteq \cup_{n=1}^\infty (T_n = \infty) \subseteq \cup_{n=1}^\infty (M^{T_n} = M)$ to argue that $P(F_c([M]) \setminus F_c(M)) = 0$. Apply a similar argument to obtain $P(F_c(M) \setminus F_c([M])) = 0$. $\circ$

Hints for exercise 2.4. Since convergent sequences are bounded, it is immediate that the inclusion $F_c(X) \subseteq (\sup_t X_t < \infty)$ holds. Therefore, in order to obtain the ersult, it suffices to argue that $P(F_c(X) \setminus (\sup_t X_t < \infty)) = 0$. To this end, define $T_n = \inf\{t \ge 0 | X_t > n\}$ and put $Z^n = n - X^{T_n}$. Show that Z^n is a supermartingale which is bounded in $\mathcal{L}^1$ and so almost surely convergent. Note that $(\sup_t X_t < \infty) \subseteq \cup_{n=1}^\infty (X = n - Z^n)$. Combine these results in order to obtain the desired result. $\circ$

Hints for exercise 2.5. The inclusion $F_c(M) \cap F_c(A) \subseteq F_c(X)$ is immediate, so it suffices to consider the other inclusion. To this end, apply Exercise 2.4 to prove the relationships $P(F_c(M)\Delta(\sup_t M_t < \infty)) = 0$ and $P(F_c(X)\Delta(\sup_t X_t < \infty)) = 0$. Use this and $M \le X$ to obtain the desired result. $\circ$

Hints for exercise 2.6. In order to show that $H \cdot W$ is in $\mathbf{c}\mathcal{M}$ if $E\int_0^t H_s^2 \,\mathrm{d}s$ is finite for all $t \ge 0$, show that the stopped process $(H \cdot W)^t$ is in $\mathbf{c}\mathcal{M}^2$. $\circ$

Hints for exercise 2.7. The issue regarding the measure μ_M being well-defined is that the family of measures induced by $[M](\omega)$ for $\omega \in \Omega$ is not uniformly bounded, and so Theorem A.1.12 and Theorem A.1.13 cannot be directly applied. Let ν_ω be the nonnegative measure induced by $[M](\omega)$ by Theorem A.1.5. The proposed definition of μ_M is then,for $A \in \mathcal{B}_+ \otimes \mathcal{F}$, to put $\mu_M(A) = \int\int 1_A(t,\omega)\,\mathrm{d}\nu_\omega(t)\,\mathrm{d}P(\omega)$. In order to show that this is well-defined, it is necessary to argue that $t \mapsto 1_A(t,\omega)$ is $\mathcal{B}_+$ measurable for all $\omega \in \Omega$ whenever $A \in \mathcal{B}_+ \otimes \mathcal{F}$, such that the inner integral is well-defined, and that $\int 1_A(t,\omega)\,\mathrm{d}[M](\omega)_t$ is $\mathcal{F}$ measurable whenever $A \in \mathcal{B}_+ \otimes \mathcal{F}$, such that the outer integral is well-defined. That the inner integral is well-defined follows from general properties of measurable mappings. In order to show that the outer integral is well-defined, let (T_n) be a localising sequence such that $[M]^{T_n} \le n$. Denote by ν_ω^n the nonnegative measure induced by $[M]^{T_n}(\omega)$ by Theorem A.1.5. Use Theorem A.1.12 to show that there exists a unique nonnegative measure λ^n on $\mathcal{B}_+ \otimes \mathcal{F}$ such that for any $B \in \mathcal{B}_+$ and $F \in \mathcal{F}$, $\lambda^n(B \times F) = \int_F \nu^n_\omega(B)\,\mathrm{d}P(\omega)$ and apply Theorem A.1.13 to show that whenever $A \in \mathcal{B}_+ \otimes \mathcal{F}$, $\omega \mapsto \int 1_A(t,\omega)\,\mathrm{d}\nu^n_\omega(t)$ is $\mathcal{F}$ measurable. Use this fact to show that for any $A \in \mathcal{B}_+ \otimes \mathcal{F}$, $\omega \mapsto 1_A(t,\omega)\,\mathrm{d}\nu_\omega(t)$ is $\mathcal{F}$ measurable.

As regards the remaining claims of the exercise, use the monotone convergence theorem to show that μ_M is a measure. Apply Lemma 1.4.10 and Lemma 2.2.7 to show the statements regarding the stochastic integral. ◦

Hints for exercise 2.8. First show that it suffices to prove the convergence to zero in probability of the variables that $(W_{t+h} - W_t)^{-1} \int_t^{t+h} (H_s - H_t) 1_{[\![t,\infty[\![}\,\mathrm{d}W_s$, where the indicator $1_{[\![t,\infty[\![}$ is included to ensure that the integrand is progressively measurable. To show this, fix $\delta > 0$ and show that

$$\frac{1_{(|W_{t+h}-W_t|>\delta)}}{W_{t+h} - W_t} \int_t^{t+h} (H_s - H_t) 1_{[\![t,\infty[\![}\,\mathrm{d}W_s \xrightarrow{P} 0,$$

using Chebychev's inequality and the results from Lemma 2.6. Then apply Lemma A.2.1 to obtain the result. ◦

Hints for exercise 2.10. To prove that X has the same distribution as a Brownian motion for $H = \frac{1}{2}$, let W be a Brownian motion and show that X and W have the same finite-dimensional distributions.

To show that X is not in $\mathbf{c}\mathcal{S}$ when $H \neq \frac{1}{2}$, first fix $t \geq 0$, put $t^n_k = tk2^{-n}$ and consider $\sum_{k=1}^{2^n} |X_{t^n_k} - X_{t^n_{k-1}}|^p$ for $p \geq 0$. Use the fact that normal distributions are determined by their mean and covariance structure to argue that the distribution of $\sum_{k=1}^{2^n} |X_{t^n_k} - X_{t^n_{k-1}}|^p$ is the same as the distribution of $2^{-npH} \sum_{k=1}^{2^n} |X_k - X_{k-1}|^p$. Show that the process $(X_k - X_{k-1})_{k\geq 1}$ is stationary. Now recall that the ergodic theorem for discrete-time stationary processes states that for a stationary process $(Y_n)_{n\geq 1}$ and a mapping $f : \mathbb{R} \to \mathbb{R}$ such that $f(Y_1)$ is integrable, it holds that $\frac{1}{n}\sum_{k=1}^n f(Y_k)$ converges almost surely and in $\mathcal{L}^1$. Use this to argue that $\frac{1}{2^n}\sum_{k=1}^{2^n} |X_k - X_{k-1}|^p$ converges almost surely and in $\mathcal{L}^1$ to a variable Z_p which is not almost surely zero.

Finally, use this to prove that X is not in $\mathbf{c}\mathcal{S}$ when $H \neq \frac{1}{2}$. To do so, first consider the case $H < \frac{1}{2}$. In this case, assume that $X \in \mathbf{c}\mathcal{S}$ and seek a contradiction. Use the result of Exercise 2.9 to show that $\sum_{k=1}^{2^n} |X_{t^n_k} - X_{t^n_{k-1}}|^{\frac{1}{H}}$ converges to zero in probability. Obtain a contradiction with the results obtained above. In the case where $H > \frac{1}{2}$, use that $\frac{1}{H} < 2$ and Exercise 2.9 to show that $\sum_{k=1}^{2^n} |X_{t^n_k} - X_{t^n_{k-1}}|^2$ converges in probability to zero, and use this to argue that $[X]$ is evanescent. Conclude that X has paths of finite variation, and use this to obtain a contradiction with our earlier results. ◦

Hints for exercise 2.12. Apply Itô's formula to the two-dimensional continuous semimartingale (t, W_t). ∘

Hints for exercise 2.13. Use Itô's formula on the two-dimensional continuous semimartingale $(M, [M])$ to prove that $\mathcal{E}(M)$ is a continuous local martingale with initial value one and that $\mathcal{E}(M)$ solves the stochastic differential equation. For uniqueness of the solution, assume given another solution $Y \in \mathbf{c}\mathcal{S}$ to the stochastic differential equation. Define $Z_t = \mathcal{E}(M)_t^{-1} Y_t$ and apply the local version of Itô's formula, Corollary 2.3.6, to $(\mathcal{E}(M), Y)$ in order to show that Z is indistinguishable from 1. ∘

Hints for exercise 2.15. Use a Taylor expansion of $x \mapsto \log(1+x)$ as in Theorem A.1.17 to rewrite Z_t^n into a form resembling $\mathcal{E}(M)$. Apply Theorem 2.3.3 and the same method of proof as in the proof of Theorem 2.3.5 to obtain control of the remainder term and prove the desired convergence in probability. ∘

Hints for exercise 2.16. Use Exercise 2.13 to conclude that $\mathcal{E}(M)$ is nonnegative and that $\mathcal{E}(M)$ is a continuous local martingale. Apply Fatou's lemma to obtain the supermartingale property. ∘

Hints for exercise 2.17. In order to obtain the criterion for $\mathcal{E}(M)$ to be a uniformly integrable martingale, note that by Exercise 2.16, $\mathcal{E}(M)$ is a nonnegative supermartingale. By applying the optional sampling theorem 1.2.7 to $\mathcal{E}(M)$, conclude by Lemma 1.2.9 that if $E\mathcal{E}(M)_\infty = 1$, then $\mathcal{E}(M)$ is a uniformly integrable martingale. To obtain the converse, use that uniformly integrable martingales are $\mathcal{L}^1$ convergent by Theorem 1.2.5.

In order to obtain the criterion for $\mathcal{E}(M)$ to be a martingale, show that $\mathcal{E}(M)$ is a martingale if and only if $\mathcal{E}(M)^t$ is a uniformly integrable martingale for all $t \geq 0$, and prove the result using the criterion for $\mathcal{E}(M)$ to be a uniformly integrable martingale. ∘

Hints for exercise 2.18. Use Theorem 2.3.2 to show that $\int_0^t f(s)\,\mathrm{d}W_s$ is the limit in probability of a sequence of variables whose distribution may be calculated explicitly. Use that convergence in probability implies weak convergence to obtain the desired result. ∘

Hints for exercise 2.20. In the case where $i = j$, use Itô's formula with the function $f(x) = x^2$, and in the case $i \neq j$, use Itô's formula with the function $f(x, y) = xy$. Afterwards, apply Lemma 2.1.8 and Lemma 2.2.7 to obtain the result. ∘

Hints for exercise 2.21. To prove integrability, first use Exercise 2.6 to show that X has the property that the stopped process X^c for any $c \geq 0$ is in $\mathbf{c}\mathcal{M}^2$. Apply this together with the Cauchy-Schwartz inequality to prove that both $X_t W_t$ and $X_t W_t^2$ are integrable. In order to obtain the mean of $X_t W_t$, use Exercise 2.6 and Lemma 1.4.10 to show that $X_t W_t - [X, W]_t$ is a martingale, and apply the properties of the stochastic integral and its quadratic covariation. In order to obtain the mean of $X_t W_t^2$, apply Itô's formula to decompose W^2 into its continuous local martingale part and continuous finite variation part and proceed as in the previous case, recalling that for a variable Z which is normally distributed with variance σ, we have $E|Z| = \sqrt{\sigma}\sqrt{2/\pi}$. ◦

Hints for exercise 2.22. Let $X = N + B$ be the decomposition of X into its continuous local martingale part and its continuous finite variation part. Apply Itô's formula to M^α with the mapping $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \exp(x - \frac{1}{2}y)$. Use Lemma 1.4.5 to show that the process $\alpha \int_0^t M_s^\alpha \,\mathrm{d}B_s + \frac{\alpha^2}{2} \int_0^t M_s^\alpha \,\mathrm{d}[X]_s - \frac{\alpha^2}{2} \int_0^t M_s^\alpha \,\mathrm{d}A_s$ is evanescent. Use this to show that $M_t^\alpha = 1 + \int_0^t M_s^\alpha \,\mathrm{d}N_s$. Use this and Corollary 2.3.6 to obtain that $[X]$ and A are indistinguishable and that $X \in \mathbf{c}\mathcal{M}_\ell$. ◦

Hints for exercise 2.23. Use Itô's formula to conclude

$$f(W_t) = \alpha + \int_0^t f'(W_s) \,\mathrm{d}W_s + \frac{1}{2} \int_0^t f''(W_s) \,\mathrm{d}s.$$

Use the growth conditions on f and Exercise 2.6 to argue that the integral with respect to W is in $\mathbf{c}\mathcal{M}$ and so $Ef(W_t) = \alpha + \frac{1}{2} E \int_0^t f''(W_s) \,\mathrm{d}s$. Again applying the growth conditions on f, show that interchange of expectation and integration is allowed, and obtain the formula for $Ef(W_t)$ from this. ◦

Hints for exercise 2.24. First note that in order to prove the result, it suffices to prove that the sequence $\sum_{k=1}^{2^n} (M_{t_k} - M_{t_{k-1}})^2$ is bounded in $\mathcal{L}^2$. To do so, write

$$E\left(\sum_{k=1}^{2^n} (M_{t_k} - M_{t_{k-1}})^2\right)^2 = E \sum_{k=1}^{2^n} (M_{t_k} - M_{t_{k-1}})^4 + E \sum_{k \neq i} (M_{t_k} - M_{t_{k-1}})^2 (M_{t_i} - M_{t_{i-1}})^2.$$

Let $C \geq 0$ be a constant such that $|M_t| \leq C$ for all $t \geq 0$. By repeated use of the martingale property, prove that the first term is less than $4C^2$ and that the second term is less than $2C^2$, thus proving boundedness in $\mathcal{L}^2$. ◦

Hints for exercise 2.26. First argue that when stopped at a deterministic timepoint, X is in $\mathbf{c}\mathcal{M}^2$. Use the properties of stochastic integrals from Lemma 2.2.7 as well as Lemma 1.4.10 to argue that for $0 \leq s \leq t$, $\operatorname{Cov}(X_s, X_t) = \int_0^s E \sin^2 W_u \,\mathrm{d}u$. Calculate $E \sin^2 W_u$ by applying the identity $\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$ and calculating $E \cos 2W_u$ using the Taylor series for the cosine function. ◦

Hints for exercise 2.27. Apply Exercise 2.12. ◦

Solution to exercise 2.1. By Theorem 1.2.1, W is a martingale. As W is continuous and has initial value zero, we obtain $W \in \mathbf{c}\mathcal{M}_\ell \subseteq \mathbf{c}\mathcal{S}$, as desired. □

Solution to exercise 2.2. As X has continuous paths, we have

$$\begin{aligned} F_c(X) &= \{\omega \in \Omega \mid X_t(\omega) \text{ is convergent as } t \to \infty\} \\ &= \{\omega \in \Omega \mid X_q(\omega) \text{ is convergent as } q \to \infty, q \in \mathbb{Q}_+\} \\ &= \{\omega \in \Omega \mid X_q(\omega) \text{ is Cauchy as } q \to \infty, q \in \mathbb{Q}_+\} \\ &= \cap_{n=1}^\infty \cup_{q \in \mathbb{Q}_+} \cap_{p,r>q,p,r\in\mathbb{Q}_+} (|X_p - X_r| < \tfrac{1}{n}), \end{aligned}$$

and this set is in $\mathcal{F}$. This proves the desired result. □

Solution to exercise 2.3. Define $T_n = \inf\{t \geq 0 \mid [M]_t > n\}$. We then have $[M]^{T_n} \leq n$, and so Lemma 1.4.10 shows that $M^{T_n} \in \mathbf{c}\mathcal{M}^2$. Therefore, M^{T_n} is almost surely convergent. Furthermore, $F_c([M]) \subseteq \cup_{n=1}^\infty (T_n = \infty) \subseteq \cup_{n=1}^\infty (M^{T_n} = M)$. Let F be the almost sure set where M^{T_n} is convergent for all n. We then obtain

$$\begin{aligned} P(F_c([M]) \setminus F_c(M)) &= P(F \cap F_c([M]) \cap F_c(M)^c) \\ &\leq P(F \cap F_c(M)^c \cap \cup_{n=1}^\infty (M^{T_n} = M)) = 0, \end{aligned}$$

since the final set is empty. We conclude $P(F_c([M]) \setminus F_c(M)) = 0$. Conversely, define the sequence of stopping times $S_n = \inf\{t \geq 0 \mid |M_t| > n\}$, so that $M^{S_n} \in \mathbf{c}\mathcal{M}^b$. By Lemma 1.4.10, $[M]^{S_n}$ is almost surely convergent, so $F_c(M) \subseteq \cup_{n=1}^\infty (S_n = \infty) \subseteq \cup_{n=1}^\infty ([M]^{S_n} = [M])$. Letting F be the almost sure set where $[M]^{S_n}$ is convergent for all n, we obtain

$$\begin{aligned} P(F_c(M) \setminus F_c([M])) &= P(F \cap F_c(M) \cap F_c([M])^c) \\ &\leq P(F \cap F_c([M])^c \cap \cup_{n=1}^\infty ([M]^{S_n} = [M])) = 0. \end{aligned}$$

We have now shown $P(F_c(M) \Delta F_c([M])) = 0$. □

Solution to exercise 2.4. As convergent sequences are bounded, we clearly have the inclusion $F_c(X) \subseteq (\sup_t X_t < \infty)$, such that $P((\sup_t X_t < \infty) \setminus F_c(X)) = 0$. We need to prove that $P(F_c(X) \setminus (\sup_t X_t < \infty)) = 0$.

Define $T_n = \inf\{t \geq 0 \mid X_t > n\}$ and put $Z^n = n - X^{T_n}$, we then have $Z^n \geq 0$. Let (S_n) be a localising sequence such that $M^{S_n} \in \mathbf{c}\mathcal{M}^b$ and A^{S_n} is bounded. We may then obtain $(Z^n)^{S_n} = n - (M+A)^{T_n \wedge S_n} = n - M^{T_n \wedge S_n} - A^{T_n \wedge S_n}$, where $M^{T_n \wedge S_n} \in \mathbf{c}\mathcal{M}^b$ and $A^{T_n \wedge S_n}$

is bounded. Therefore, $(Z^n)^{S_n}$ is bounded, in particular $(Z^n)^{S_n}_t$ is integrable for all $t \geq 0$. Since $A^{T_n \wedge S_n}$ is increasing, we have for $0 \leq s \leq t$ that

$$\begin{aligned} E((Z^n)^{S_n}_t|\mathcal{F}_s) &= E(n - M^{T_n\wedge S_n}_t - A^{T_n\wedge S_n}_t|\mathcal{F}_s) \\ &= n - M^{T_n\wedge S_n}_s - E(A^{T_n\wedge S_n}_t|\mathcal{F}_s) \\ &\leq n - M^{T_n\wedge S_n}_s - E(A^{T_n\wedge S_n}_s|\mathcal{F}_s) \\ &= n - M^{T_n\wedge S_n}_s - A^{T_n\wedge S_n}_s = (Z^n)^{S_n}_s. \end{aligned}$$

Thus, $(Z^n)^{S_n}$ is a nonnegative supermartingale. For $0 \leq s \leq t$, we then obtain using Fatou's lemma that

$$E(Z^n_t|\mathcal{F}_s) = E(\liminf_{n\to\infty}(Z^n)^{S_n}_t|\mathcal{F}_s) \leq \liminf_{n\to\infty} E((Z^n)^{S_n}_t|\mathcal{F}_s) \leq \liminf_{n\to\infty}(Z^n)^{S_n}_s = Z^n_s,$$

so Z^n is also a nonnegative supermartingale. As $Z^n_0 = n$, we have $E|Z^n_t| = EZ^n_t \leq n$, so Z^n is bounded in $\mathcal{L}^1$, and thus almost surely convergent by Theorem 1.2.4. Now note that $(\sup_t X_t < \infty) \subseteq \cup_{n=1}^\infty (T_n = \infty) \subseteq \cup_{n=1}^\infty (X = n - Z^n)$. As the set such that Z^n is almost surely convergent for all n is an almost sure set, we conclude

$$\begin{aligned} P((\sup_t X_t < \infty) \setminus F_c(X)) &= P((\cap_{n=1}^\infty F_c(Z^n)) \cap (\sup_t X_t < \infty) \cap F_c(X)^c) \\ &\leq P((\cap_{n=1}^\infty F_c(Z^n)) \cap F_c(X)^c \cap \cup_{n=1}^\infty (X = n - Z^n)) = 0. \end{aligned}$$

Combining our results, we have now shown $P(F_c(X)\Delta(\sup_t X_t < \infty)) = 0$. □

Solution to exercise 2.5. Clearly, $F_c(M) \cap F_c(A) \subseteq F_c(X)$, we need to prove the converse inclusion. Note that both X and M is in $\mathbf{c}\mathcal{S}$ and satisfy the criterion from Exercise 2.4. Therefore, $P(F_c(M)\Delta(\sup_t M_t < \infty)) = 0$ and $P(F_c(X)\Delta(\sup_t X_t < \infty)) = 0$. This in particular implies that $F_c(X) \subseteq (\sup_t X_t < \infty) \cup N$ and $(\sup_t M_t < \infty) \subseteq F_c(M) \cup N$, where N is some null set. Also, as $A \geq 0$, we have $M = X - A \leq X$, so whenever $\sup_t X_t$ is finite, so is $\sup_t M_t$. Thus, $F_c(X) \subseteq (\sup_t X_t < \infty) \cup N \subseteq (\sup_t M_t < \infty) \cup N \subseteq F_c(M) \cup N$.

Whenever X and M are both convergent with finite limits, so is A. Therefore, we finally conclude $F_c(X) \subseteq (F_c(X) \cap F_c(M)) \cup N \subseteq (F_c(M) \cap F_c(A)) \cup N$. As N is a null set, this implies $P(F_c(X) \setminus (F_c(M) \cap F_c(A))) = 0$. We conclude $P(F_c(X)\Delta(F_c(M) \cap F_c(A))) = 0$. □

Solution to exercise 2.6. As $[W]_t = t$ by Theorem 1.4.8, we obtain $[H \cdot W]_\infty = \int_0^\infty H_s^2 \,\mathrm{d}s$. Therefore, Lemma 1.4.10 shows that $H \cdot W \in \mathbf{c}\mathcal{M}^2$ if and only if $E\int_0^\infty H_s^2 \,\mathrm{d}s$ is finite. As regards the second claim, assume that for any $t \geq 0$, $E\int_0^t H_s^2 \,\mathrm{d}s$ is finite. We have $[(H \cdot W)^t]_\infty = [H \cdot W]^t_\infty = \int_0^t H_s^2 \,\mathrm{d}s$, so our assumptions yields that $(H \cdot W)^t$ is in $\mathbf{c}\mathcal{M}^2$ for all $t \geq 0$, and Lemma 1.4.10 shows that $((H \cdot W)^t)^2 - [(H \cdot W)^t]$ is in $\mathbf{c}\mathcal{M}^u$.

In order to obtain that $H \cdot W$ is a martingale, let T be a bounded stopping time, assume that $T \leq c$. We then have $(H \cdot W)_T = (H \cdot W)^c_T$, so as $(H \cdot W)^c$ is in $\mathbf{c}\mathcal{M}^2$, $(H \cdot W)_T$ is integrable and has mean zero. Therefore, 1.2.9 shows that $H \cdot W$ is in $\mathbf{c}\mathcal{M}$. Furthermore, we have $E(H \cdot W)^2_t = E((H \cdot W)^t)^2_\infty = E[(H \cdot W)^t]_\infty = E \int_0^t H_s^2 \, \mathrm{d}s$. $\square$

Solution to exercise 2.7. Let ν_ω be the nonnegative measure induced by $[M](\omega)$ by Theorem A.1.5. The proposed definition of μ_M is then $\mu_M(A) = \int \int 1_A(t,\omega) \, \mathrm{d}\nu_\omega(t) \, \mathrm{d}P(\omega)$ for sets $A \in \mathcal{B}_+ \otimes \mathcal{F}$. In order to show that this is well-defined, we need to argue that $t \mapsto 1_A(t,\omega)$ is $\mathcal{B}_+$ measurable for all $\omega \in \Omega$ whenever $A \in \mathcal{B}_+ \otimes \mathcal{F}$ and that $\int 1_A(t,\omega) \, \mathrm{d}[M](\omega)_t$ is $\mathcal{F}$ measurable whenever $A \in \mathcal{B}_+ \otimes \mathcal{F}$. The first claim follows since the mapping $(t,\omega) \mapsto 1_A(t,\omega)$ is $\mathcal{B}_+ \otimes \mathcal{F}$-$\mathcal{B}$ measurable and therefore, for any $\omega \in \Omega$, $t \mapsto 1_A(t,\omega)$ is $\mathcal{B}_+$-$\mathcal{B}$ measurable.

In order to prove the remaining claims, let (T_n) be a localising sequence such that $[M]^{T_n} \leq n$. By ν^n_ω, we denote the nonnegative measure induced by $[M]^{T_n}(\omega)$ by Theorem A.1.5. We wish to argue that the family $(\nu^n_\omega)_{\omega \in \Omega}$ satisfies the requirements in Theorem A.1.12. As ν^n_ω is bounded by n, the family $(\nu^n_\omega)_{\omega \in \Omega}$ is uniformly bounded. Let $\mathbb{D}$ be the family of sets $B \in \mathcal{B}_+$ such that $\omega \mapsto \nu^n_\omega(B)$ is $\mathcal{F}$ measurable. We will show that $\mathbb{D}$ is a Dynkin class containing a generating family for $\mathcal{B}_+$ which is stable under intersections, Lemma A.1.19 will then yield that $\mathbb{D}$ is equal to $\mathcal{B}_+$.

For any fixed $0 \leq s \leq t$, we find that $\nu^n_\omega((s,t]) = [M]^{T_n}(\omega)_t - [M]^{T_n}(\omega)_s$. As $[M]^{T_n}_t$ is $\mathcal{F}$ measurable for all $t \geq 0$, we find that $\omega \mapsto \nu^n_\omega((s,t])$ is $\mathcal{F}$ measurable. Thus, $\mathbb{D}$ contains a generating family for $\mathcal{B}_+$ which is stable under intersections. In order to show that $\mathbb{D}$ is a Dynkin class, first note that $\nu^n_\omega(\mathbb{R}_+) = [M]^{T_n}_\infty$, so the mapping $\omega \mapsto \nu^n_\omega(\mathbb{R}^+)$ is also $\mathcal{F}$ measurable. If $A, B \in \mathbb{D}$ with $A \subseteq B$, we find that $\nu^n_\omega(B \setminus A) = \nu^n_\omega(B) - \nu^n_\omega(B)$, so $B \setminus A \in \mathbb{D}$. And if (B_n) is an increasing sequence in $\mathbb{D}$, upper continuity of measures shows that $\nu^n_\omega(\cup_{k=1}^\infty B_n) = \lim_k \nu^n_\omega(B_k)$. Thus, $\cup_{k=1}^\infty B_k \in \mathbb{D}$. We have now shown that $\mathbb{D}$ is a Dynkin class. Therefore, Lemma A.1.19 shows that $\omega \mapsto \nu^n_\omega(B)$ is $\mathcal{F}$ measurable for all $B \in \mathcal{B}_+$. The requirements given in Theorem A.1.12 are therefore satisfied, and we conclude that there exists a unique nonnegative measure λ^n on $\mathcal{B}_+ \otimes \mathcal{F}$ such that for any $B \in \mathcal{B}_+$ and $F \in \mathcal{F}$, $\lambda^n(B \times F) = \int_F \nu^n_\omega(B) \, \mathrm{d}P(\omega)$.

Now, by Theorem A.1.13, whenever $A \in \mathcal{B}_+ \otimes \mathcal{F}$, it holds that $\omega \mapsto \int 1_A(t,\omega) \, \mathrm{d}\nu^n_\omega(t)$ is $\mathcal{F}$ measurable. We will use this to argue that $\omega \mapsto \int 1_A(t,\omega) \, \mathrm{d}\nu_\omega(t)$ is $\mathcal{F}$ measurable as well. To obtain this, fix $A \in \mathcal{B}_+ \otimes \mathcal{F}$ and define A_n by putting $A_n = A \cap \{(t,\omega) \mid t \leq T_n(\omega)\}$. As $T_n(\omega)$ converges to infinity for all ω, we find that (A_n) is increasing with $A = \cup_{n=1}^\infty A_n$. Furthermore, for each fixed ω, it holds that the measures ν_ω and ν^n_ω agree on $[0, T_n(\omega)]$. Therefore, we obtain $\int 1_A(t,\omega) \, \mathrm{d}\nu_\omega(t) = \lim_n \int 1_{A_n}(t,\omega) \, \mathrm{d}\nu_\omega(t) = \lim_n \int 1_{A_n}(t,\omega) \, \mathrm{d}\nu^n_\omega(t)$ by the monotone convergence theorem. As the mappings in the latter pointwise limit are $\mathcal{F}$ measurable, we conclude that the mapping $\omega \mapsto \int 1_A(t,\omega)\,\mathrm{d}\nu_\omega(t)$ is $\mathcal{F}$ measurable. We have now shown that the proposed definition of μ_M is well-defined, and so μ_M is a well-defined mapping from $\mathcal{B}_+ \otimes \mathcal{F}$ to $[0,\infty]$.

That μ_M is a measure follows as we clearly have $\mu_M(\emptyset) = 0$ and for any sequence of disjoint sets (A_n) in $\mathcal{B}_+ \otimes \mathcal{F}$, two applications of the monotone convergence theorem yield

$$\begin{aligned}\mu_M(\cup_{n=1}^\infty A_n) &= \int\int 1_{\cup_{n=1}^\infty A_n}(t,\omega)\,\mathrm{d}[M](\omega)_t\,\mathrm{d}P(\omega) = \int\int \sum_{n=1}^\infty 1_{A_n}(t,\omega)\,\mathrm{d}[M](\omega)_t\,\mathrm{d}P(\omega)\\ &= \sum_{n=1}^\infty \int\int 1_{A_n}(t,\omega)\,\mathrm{d}[M](\omega)_t\,\mathrm{d}P(\omega) = \sum_{n=1}^\infty \mu_M(A_n).\end{aligned}$$

We conclude that μ_M is a measure. Furthermore, as $\mu_M(\{(t,\omega) \in \mathbb{R}_+ \times \Omega \mid T_n(\omega) \le t\}) \le n$, we find that μ_M is σ-finite. In the case where $M \in \mathbf{c}\mathcal{M}^2$, we find $\mu_M(\mathbb{R}_+ \times \Omega) = E[M]_\infty$, which is finite, so μ_M is a finite measure in this case.

Finally, we consider the claims regarding the stochastic integral. Let $H \in \mathfrak{I}$, we need to show that $H \cdot M \in \mathbf{c}\mathcal{M}^2$ if and only if $H \in \mathcal{L}^2(M)$. By construction, we know that we always have $H \cdot M \in \mathbf{c}\mathcal{M}_\ell$. By Lemma 1.4.10, $H \cdot M$ is in $\mathbf{c}\mathcal{M}^2$ if and only if $[H \cdot M]_\infty$ is integrable. From Lemma 2.2.7, we know that H^2 is integrable with respect to $[M]$ and $[H \cdot M]_t = \int_0^t H_s^2\,\mathrm{d}[M]_s$ for all $t \ge 0$. In particular, $E[H \cdot M]_\infty = E\int_0^\infty H_s^2\,\mathrm{d}[M]_s = \int\int H_s^2(\omega)\,\mathrm{d}[M](\omega)_s\,\mathrm{d}P(\omega)$, which is equal to $\|H\|_M$. Therefore, $H \cdot M$ is in $\mathbf{c}\mathcal{M}^2$ if and only if $\|H\|_M$ is finite, which is the case if and only if $H \in \mathcal{L}^2(M)$. Finally, in the affirmative case, Lemma 1.4.10 shows that $\|H \cdot M\|_2^2 = E(H \cdot M)_\infty^2 = E[H \cdot M]_\infty = \|H\|_M^2$ by what we already have shown, proving the final claim of the exercise. □

Solution to exercise 2.8. Note that the conclusion is well-defined, as $W_{t+h} - W_t$ is almost surely never zero. To show the result, first note that Lemma 2.2.7 yields, up to indistinguishability,

$$\begin{aligned}\int_t^{t+h} H_s\,\mathrm{d}W_s &= \int_t^{t+h} H_s 1_{[\![t,\infty[\![}\,\mathrm{d}W_s = \int_t^{t+h} H_t 1_{[\![t,\infty[\![}\,\mathrm{d}W_s + \int_t^{t+h} (H_s - H_t) 1_{[\![t,\infty[\![}\,\mathrm{d}W_s\\ &= H_t(W_{t+h} - W_t) + \int_t^{t+h} (H_s - H_t) 1_{[\![t,\infty[\![}\,\mathrm{d}W_s,\end{aligned}$$

where the indicators $[\![t,\infty[\![$ are included as a formality to ensure that the integrals are well-defined. Therefore, it suffices to show that $(W_{t+h} - W_t)^{-1}\int_t^{t+h}(H_s - H_t)1_{[\![t,\infty[\![}\,\mathrm{d}W_s$ converges in probability to zero as h tends to zero. To this end, let c be a bound for H and note that by Exercise 2.6, we have

$$E\left(\frac{1}{\sqrt{h}}\int_t^{t+h}(H_s-H_t)1_{[\![t,\infty[\![}\,\mathrm{d}W_s\right)^2=\frac{1}{h}E\int_t^{t+h}(H_s-H_t)^2\,\mathrm{d}s.$$

As H is bounded and continuous, the dominated convergence theorem shows that the above tends to zero as h tends to zero. Thus, $\frac{1}{\sqrt{h}}\int_t^{t+h}(H_s-H_t)1_{[\![t,\infty[\![}\,\mathrm{d}W_s$ tends to zero in $\mathcal{L}^2$. Now fix $\delta, M>0$. With $Y_h=\frac{1}{\sqrt{h}}\int_t^{t+h}(H_s-H_t)1_{[\![t,\infty[\![}\,\mathrm{d}W_s$, we have

$$\begin{aligned}
&P\left(\left|(W_{t+h}-W_t)^{-1}\int_t^{t+h}(H_s-H_t)1_{[\![t,\infty[\![}\,\mathrm{d}W_s\right|>\delta\right)\\
\leq\;& P\left(\left|\frac{\sqrt{h}}{W_{t+h}-W_t}Y_h\right|>\delta,\left|\frac{\sqrt{h}}{W_{t+h}-W_t}\right|\leq M\right)+P\left(\left|\frac{\sqrt{h}}{W_{t+h}-W_t}\right|>M\right)\\
\leq\;& P\left(|Y_h|>\frac{\delta}{M}\right)+P\left(\left|\frac{\sqrt{h}}{W_{t+h}-W_t}\right|>M\right).
\end{aligned}$$

Here, $P(\sqrt{h}(W_{t+h}-W_t)^{-1}>M)$ does not depend on h, as $(W_{t+h}-W_t)(\sqrt{h})^{-1}$ is a standard normal distribution. For definiteness, we define $\varphi(M)=P(\sqrt{h}(W_{t+h}-W_t)^{-1}>M)$. The above then allows us to conclude

$$\limsup_{h\to\infty}P\left(\left|(W_{t+h}-W_t)^{-1}\int_t^{t+h}(H_s-H_t)1_{[\![t,\infty[\![}\,\mathrm{d}W_s\right|>\delta\right)\leq\varphi(M),$$

and letting M tend to infinity, we obtain the desired result. □

Solution to exercise 2.9. For $q>p$, we have

$$\sum_{k=1}^{2^n}|X_{t_k^n}-X_{t_{k-1}^n}|^q\leq\left(\max_{k\leq 2^n}|X_{t_k^n}-X_{t_{k-1}^n}|^{q-p}\right)\sum_{k=1}^{2^n}|X_{t_k^n}-X_{t_{k-1}^n}|^p.$$

As X has continuous paths, the paths of X are uniformly continuous on $[0,t]$. In particular, $\max_{k\leq 2^n}|X_{t_k^n}-X_{t_{k-1}^n}|^{q-p}$ converges almost surely to zero. Therefore, this variable also converges to zero in probability, and so $\sum_{k=1}^{2^n}|X_{t_k^n}-X_{t_{k-1}^n}|^q$ converges in probability to zero, as was to be proven. □

Solution to exercise 2.10. First, consider the case where $H=\frac{1}{2}$. In this case, X is a process whose finite-dimensional distributions are normally distributed with mean zero and with the property that for any $s,t\geq 0$, $EX_sX_t=\frac{1}{2}(t+s-|t-s|)$. For a Brownian motion W, we have that when $0\leq s\leq t$, $EW_sW_t=EW_s(W_t-W_s)+EW_s^2=EW_sE(W_t-W_s)+EW_s^2=s$. In this case, $|t-s|=t-s$, and so $EW_sW_t=\frac{1}{2}(t+s-|t-s|)$. In the case where $0\leq t\leq s$,

$EW_sW_t = t = \frac{1}{2}(t+s+(t-s)) = \frac{1}{2}(t+s-|t-s|)$ as well. Thus, X and W are processes whose finite-dimensional distributions are normally distributed and have the same mean and covariance structure. Therefore, their distributions are the same, and so X has the distribution of a Brownian motion.

In order to show that X is not in $\mathbf{c}\mathcal{S}$ when $H \neq \frac{1}{2}$, we first fix $t \geq 0$ and consider the sum $\sum_{k=1}^{2^n} |X_{t_k^n} - X_{t_{k-1}^n}|^p$ for $p \geq 0$. Understanding the convergence of such sums will allow us to prove our desired result. We know that the collection of variables $X_{t_k^n} - X_{t_{k-1}^n}$ follows a multivariate normal distribution with $E(X_{t_k^n} - X_{t_{k-1}^n}) = 0$ and, using the property that $EX_sX_t = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$, we obtain

$$\begin{aligned} E(X_{t_k^n} - X_{t_{k-1}^n})(X_{t_i^n} - X_{t_{i-1}^n}) &= EX_{t_k^n}X_{t_i^n} - EX_{t_k^n}X_{t_{i-1}^n} - EX_{t_{k-1}^n}X_{t_i^n} + EX_{t_{k-1}^n}X_{t_{i-1}^n} \\ &= 2^{-2nH}\tfrac{1}{2}(|k-i+1|^{2H} + |k-1-i|^{2H} - 2|k-i|^{2H}). \end{aligned}$$

Here, the parameter n only enters the expression through the constant multiplicative factor 2^{-2nH}. Therefore, as normal distributions are determined by their covariance structure, it follows that the distribution of the variables $(X_{t_k^n} - X_{t_{k-1}^n})$ for $k \leq 2^n$ is the same as the distribution of the variables $2^{-nH}(X_k - X_{k-1})$ for $k \leq 2^n$. In particular, it follows that the distributions of $\sum_{k=1}^{2^n} |X_{t_k^n} - X_{t_{k-1}^n}|^p$ and $2^{-npH}\sum_{k=1}^{2^n} |X_k - X_{k-1}|^p$ are the same. We wish to apply the ergodic theorem for stationary processes to the sequence $(X_k - X_{k-1})_{k\geq 1}$. To this end, we first check that this sequence is in fact stationary. To do so, we need to check for any $m \geq 1$ that the variables $X_k - X_{k-1}$ for $k \leq n$ have the same distribution as the variables $X_{m+k} - X_{m+k-1}$ for $k \leq n$. As both families of variables are normally distributed with mean zero, it suffices to check that the covariance structure is the same. However, by what we already have shown,

$$\begin{aligned} & E(X_k - X_{k-1})(X_{t_i^n} - X_{i-1}) \\ = \; & \tfrac{1}{2}(|k-i+1|^{2H} + |k-1-i|^{2H} - 2|k-i|^{2H}) \\ = \; & E(X_{m+k} - X_{m+k-1})(X_{t_{m+i}^n} - X_{m+i-1}). \end{aligned}$$

This allows us to conclude that the sequence $(X_k - X_{k-1})_{k\geq 1}$ is stationary. As $E|X_k - X_{k-1}|^p$ is finite, the ergodic theorem shows that $\frac{1}{n}\sum_{k=1}^{n} |X_k - X_{k-1}|^p$ converges almost surely and in $\mathcal{L}^1$ to some variable Z_p, where Z_p is integrable and $EZ_p = E|X_1 - X_0|^p = E|X_1|^p > 0$. This property ensures that Z_p is not almost surely zero. Next, we observe that we have $2^{-npH}\sum_{k=1}^{2^n} |X_k - X_{k-1}|^p = 2^{n(1-pH)}(\frac{1}{2^n}\sum_{k=1}^{2^n} |X_k - X_{k-1}|^p)$, where we have just checked that the latter factor always converges almost surely and in $\mathcal{L}^1$ to Z_p. Having this result at hand, we are ready to prove that X is not in $\mathbf{c}\mathcal{S}$ when $H \neq \frac{1}{2}$.

First consider the case where $H < \frac{1}{2}$. In this case, $\frac{1}{H} > 2$. If $X \in \mathbf{c}\mathcal{S}$, we have that $\sum_{k=1}^{2^n} |X_{t_k^n} - X_{t_{k-1}^n}|^2$ converges in probability to $[X]_t$. Therefore, by Exercise, 2.9, we find that $\sum_{k=1}^{2^n}|X_{t_k^n}-X_{t_{k-1}^n}|^{\frac{1}{H}}$ converges to zero in probability. As $\sum_{k=1}^{2^n}|X_{t_k^n}-X_{t_{k-1}^n}|^{\frac{1}{H}}$ has the same distribution as $\frac{1}{2^n}\sum_{k=1}^{2^n}|X_k-X_{k-1}|^{\frac{1}{H}}$, we conclude that this sequence converges to zero in probability. However, this is in contradiction with what we have already shown, namely that this sequence converges in probability to a variable $Z_{\frac{1}{H}}$ which is not almost surely zero. We conclude that in the case $H<\frac{1}{2}$, X cannot be in $\mathbf{c}\mathcal{S}$.

Next, consider the case $H>\frac{1}{2}$. Again, we assume that $X\in\mathbf{c}\mathcal{S}$ and hope for a contradiction. In this case, $1-2H<0$, so $2^{n(1-2H)}$ converges to zero and so, by our previous results, $2^{n(1-2H)}(\frac{1}{2^n}\sum_{k=1}^{2^n}|X_k-X_{k-1}|^2)$ converges to zero in probability. Therefore, we find that $\sum_{k=1}^{2^n}|X_{t_k^n}-X_{t_{k-1}^n}|^2$ converges to zero in probability as well, since this sequence has the same distribution as the previously considered sequence. By Theorem 2.3.3, this implies $[X]_t=0$ almost surely. As $t\geq 0$ was arbitrary, we conclude that $[X]$ is evanescent. With $X=M+A$ being the decomposition of X into its continuous local martingale part and its continuous finite variation part, we have $[X]=[M]$, so $[M]$ is evanescent and so by Lemma 1.4.9, M is evanescent. Therefore, X almost surely has paths of finite variation. In particular, $\sum_{k=1}^{2^n}|X_{t_k^n}-X_{t_{k-1}^n}|$ is almost surely convergent, in particular convergent in probability. As $H<1$, we have $\frac{1}{H}>1$, so by Exercise 2.9, $\sum_{k=1}^{2^n}|X_{t_k^n}-X_{t_{k-1}^n}|^{\frac{1}{H}}$ converges in probability to zero. Therefore, $\frac{1}{2^n}\sum_{k=1}^{2^n}|X_k-X_{k-1}|^{\frac{1}{H}}$ converges to zero in probability as well. As in the previous case, this is in contradiction with the fact that that this sequence converges in probability to a variable $Z_{\frac{1}{H}}$ which is not almost surely zero. We conclude that in the case $H<\frac{1}{2}$, X cannot be in $\mathbf{c}\mathcal{S}$ either. □

Solution to exercise 2.11. By Itô's formula of Theorem 2.3.5 and Theorem 1.4.8, we have

$$\begin{aligned} f(W_t)-f(0) &= \sum_{i=1}^p\int_0^t\frac{\partial f}{\partial x_i}(W_s)\,\mathrm{d}W_s^i+\frac{1}{2}\sum_{i=1}^p\sum_{j=1}^p\int_0^t\frac{\partial^2 f}{\partial x_i\partial x_j}(W_s)\,\mathrm{d}[W^i,W^j]_s \\ &= \sum_{i=1}^p\int_0^t\frac{\partial f}{\partial x_i}(W_s)\,\mathrm{d}W_s^i+\frac{1}{2}\sum_{i=1}^p\int_0^t\frac{\partial^2 f}{\partial x_i\partial x_j}(W_s)\,\mathrm{d}s, \end{aligned}$$

and by our assumptions on f, this is equal to $\sum_{i=1}^p\int_0^t\frac{\partial f}{\partial x_i}(W_s)\,\mathrm{d}W_s^i$, since the second term vanishes. Here, $\sum_{i=1}^p\int_0^t\frac{\partial f}{\partial x_i}(W_s)\,\mathrm{d}W_s^i$ is in $\mathbf{c}\mathcal{M}_\ell$. Therefore, $f(W_t)-f(0)$ is in $\mathbf{c}\mathcal{M}_\ell$ and so $f(W_t)$ is a continuous local martingale. □

Solution to exercise 2.12. Define the two-dimensional process X by putting $X_t=(t,W_t)$. With $A_t=t$, we have $[A,W]_t=0$, so Itô's formula of Theorem 2.3.5 shows

$$\begin{aligned} f(t,W_t)-f(0,0) &= \int_0^t\frac{\partial f}{\partial t}(s,W_s)\,\mathrm{d}s+\int_0^t\frac{\partial f}{\partial x}(s,W_s)\,\mathrm{d}W_s+\frac{1}{2}\int_0^t\frac{\partial^2 f}{\partial x^2}(s,W_s)\,\mathrm{d}s \\ &= \int_0^t\frac{\partial f}{\partial t}(s,W_s)+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,W_s)\,\mathrm{d}s+\int_0^t\frac{\partial f}{\partial x}(s,W_s)\,\mathrm{d}W_s, \end{aligned}$$

which is equal to $\int_0^t \frac{\partial f}{\partial x}(s, W_s)\,\mathrm{d}W_s$ by our assumptions on f, and this is in $\mathbf{c}\mathcal{M}_\ell$. Therefore, $f(t, W_t)$ is a continuous local martingale, and $f(t, W_t) = f(0,0) + \int_0^t \frac{\partial f}{\partial x}(s, W_s)\,\mathrm{d}W_s$. $\square$

Solution to exercise 2.13. Define $X_t = (M_t, [M]_t)$ and put $f(x, y) = \exp(x - \frac{1}{2}y)$. Then f is C^2 with $\frac{\partial^2 f}{\partial x^2}(x, y) = \frac{\partial f}{\partial x}(x, y) = f(x, y)$ and $\frac{\partial f}{\partial y}(x, y) = -\frac{1}{2}f(x, y)$, and $\mathcal{E}(M)_t = f(X_t)$, so Itô's formula yields

$$\begin{aligned}
\mathcal{E}(M)_t &= 1 + \int_0^t \mathcal{E}(M)_s\,\mathrm{d}M_s - \frac{1}{2}\int_0^t \mathcal{E}(M)_s\,\mathrm{d}[M]_s + \frac{1}{2}\int_0^t \mathcal{E}(M)_s\,\mathrm{d}[M]_s \\
&= 1 + \int_0^t \mathcal{E}(M)_s\,\mathrm{d}M_s.
\end{aligned}$$

This proves that $\mathcal{E}(M)$ as defined satisfies the stochastic differential equation given, in particular $\mathcal{E}(M)$ is a continuous local martingale with initial value one as the stochastic integral of a process with respect to an element of $\mathbf{c}\mathcal{M}_\ell$ is in $\mathbf{c}\mathcal{M}_\ell$. It remains to show that $\mathcal{E}(M)$ is the unique solution to the stochastic differential equation. Assume therefore given a process $Y \in \mathbf{c}\mathcal{S}$ satisfying $Y_t = 1 + \int_0^t Y_s\,\mathrm{d}M_s$ up to indistinguishability. In particular, $Y \in \mathbf{c}\mathcal{M}_\ell$. Define $Z_t = \mathcal{E}(M)_t^{-1} Y_t$, we wish to show that Z is indistinguishable from one. Define $f(x, y) = \frac{y}{x}$. The function f is C^2 on the open set $(0, \infty) \times \mathbb{R}$, and the two-dimensional semimartingale $(\mathcal{E}(M), Y)$ takes its values in this set. Therefore, we may apply the local version of Itô's formula, Corollary 2.3.6, to $(\mathcal{E}(M), Y)$ and f. First note that as $\mathcal{E}(M) = 1 + \mathcal{E}(M) \cdot M$, we find $[\mathcal{E}(M)] = [\mathcal{E}(M) \cdot M] = \mathcal{E}(M)^2 \cdot [M]$ and also

$$[\mathcal{E}(M), Y] = [\mathcal{E}(M) \cdot M, Y \cdot M] = \mathcal{E}(M) \cdot [M, Y \cdot M] = \mathcal{E}(M)Y \cdot [M].$$

All in all, this yields

$$\begin{aligned}
Z_t &= f(\mathcal{E}(M)_t, Y_t) \\
&= 1 - \int_0^t \frac{Y_s}{\mathcal{E}(M)_s^2}\,\mathrm{d}\mathcal{E}(M)_s + \int_0^t \frac{1}{\mathcal{E}(M)_s}\,\mathrm{d}Y_s \\
&+ \frac{1}{2}\int_0^t \frac{2Y_s}{\mathcal{E}(M)_s^3}\,\mathrm{d}[\mathcal{E}(M)]_s - \frac{1}{2}\int_0^t \frac{1}{\mathcal{E}(M)_s^2}\,\mathrm{d}[\mathcal{E}(M), Y]_s - \frac{1}{2}\int_0^t \frac{1}{\mathcal{E}(M)_s^2}\,\mathrm{d}[Y, \mathcal{E}(M)]_s \\
&= 1 - \int_0^t \frac{Y_s}{\mathcal{E}(M)_s}\,\mathrm{d}M_s + \int_0^t \frac{Y_s}{\mathcal{E}(M)_s}\,\mathrm{d}M_s + \frac{1}{2}\int_0^t \frac{2Y_s}{\mathcal{E}(M)_s}\,\mathrm{d}[M]_s - \frac{1}{2}\int_0^t \frac{2Y}{\mathcal{E}(M)_s}\,\mathrm{d}[M]_s,
\end{aligned}$$

which is one. Thus, we conclude that almost surely, $Z_t = 1$ for all $t \geq 0$, and so $\mathcal{E}(M)$ and Y are indistinguishable. This proves that up to indistinguishability, $\mathcal{E}(M)$ is the only solution in $Y \in \mathbf{c}\mathcal{S}$ satisfying $Y_t = 1 + \int_0^t Y_s\,\mathrm{d}[M]_s$. $\square$

Solution to exercise 2.14. Clearly, $\mathcal{E}(M)$ and $\mathcal{E}(N)$ are indistinguishable if M and N are indistinguishable. Conversely, assume that $\mathcal{E}(M)$ and $\mathcal{E}(N)$ are indistinguishable. In particular, $M - \frac{1}{2}[M]$ and $N - \frac{1}{2}[N]$ are indistinguishable. This yields $M - N = \frac{1}{2}([M] - [N])$ up to indistinguishability, so $M-N$ is almost surely equal to an element of $\mathbf{c}\mathcal{M}_\ell$ with paths of finite variation. Therefore, Lemma 1.4.5 shows that M and N are indistinguishable. $\square$

Solution to exercise 2.15. Defining $f(x)=\log(1+x)$, we find that $f'(x)=\frac{1}{1+x}$ and that $f''(x)=-\frac{1}{(1+x)^2}$. By Theorem A.1.17 we then obtain $\log(1+x)=x+\frac{1}{2}x^2+s(x)$, where the remainder term satisfies $s(x)=(f''(\xi)-f''(0))x^2=(1+f''(\xi))x^2$ for some ξ between zero and x, depending on x. We may use this to obtain

$$\begin{aligned}
Z^n_t &= \exp\left(\sum_{k=1}^{2^n}\log(M_{t^n_k}-M_{t^n_{k-1}})\right)\\
&= \exp\left(\sum_{k=1}^{2^n} M_{t^n_k}-M_{t^n_{k-1}}+\frac{1}{2}(M_{t^n_k}-M_{t^n_{k-1}})^2+s(M_{t^n_k}-M_{t^n_{k-1}})\right)\\
&= \exp\left(M_t+\frac{1}{2}\sum_{k=1}^{2^n}(M_{t^n_k}-M_{t^n_{k-1}})^2+\sum_{k=1}^{2^n}s(M_{t^n_k}-M_{t^n_{k-1}})\right).
\end{aligned}$$

By Theorem 2.3.3, $\frac{1}{2}\sum_{k=1}^{2^n}(M_{t^n_k}-M_{t^n_{k-1}})^2$ converges in probability to $\frac{1}{2}[M]_t$. In order to show the result, it therefore suffices to show that $\sum_{k=1}^{2^n}s(M_{t^n_k}-M_{t^n_{k-1}})$ converges to zero in probability. To this end, note that for some ξ^n_k on the line segment between zero and $M_{t^n_k}-M_{t^n_{k-1}}$, we have

$$\begin{aligned}
\sum_{k=1}^{2^n}s(M_{t^n_k}-M_{t^n_{k-1}}) &= \sum_{k=1}^{2^n}\left(1-\frac{1}{(1+\xi^n_k)^2}\right)(M_{t^n_k}-M_{t^n_{k-1}})^2\\
&\le \sum_{k=1}^{2^n}\left(1-\frac{1}{(1+|M_{t^n_k}-M_{t^n_{k-1}}|)^2}\right)(M_{t^n_k}-M_{t^n_{k-1}})^2\\
&\le \max_{k\le 2^n}(1-(1+|M_{t^n_k}-M_{t^n_{k-1}}|)^{-2})\sum_{k=1}^{2^n}(M_{t^n_k}-M_{t^n_{k-1}})^2
\end{aligned}$$

Therefore, it suffices to prove that the latter converges to zero in probability. To this end, let $\varepsilon>0$ and fix ω. As M has continuous paths, M has uniformly continuous paths on $[0,t]$, and therefore, for some n large enough, $|M_{t^n_k}-M_{t^n_{k-1}}|\le\varepsilon$ for all $k\le 2^n$, showing that $\max_{k\le 2^n}(1-(1+|M_{t^n_k}-M_{t^n_{k-1}}|)^{-2})\le 1-(1-\varepsilon)^{-2}$. Thus, the maximum tends pointwise to zero, in particular it converges to zero in probability. As $\sum_{k=1}^{2^n}(M_{t^n_k}-M_{t^n_{k-1}})^2$ tends to $[M]_t$ by Theorem 2.3.3, we conclude that $\sum_{k=1}^{2^n}s(M_{t^n_k}-M_{t^n_{k-1}})$ converges in probability to zero, and the result follows. $\square$

Solution to exercise 2.16. From the expression $\mathcal{E}(M)_t=\exp(M_t-\frac{1}{2}[M]_t)$, we see that $\mathcal{E}(M)$ is nonnegative. From Exercise 2.13, we know that $\mathcal{E}(M)$ is a continuous local martingale.

Letting (T_n) be a localising sequence such that $\mathcal{E}(M)^{T_n}$ is a martingale, we may apply Fatou's lemma to obtain

$$\begin{aligned} E(\mathcal{E}(M)_t|\mathcal{F}_s) &= E(\liminf_n \mathcal{E}(M)^{T_n}_t|\mathcal{F}_s) \le \liminf_n E(\mathcal{E}(M)^{T_n}_t|\mathcal{F}_s) \\ &= \liminf_n \mathcal{E}(M)^{T_n}_s = \mathcal{E}(M)_s. \end{aligned}$$

Therefore, $\mathcal{E}(M)$ is a supermartingale. In particular, since $\mathcal{E}(M)$ has initial value 1, we conclude $E\mathcal{E}(M)_t \le 1$. Therefore, $\mathcal{E}(M)$ is bounded in $\mathcal{L}^1$, so from Theorem 1.2.4, we find that $\mathcal{E}(M)$ is almost surely convergent. By Fatou's lemma, $E\mathcal{E}(M)_\infty \le \liminf_t E\mathcal{E}(M)_t = 1$, as desired. $\square$

Solution to exercise 2.17. We first prove the criterion for $\mathcal{E}(M)$ to be a uniformly integrable martingale. Assume that $E\mathcal{E}(M)_\infty = 1$, we will show that $\mathcal{E}(M)$ is uniformly integrable. By Exercise 2.16, $\mathcal{E}(M)$ is a nonnegative supermartingale. Therefore, Theorem 1.2.7 shows that for any stopping time T that $1 = E\mathcal{E}(M)_\infty \le E\mathcal{E}(M)_T \le 1$. Thus, $E\mathcal{E}(M)_T = 1$ for all stopping times T, and so, Lemma 1.2.9 shows that $\mathcal{E}(M)$ is a uniformly integrable martingale. Conversely, assume that $\mathcal{E}(M)$ is a uniformly integrable martingale. Theorem 1.2.5 then shows that $\mathcal{E}(M)_t$ converges to $\mathcal{E}(M)_\infty$ in $\mathcal{L}^1$, and so $E\mathcal{E}(M)_\infty = \lim_t E\mathcal{E}(M)_t = 1$. This proves the criterion for $\mathcal{E}(M)$ to be a uniformly integrable martingale.

Next, we consider the martingale result. From Lemma 1.2.9, we find that $\mathcal{E}(M)$ is a martingale if and only if $\mathcal{E}(M)^t$ is a uniformly integrable martingale for all $t \ge 0$. From $\mathcal{E}(M)_t = \exp(M_t - \frac{1}{2}[M]_t)$, we note that $\mathcal{E}(M)^t = \mathcal{E}(M^t)$. Therefore, $\mathcal{E}(M)$ is a martingale if and only if $\mathcal{E}(M^t)$ is a uniformly integrable martingale for all $t \ge 0$. This is the case if and only if $E\mathcal{E}(M^t)_\infty = 1$ for all $t \ge 0$. As $\mathcal{E}(M^t)_\infty = \mathcal{E}(M)_t$, the result follows. $\square$

Solution to exercise 2.18. As f is adapted and continuous, $f \in \mathfrak{I}$ by Lemma 2.2.5. Put $t^n_k = kt2^{-n}$. By Theorem 2.3.2, we find that $\sum_{k=1}^{2^n} f(t^n_{k-1})(W_{t^n_k} - W_{t^n_{k-1}})$ converges in probability to $\int_0^t f(s)\,\mathrm{d}W_s$. However, the finite sequence of variables $W_{t^n_k} - W_{t^n_{k-1}}$ for $k = 1, \ldots, 2^n$ are independent and normally distributed with mean zero and variance $t2^{-n}$. Therefore, we find that $\sum_{k=1}^{2^n} f(t^n_{k-1})(W_{t^n_k} - W_{t^n_{k-1}})$ is normally distributed with mean zero and variance $t2^{-n}\sum_{k=1}^{2^n} f(t^n_{k-1})^2$. As f is continuous, this converges to $\int_0^t f(s)^2\,\mathrm{d}s$. Therefore, $\sum_{k=1}^{2^n} f(t^n_{k-1})(W_{t^n_k} - W_{t^n_{k-1}})$ converges weakly to a normal distribution with mean zero and variance $\int_0^t f(s)^2\,\mathrm{d}s$. As this sequence of variables also converges in probability to $\int_0^t f(s)\,\mathrm{d}W_s$, and convergence in probability implies weak convergence, we conclude by uniqueness of limits that $\int_0^t f(s)\,\mathrm{d}W_s$ follows a normal distribution with mean zero and variance $\int_0^t f(s)^2\,\mathrm{d}s$. $\square$

Solution to exercise 2.19. By Itô's formula of Theorem 2.3.5, as well as Lemma 2.1.8 and

Lemma 2.2.7, we have

$$\begin{aligned}[f(X), g(Y)]_t &= [f'(X)\cdot X + \tfrac{1}{2}f''(X)\cdot[X], g'(Y)\cdot Y + \tfrac{1}{2}g''(Y)\cdot[Y]]_t \\ &= [f'(X)\cdot X, g'(Y)\cdot Y]_t = \int_0^t f'(X_s)g'(Y_s)\,\mathrm{d}[X,Y]_s,\end{aligned}$$

up to indistinguishability. This shows in particular that with W an $\mathcal{F}_t$ Brownian motion, $[W^p]_t = \int_0^t (pW_s^{p-1})^2\,\mathrm{d}[W]_s = p^2\int_0^t W_s^{2(p-1)}\,\mathrm{d}s$. □

Solution to exercise 2.20. In the case where $i = j$, we may apply Itô's formula with the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ and obtain $(W^i)_t^2 = 2\int_0^t W_s^i\,\mathrm{d}W_s^i + t$. Lemma 2.1.8 then shows that $[(W^i)^2]_t = 4[W^i\cdot W^i]_t = 4\int_0^t (W_s^i)^2\,\mathrm{d}s$. Next, consider the case where $i \neq j$. Applying Itô's formula with the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = xy$, we have

$$W_t^i W_t^j = \int_0^t W_s^i\,\mathrm{d}W_s^j + \int_0^t W_s^j\,\mathrm{d}W_s^i.$$

Using Lemma 2.1.8 and Lemma 2.2.7, we then obtain

$$\begin{aligned}[W^iW^j]_t &= [W^i\cdot W^j]_t + 2[W^i\cdot W^j, W^j\cdot W^i]_t + [W^j\cdot W^i]_t \\ &= \int_0^t (W_s^i)^2\,\mathrm{d}s + 2\int_0^t W_s^iW_s^j\,\mathrm{d}[W^i,W^j]_s + \int_0^t (W_s^j)^2\,\mathrm{d}s \\ &= \int_0^t (W_s^i)^2\,\mathrm{d}s + \int_0^t (W_s^j)^2\,\mathrm{d}s.\end{aligned}$$

□

Solution to exercise 2.21. Define $H_t = \operatorname{sgn}(W_t)$, we then have $X = H\cdot W$. Fix $t \geq 0$. By Exercise 2.6, $H\cdot W$ stopped at any deterministic timepoint is in $\mathbf{c}\mathcal{M}^2$. As X_t thus is square integrable and W has moments of all orders, the Cauchy-Schwartz inequality shows that both X_tW_t and $X_tW_t^2$ are integrable.

As for the values of the integral, first note that as $H\cdot W$ and W are both in $\mathbf{c}\mathcal{M}_\ell$, it holds that $(H\cdot W)W - [H\cdot W, W]$ is in $\mathbf{c}\mathcal{M}_\ell$ as well. This same property holds for W as well. Therefore, using Lemma 1.4.10, $(H\cdot W)W - [H\cdot W, W]$ is in $\mathbf{c}\mathcal{M}$, and so we obtain

$$EX_tW_t = E(H\cdot W)_tW_t = E[H\cdot W, W]_t = E\int_0^t H_s\,\mathrm{d}[W,W]_s = E\int_0^t H_s\,\mathrm{d}s.$$

Now, as H is bounded by the constant one, we may interchange integration and expectation in the above. And as W_t for any $t \geq 0$ has a distribution which is symmetric around zero, we find $E\int_0^t H_s\,\mathrm{d}s = \int_0^t EH_s\,\mathrm{d}s = 0$, proving that $EX_tW_t = 0$.

As regards the other expectation, we have by Itô's formula that $W_t^2 = 2\int_0^t W_s \, \mathrm{d}W_s + t$ up to indistinguishability. Again by Exercise 2.6, we find that $W \cdot W \in \mathbf{c}\mathcal{M}$ and that this process is in $\mathbf{c}\mathcal{M}^2$ when stopped at any deterministic timepoint. Therefore, by Lemma 1.4.10, $(H \cdot W)(W \cdot W) - [H \cdot W, W \cdot W]$ is in $\mathbf{c}\mathcal{M}$, and so, using Lemma 2.2.7 and that $H_t W_t = |W_t|$,

$$EX_t W_t^2 = 2E(H \cdot W)_t (W \cdot W)_t + tEX_t = 2E[H \cdot W, W \cdot W]_t = 2\int_0^t E|W_s| \, \mathrm{d}s.$$

Now, W_s follows a normal distribution with mean zero and variance s, so $E|W_s| = \sqrt{s}\sqrt{2/\pi}$. Therefore, we obtain $2\int_0^t E|W_s| \, \mathrm{d}s = 2\sqrt{2/\pi}\int_0^t \sqrt{s} \, \mathrm{d}s = 2\sqrt{2/\pi}\frac{2}{3}t^{\frac{3}{2}} = 2^{\frac{5}{2}}(3\sqrt{\pi})^{-1}t^{\frac{3}{2}}$, which proves the result. □

Solution to exercise 2.22. Let $X = N + B$ be the decomposition of X into its continuous local martingale and continuous finite variation parts. We first define $f : \mathbb{R}^2 \to \mathbb{R}$ by putting $f(x, y) = \exp(x - \frac{1}{2}y)$, f is then C^2 and satisfies $\frac{\partial^2 f}{\partial x^2} f(x, y) = \frac{\partial f}{\partial x} = f(x, y)$ and $\frac{\partial f}{\partial y} f(x, y) = -\frac{1}{2} f(x, y)$. Then $M_t^\alpha = f(\alpha X_t, \alpha^2 A_t)$, so Applying Itô's formula, we obtain

$$\begin{aligned} M_t^\alpha &= f(\alpha X_t, \alpha^2 A_t) \\ &= 1 + \alpha \int_0^t M_s^\alpha \, \mathrm{d}X_s - \frac{\alpha^2}{2}\int_0^t M_s^\alpha \, \mathrm{d}A_s + \frac{\alpha^2}{2}\int_0^t M_s^\alpha \, \mathrm{d}[X]_s \\ &= 1 + \alpha \int_0^t M_s^\alpha \, \mathrm{d}N_s + \left(\alpha \int_0^t M_s^\alpha \, \mathrm{d}B_s + \frac{\alpha^2}{2}\int_0^t M_s^\alpha \, \mathrm{d}[X]_s - \frac{\alpha^2}{2}\int_0^t M_s^\alpha \, \mathrm{d}A_s\right). \end{aligned}$$

As M^α and N both are in $\mathbf{c}\mathcal{M}_\ell$, we may apply Lemma 1.4.5 to show that the process $\alpha\int_0^t M_s^\alpha \, \mathrm{d}B_s + \frac{\alpha^2}{2}\int_0^t M_s^\alpha \, \mathrm{d}[X]_s - \frac{\alpha^2}{2}\int_0^t M_s^\alpha \, \mathrm{d}A_s$ is evanescent, so that $M_t^\alpha = 1 + \alpha\int_0^t M_s^\alpha \, \mathrm{d}N_s$ and $[M^\alpha]_t = \alpha^2 \int_0^t (M_s^\alpha)^2 \, \mathrm{d}[N]_s$. Applying the local version of Itô's formula, Corollary 2.3.6, with the mapping $x \mapsto \log x$ defined on $(0, \infty)$, we then obtain

$$\alpha X_t - \tfrac{\alpha^2}{2} A_t = \int_0^t \frac{1}{M_s^\alpha} \, \mathrm{d}M_s^\alpha - \frac{1}{2}\int_0^t \frac{1}{(M^\alpha)_s^2} \, \mathrm{d}[M^\alpha]_s = \alpha N_t - \tfrac{\alpha^2}{2}[N]_t.$$

This shows $\alpha B = \frac{\alpha^2}{2}(A - [N]) = \frac{\alpha^2}{2}(A - [X])$. In particular, for any $\alpha \neq 0$, $B = \frac{\alpha}{2}(A - [X])$. This implies that $[X]$ and A are indistinguishable and that B is evanescent, so $X \in \mathbf{c}\mathcal{M}_\ell$. □

Solution to exercise 2.23. By Itô's formula, $f(W_t) = \alpha + \int_0^t f'(W_s) \, \mathrm{d}W_s + \frac{1}{2}\int_0^t f''(W_s) \, \mathrm{d}s$. We will prove the formula for $Ef(W_t)$ by arguing that the integral with respect to Brownian motion is in $\mathbf{c}\mathcal{M}$ and that interchange of integration and expectation is allowed in the final term. To prove the first claim, first note that

$$\begin{aligned} |f'(x)| &\leq \left|\int_0^x f''(y) \, \mathrm{d}y\right| \leq \int_0^{|x|} |f''(y)| \, \mathrm{d}y \\ &\leq \int_0^{|x|} C + \exp(\beta y) \, \mathrm{d}y = C|x| + \frac{1}{\beta}(\exp(\beta|x|) - 1), \end{aligned}$$

so we also have $|f'(x)| \leq C^* + \exp(\beta^*|x|)$ for some $C^*, \beta^* > 0$. Now, using that for any variable X which is standard normally distributed, $E\exp(tX) = \exp(\frac{1}{2}t^2)$ for $t \in \mathbb{R}$, we obtain by Tonelli's theorem that

$$\begin{aligned} E\int_0^t f'(W_s)^2 \,\mathrm{d}s &= \int_0^t Ef'(W_s)^2 \,\mathrm{d}s \leq \int_0^t E(C + \exp(\beta|W_s|))^2 \,\mathrm{d}s \\ &\leq \int_0^t 4C^2 + 4E\exp(2\beta|W_s|) \,\mathrm{d}s \leq \int_0^t 4(1+C^2) + 4E\exp(2\beta W_s) \,\mathrm{d}s \\ &= 4(1+C^2)t + \int_0^t 4\exp(2s\beta^2) \,\mathrm{d}s, \end{aligned}$$

which is finite, and so Exercise 2.6 shows that $\int_0^t f'(W_s)\,\mathrm{d}W_s$ is in $\mathbf{c}\mathcal{M}$. From this we conclude $Ef(W_t) = \alpha + \frac{1}{2}E\int_0^t f''(W_s)\,\mathrm{d}s$. By calculations analogous to those above, we also conclude that $E\int_0^t |f''(W_s)|\,\mathrm{d}s$ is finite, and so we finally obtain $Ef(W_t) = \alpha + \int_0^t Ef''(W_s)\,\mathrm{d}s$, as desired.

As for the formula for EW_t^p, we note that for $p \geq 2$, the mappings $x \mapsto x^p$ clearly satisfy the criteria for exponential growth considered above. We therefore obtain for any $p \in \mathbb{N}$ with $p \geq 2$ that $EW_t^p = \frac{1}{2}p(p-1)\int_0^t EW_t^{p-2}\,\mathrm{d}s$. We prove the formulas for EW_t^p by recursion. As $EW_t = 0$ for all $t \geq 0$, it follows from the formula above that $EW_t^{2p-1} = 0$ for all $p \in \mathbb{N}$ and $t \geq 0$. As for the moments of even order, we know that $EW_t^0 = 1$ for all $t \geq 0$. Assume inductively that $EW_t^{2p} = t^p \prod_{i=1}^p (2i-1)$ for all $t \geq 0$. The formula above then shows

$$\begin{aligned} EW_t^{2(p+1)} &= \frac{1}{2}(2p+2)(2p+1)\int_0^t s^p \prod_{i=1}^p (2i-1)\,\mathrm{d}s \\ &= \left(\prod_{i=1}^p (2i-1)\right)\frac{1}{2}(2p+2)(2p+1)\frac{1}{p+1}t^{p+1} \\ &= t^{p+1}\left(\prod_{i=1}^p (2i-1)\right)(2p+1) = t^{p+1}\prod_{i=1}^{p+1}(2i-1), \end{aligned}$$

as desired. □

Solution to exercise 2.24. By Theorem 2.3.3, we know that $\sum_{k=1}^{2^n}(M_{t_k} - M_{t_{k-1}})^2$ converges to $[M]_t$ in probability. Therefore, by Lemma A.2.5, we have convergence in $\mathcal{L}^1$ if and only if the sequence of variables is uniformly integrable. To show uniform integrability, it suffices by Lemma A.2.4 to show boundedness in $\mathcal{L}^2$. To prove this, we first note the relationship $E(\sum_{k=1}^{2^n}(M_{t_k} - M_{t_{k-1}})^2)^2 = E\sum_{k=1}^{2^n}(M_{t_k} - M_{t_{k-1}})^4 + E\sum_{k\neq i}(M_{t_k} - M_{t_{k-1}})^2(M_{t_i} - M_{t_{i-1}})^2$. With $C \geq 0$ being a constant such that $|M_t| \leq C$ for all $t \geq 0$, we may use the martingale property to obtain

$$\begin{aligned} E\sum_{k=1}^{2^n}(M_{t_k}-M_{t_{k-1}})^4 &\le 4C^2\sum_{k=1}^{2^n}E(M_{t_k}-M_{t_{k-1}})^2 \\ &= 4C^2\sum_{k=1}^{2^n}EM_{t_k}^2+EM_{t_{k-1}}^2-2EM_{t_k}M_{t_{k-1}} \\ &= 4C^2\sum_{k=1}^{2^n}EM_{t_k}^2-EM_{t_{k-1}}^2\le 4C^4. \end{aligned}$$

Furthermore, we have by symmetry that

$$E\sum_{k\neq i}(M_{t_k}-M_{t_{k-1}})^2(M_{t_i}-M_{t_{i-1}})^2=2E\sum_{k=1}^{2^n-1}\sum_{i=k+1}^{2^n}(M_{t_k}-M_{t_{k-1}})^2(M_{t_i}-M_{t_{i-1}})^2,$$

and this is equal to $2E\sum_{k=1}^{2^n-1}(M_{t_k}-M_{t_{k-1}})^2\sum_{i=k+1}^{2^n}E((M_{t_i}-M_{t_{i-1}})^2|\mathcal{F}_{t_k})$, since M is adapted. Here, we may apply the martingale property to obtain

$$\begin{aligned} \sum_{i=k+1}^{2^n}E((M_{t_i}-M_{t_{i-1}})^2|\mathcal{F}_{t_k}) &= \sum_{i=k+1}^{2^n}E(M_{t_i}^2-2M_{t_i}M_{t_{i-1}}+M_{t_{i-1}}^2|\mathcal{F}_{t_k}) \\ &= \sum_{i=k+1}^{2^n}E(M_{t_i}^2-M_{t_{i-1}}^2|\mathcal{F}_{t_k})=E(M_t^2-M_{t_k^n}^2|\mathcal{F}_{t_k})\le C^2, \end{aligned}$$

which finally yields

$$\begin{aligned} E\sum_{k\neq i}(M_{t_k}-M_{t_{k-1}})^2(M_{t_i}-M_{t_{i-1}})^2 &\le 2C^2E\sum_{k=1}^{2^n-1}(M_{t_k}-M_{t_{k-1}})^2 \\ &= 2C^2\sum_{k=1}^{2^n-1}E(M_{t_k}^2-2M_{t_k}M_{t_{k-1}}+M_{t_{k-1}}^2) \\ &= 2C^2\sum_{k=1}^{2^n-1}EM_{t_k}^2-EM_{t_{k-1}}^2\le 2C^4. \end{aligned}$$

Thus, we conclude $E(\sum_{k=1}^{2^n}(M_{t_k}-M_{t_{k-1}})^2)^2\le 6C^4$, and so the sequence is bounded in $\mathcal{L}^2$. From our previous deliberations, we may now conclude that $\sum_{k=1}^{2^n}(M_{t_k}-M_{t_{k-1}})^2$ converges in $\mathcal{L}^1$ to $[M]_t$. □

Solution to exercise 2.25. As $\sin W_s+\cos W_s$ is continuous and adapted, it is in $\mathfrak{I}$ by Lemma 2.2.5, so the stochastic integral is well-defined. As $|\sin x+\cos x|\le 2$, $\int_0^t(\sin W_s+\cos W_s)^2\,\mathrm{d}s$ is finite, so by Exercise 2.6, we find that X is in $\mathbf{c}\mathcal{M}$. Therefore, $EX_t=0$. Furtheremore, Exercise 2.6 also shows that $EX_t^2 = E\int_0^t (\sin W_s + \cos W_s)^2 \,\mathrm{d}s$. For any $t \geq 0$, the Fubini Theorem yields

$$\begin{aligned} E\int_0^t (\sin W_s + \cos W_s)^2 \,\mathrm{d}s &= E\int_0^t \sin^2 W_s + \cos^2 W_s + 2\sin W_s \cos W_s \,\mathrm{d}s \\ &= t + \int_0^t E \sin W_s \cos W_s \,\mathrm{d}s. \end{aligned}$$

Noting that $\sin(-x)\cos(-x) = -\sin x \cos x$, the function $x \mapsto \sin x \cos x$ is odd, and therefore $E\sin W_s \cos W_s = 0$. Thus, we conclude that $EX_t = 0$ and $EX_t^2 = t$. □

Solution to exercise 2.26. As $\sin W_s$ is continuous and adapted, Lemma 2.2.5 shows that the stochastic integral is well-defined. As $|\sin W_s| \leq 1$, we find that $\int_0^t \sin^2 W_s \,\mathrm{d}s$ is finite. Therefore, Exercise 2.6 shows that $X \in \mathbf{c}\mathcal{M}$ and that the stopped process X^t is in $\mathbf{c}\mathcal{M}^2$ for all $t \geq 0$. In particular, the covariance between X_s and X_t exists. As $X \in \mathbf{c}\mathcal{M}$, we have $\mathrm{Cov}(X_s, X_t) = EX_sX_t$. Now note that by Lemma 2.2.7, $[H \cdot W, K \cdot W]_t = \int_0^t H_s K_s \,\mathrm{d}s$ for any $H, K \in \mathfrak{I}$. If $H \cdot W$ and $K \cdot W$ are in $\mathbf{c}\mathcal{M}^2$ when stopped at $t \geq 0$, we may then use Lemma 1.4.10 to obtain $E(H\cdot W)_t(K\cdot W)_t = E[H\cdot W, K\cdot W]_t = E\int_0^t H_sK_s \,\mathrm{d}s$. Applying this reasoning to the case at hand, we find for $0 \leq s \leq t$ that

$$\begin{aligned} EX_sX_t &= E\left(\int_0^t \sin W_u \,\mathrm{d}W_u\right)\left(\int_0^s \sin W_u \,\mathrm{d}W_u\right) \\ &= E\left(\int_0^t \sin W_u \,\mathrm{d}W_u\right)\left(\int_0^t 1_{[\![0,s]\!]} \sin W_u \,\mathrm{d}W_u\right) \\ &= E\int_0^t \sin W_u 1_{[\![0,s]\!]} \sin W_u \,\mathrm{d}u = \int_0^s E\sin^2 W_u \,\mathrm{d}u. \end{aligned}$$

By the double-angle formula, $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$, so $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$. Therefore, we obtain $E\sin^2 W_u = \frac{1}{2} - \frac{1}{2}E\cos 2W_u$. Now recall that cosine is given by its Taylor series $\cos x = \sum_{n=0}^\infty (-1)^n x^{2n}((2n)!)^{-1}$, with the series converging absolutely for all $x \in \mathbb{R}$. We have that $|\sum_{n=0}^k (-1)^n x^{2n}((2n)!)^{-1}| \leq \sum_{n=0}^\infty x^{2n}((2n)!)^{-1} \leq \exp(|x|)$, so as $E\exp(|W_u|)$ is finite, we obtain by the dominated convergence theorem that

$$\begin{aligned} E\cos 2W_u &= E\lim_{k\to\infty}\sum_{n=0}^k \frac{(-1)^n(2W_u)^{2n}}{(2n)!} \\ &= \lim_{k\to\infty} E\sum_{n=0}^k \frac{(-1)^n(2W_u)^{2n}}{(2n)!} = \sum_{n=0}^\infty E\frac{(-1)^n(2W_u)^{2n}}{(2n)!}. \end{aligned}$$

With the notational convention that the product of zero factors equals one, we have

$$\begin{aligned}\sum_{n=0}^{\infty} E\frac{(-1)^n(2W_u)^{2n}}{(2n)!} &= \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{(2n)!}EW_u^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{(2n)!}u^n\prod_{i=1}^{n}(2i-1)\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{(2n)!}u^n\frac{(2n)!}{\prod_{i=1}^{n}(2i)} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{n!2^n}u^n = \exp(-2u).\end{aligned}$$

All in all, we conclude $E\sin^2 W_u = \frac{1}{2} - \frac{1}{2}\exp(-2u)$, and we then finally obtain

$$\begin{aligned}\operatorname{Cov}(X_s, X_t) &= \frac{1}{2}\int_0^s 1 - \exp(-2u)\,\mathrm{d}u = \frac{1}{2}\left(s - \left(\frac{1}{2} - \frac{1}{2}\exp(-2s)\right)\right)\\ &= \frac{s}{2} - \frac{1}{4} + \frac{1}{4}\exp(-2s).\end{aligned}$$

□

Solution to exercise 2.27. Since we have

$$\left(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\exp(\tfrac{1}{2}t)\sin x = \frac{1}{2}\exp(\tfrac{1}{2}t)\sin x - \frac{1}{2}\exp(\tfrac{1}{2}t)\sin x = 0,$$

it follows from Exercise 2.12 that the first process is in $\mathbf{c}\mathcal{M}_\ell$. As for the second process, we obtain by direct calculations that

$$\begin{aligned}\frac{\partial}{\partial t}(x+t)\exp(-x-\tfrac{1}{2}t) &= -\tfrac{1}{2}x\exp(-x-\tfrac{1}{2}t) + \frac{\partial}{\partial t}t\exp(-x-\tfrac{1}{2}t)\\ &= -\tfrac{1}{2}x\exp(-x-\tfrac{1}{2}t) + \exp(-x-\tfrac{1}{2}t) - \tfrac{1}{2}t\exp(-x-\tfrac{1}{2}t)\\ &= \exp(-x-\tfrac{1}{2}t) - \tfrac{1}{2}(x+t)\exp(-x-\tfrac{1}{2}t),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial x^2}(x+t)\exp(-x-\tfrac{1}{2}t) &= \left(\frac{\partial^2}{\partial x^2}x\exp(-x-\tfrac{1}{2}t)\right) + t\exp(-x-\tfrac{1}{2}t)\\ &= \frac{\partial}{\partial x}\left(\exp(-x-\tfrac{1}{2}t) - x\exp(-x-\tfrac{1}{2}t)\right) + t\exp(-x-\tfrac{1}{2}t)\\ &= -2\exp(-x-\tfrac{1}{2}t) + (t+x)\exp(-x-\tfrac{1}{2}t),\end{aligned}$$

so we conclude that $\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}(x+t)\exp(-x-\frac{1}{2}t)$ is identically zero as well, and so Exercise 2.12 shows that $(B_t+t)\exp(-B_t-\frac{1}{2}t)$ is in $\mathbf{c}\mathcal{M}_\ell$ as well. □

Solution to exercise 2.28. Defining $p(t,x) = x^3 - 3tx$ and $q(t,x) = x^4 - 6tx^2 + 3t^2$, we have

$$\begin{aligned}\frac{\partial p}{\partial t} + \frac{1}{2}\frac{\partial^2 p}{\partial x^2} &= -3x + \tfrac{1}{2}(6x) = 0\\ \frac{\partial q}{\partial t} + \frac{1}{2}\frac{\partial^2 q}{\partial x^2} &= -6x^2 + 6t + \tfrac{1}{2}(12x^2 - 12t) = 0,\end{aligned}$$

so it follows directly from Exercise 2.12 that both processes are in $\mathbf{c}\mathcal{M}_\ell$. In order to show that the processes are in $\mathbf{c}\mathcal{M}$, we recall that Exercise 2.12 also yields

$$\begin{aligned} p(t, W_t) &= \int_0^t \frac{\partial p}{\partial x}(s, W_s)\,\mathrm{d}W_s \\ q(t, W_t) &= \int_0^t \frac{\partial q}{\partial x}(s, W_s)\,\mathrm{d}W_s. \end{aligned}$$

Using Exercise 2.6, we then obtain the result. □

B.3 Hints and solutions for Appendix A

Hints for exercise A.1. For the equivalence between uniform integrability and $\mathcal{L}^1$ boundedness and $c = 0$, use Lemma A.2.3. For the asymptotic bound on $E|X_n - X|$ when $X_n \xrightarrow{P} X$ and X is integrable, fix $\varepsilon > 0$ and write $E|X_n - X| = E1_{(|X_n - X| \le \varepsilon)}|X_n - X| + E1_{(|X_n - X| > \varepsilon)}|X_n - X|$. Prove the result by considering n so large that $P(|X_n - X| > \varepsilon) \le \varepsilon$. ∘

Hints for exercise A.2. Define $M_n = \frac{1}{n}\sum_{k=1}^n X_k$ and put $\mathcal{F}_n = \sigma(M_n, M_{n+1}, \ldots)$. Show that (M_n) is a backwards martingale. To do so, first note that we have the relationship $E(nM_n|\mathcal{F}_{n+1}) = (n+1)M_{n+1} - nE(X_{n+1}|\mathcal{F}_{n+1})$, so that the backwards martingale property may be obtained by calculating $E(X_{n+1}|\mathcal{F}_{n+1})$. To obtain an experession for the conditional expectation, first show $E(X_{n+1}|\mathcal{F}_{n+1}) = E(X_{n+1}|M_{n+1})$. Afterwards, prove that $E(X_{n+1}|M_{n+1}) = E(X_k|M_{n+1})$ almost surely for all $k \le n+1$ and use this to identify $E(X_{n+1}|M_{n+1})$.

Having shown that (M_n) is a backwards martingale, the backwards Martingale Convergence Theorem shows that M_n converges almost surely and in $\mathcal{L}^1$ to some M_∞. Use Kolmogorov's zero-one law to argue that M_∞ is almost surely equal to ξ. ∘

Hints for exercise A.3. For any $\varepsilon > 0$, use Lemma A.2.3 to obtain a $\delta > 0$ such that whenever $P(F) \le \delta$, $E1_F|X_i| \le \varepsilon$. Show that this δ satisfies that for any $X \in \mathbb{A}$ and $P(F) \le \delta$, $E1_F|X| \le 2\varepsilon$. ∘

Hints for exercise A.4. Use uniform integrability to find an increasing sequence (λ_k) of positive numbers such that with $X_n^k = X_n 1_{(|X_n| \le \lambda_k)}$, it holds that $\sup_n E|X_n - X_n^k| \le 2^{-k}$. Use Lemma A.2.7 to recursively define sequences $(Y_n^k)_{n \ge 1}$ with Y_n^k being a finite convex combination of elements in $\{X_n^k, X_{n+1}^k, \ldots\}$ such that Y_n^k converges to Y^k. By recursively using the convex weights from the k'th sequence to obtain the weights for the $k+1$'th sequence, ensure that $E|Y^k - Y^{k+1}| \leq 2^{1-k}$ and show that this implies that (Y^k) converges to Y in $\mathcal{L}^1$ for some integrable Y. Having now defined sequences (Y_n^k) such that Y_n^k is a finite convex combination of elements from $\{X_n, X_{n+1}, \ldots\}$ converging to Y^k such that Y^k also converges to Y, use uniform integrability again to obtain a sequence of variables (Y_n) with Y_n being a finite convex combination of elements from $\{X_n, X_{n+1}, \ldots\}$ such that Y_n converges to Y. ◦

Hints for exercise A.5. Consider first the square bracket process $[M]$. To prove the martingale property of $M^2 - [M]$, apply the telescoping sum equality $M_{n+1}^2 = \sum_{k=1}^{n+1} M_k^2 - M_{k-1}^2$ and calculate $E(M_{n+1}^2 - [M]_{n+1}|\mathcal{F}_n)$ manually. In order to obtain uniform integrability, use Lemma A.3.4 to obtain that M^2 is uniformly integrable. Use the monotone convergence theorem and Lemma A.2.4 to show that $[M]$ is uniformly integrable as well, leading to uniform integrability of $M^2 - [M]$. As regards the angle bracket process $\langle M \rangle$, show that $[M] - \langle M \rangle$ is a martingale and use this to obtain the desired results. ◦

Hints for exercise A.6. Consider first the case $p = \frac{1}{2}$. Prove that $P(T_a = T_{-a}) = 0$ by showing that $P(T_a = T_{-a} = \infty)$ is zero. This may be done by first applying the Borel-Cantelli Lemma to show that $|Z_n| \geq k$ infinitely often with probability one. Having obtained this, we find $P(T_a > T_{-a}) = 1 - P(T_a < T_{-a})$. Use a symmetry argument to show that $P(T_a < T_{-a}) = P(T_a > T_{-a})$, implying $P(T_a > T_{-a}) = \frac{1}{2}$.

Regarding the case of $p \neq \frac{1}{2}$, define $M_n = c^{Z_n}$ and $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Show that M is a martingale and use the strong law of large numbers to show that M is almost surely convergent to zero. Use the optional sampling theorem on $T_a \wedge T_{-a}$ to obtain an equation for $P(T_a > T_{-a})$ yielding the desired result. ◦

Hints for exercise A.7. In order to prove the implication that whenever X_t converges in $\mathcal{L}^1$ to X, (X_t) is uniformly integrable, apply Lemma A.2.3. To do so, pick $\varepsilon > 0$. Take $u \geq 0$ so large that $E|X_t - X| \leq \frac{\varepsilon}{2}$. Use uniform continuity of $t \mapsto X_t$ on compact intervals to obtain $\eta > 0$ such that whenever $s, t \in [0, u]$ with $|t - s| \leq \eta$, then $E|X_t - X_s| \leq \frac{\varepsilon}{2}$. Combine these results and the fact that finite families of variables are uniformly integrable to obtain the result. ◦

Hints for exercise A.8. Fix $p \geq 1$. With X being the limit of X_n in probability, show that X can be taken to be bounded and use Lemma A.2.4 and Lemma A.2.5 to obtain convergence in $\mathcal{L}^p$. ◦

Hints for exercise A.9. Apply Fatou's lemma to the sequence $(\lambda - X_n 1_{(X_n \leq \lambda)})_{n \geq 1}$, and rearrange to obtain a result close to the objective. Apply uniform integrability to finalize the proof. ∘

Solution to exercise A.1. First assume that the integrability index is zero and that (X_n) is bounded in $\mathcal{L}^1$. Then, for any $\varepsilon > 0$, it holds that $c < \varepsilon$ and so there exists δ such that $\sup_{F:P(F)\leq\delta}\sup_{n\geq 1} E1_F|X_n| \leq \varepsilon$. In particular, for any $F \in \mathcal{F}$ with $P(F) \leq \delta$, we have $E1_F|X_n| \leq \varepsilon$. By Lemma A.2.3, (X_n) is uniformly integrable. Conversely, assume that (X_n) is uniformly integrable. Using Lemma A.2.3, we know that for any $\varepsilon > 0$, we may find $\delta \geq 0$ such that whenever $P(F) \leq \delta$, $E1_F|X_n| \leq \varepsilon$ for all $n \geq 1$, so that in particular $\sup_{P(F)\leq\delta}\sup_n E1_F|X_n| \leq \varepsilon$. Therefore, $c \leq \varepsilon$, and as $\varepsilon > 0$ was arbitrary, $c = 0$. Lemma A.2.3 also shows that (X_n) is bounded in $\mathcal{L}^1$. This shows that (X_n) is uniformly integrable if and only if $c = 0$ and (X_n) is bounded in $\mathcal{L}^1$.

Now assume that $X_n \xrightarrow{P} X$. Fix $\varepsilon > 0$, we then obtain

$$\begin{aligned}
\limsup_n E|X_n - X| &= \limsup_n E1_{(|X_n-X|\leq\varepsilon)}|X_n - X| + \limsup_n E1_{(|X_n-X|>\varepsilon)}|X_n - X| \\
&\leq \varepsilon + \limsup_n E1_{(|X_n-X|>\varepsilon)}|X_n - X| \\
&\leq \varepsilon + \limsup_n E1_{(|X_n-X|>\varepsilon)}|X| + \limsup_n E1_{(|X_n-X|>\varepsilon)}|X_n|.
\end{aligned}$$

Now, as $X_n \xrightarrow{P} X$, it holds that $P(|X_n - X| > \varepsilon)$ tends to zero. And by Lemma A.2.4, as X is integrable, the one-variable family $\{X\}$ is uniformly integrable. Lemma A.2.3 then shows that $\limsup_n E1_{(|X_n-X|>\varepsilon)}|X| = 0$. As regards the final term in the above, we know that for n large enough, $P(|X_n - X| > \varepsilon) \leq \varepsilon$, and so $\limsup_n E1_{(|X_n-X|>\varepsilon)}|X_n| \leq \sup_{P(F)\leq\varepsilon} E1_F|X_n|$. All in all, we obtain $\limsup_n E|X_n - X| \leq \varepsilon + \sup_{P(F)\leq\varepsilon} E1_F|X_n|$ for all positive ε. Taking infimum over all such ε, we obtain $\limsup_n E|X_n - X| \leq c$, as was to be proven. □

Solution to exercise A.2. Define $M_n = \frac{1}{n}\sum_{k=1}^n X_k$ and put $\mathcal{F}_n = \sigma(M_n, M_{n+1}, \ldots)$. We claim that (M_n) is a backwards martingale with respect to $(\mathcal{F}_n)$. Clearly, M_n is $\mathcal{F}_n$ measurable, and we have

$$\begin{aligned}
E(nM_n|\mathcal{F}_{n+1}) &= E((n+1)M_{n+1} - nX_{n+1}|\mathcal{F}_{n+1}) \\
&= (n+1)M_{n+1} - nE(X_{n+1}|\mathcal{F}_{n+1})
\end{aligned}$$

In order to calculate the conditional expectation, note that $M_{n+1} = \frac{n}{n+1}M_n + \frac{1}{n+1}X_{n+1}$. Therefore, we obtain $\mathcal{F}_n = \sigma(M_n, M_{n+1}, \ldots) = \sigma(M_n, X_{n+1}, X_{n+2}, \ldots)$ and in particualar $\mathcal{F}_{n+1} = \sigma(M_{n+1}, X_{n+2}, X_{n+3}, \ldots)$. Therefore, the sets of type $A \cap B$ where $A \in \sigma(M_{n+1})$ and $B \in \sigma(X_{n+2}, X_{n+3}, \ldots)$ form a generating class for $\mathcal{F}_{n+1}$, stable under intersections, and for such A and B,

$$\begin{aligned} E1_{A\cap B}E(X_{n+1}|\mathcal{F}_{n+1}) &= E1_{A\cap B}X_{n+1} = E1_B 1_A X_{n+1} = P(B)E1_A X_{n+1} \\ &= P(B)E1_A E(X_{n+1}|M_{n+1}) = E1_{A\cap B}E(X_{n+1}|M_{n+1}). \end{aligned}$$

As the set of $F \in \mathcal{F}_{n+1}$ such that $E1_F E(X_{n+1}|\mathcal{F}_{n+1}) = E1_F E(X_{n+1}|M_{n+1})$ is a Dynkin class, Lemma A.1.19 shows that for all $F \in \mathcal{F}_{n+1}$, $E1_F E(X_{n+1}|\mathcal{F}_{n+1}) = E1_F E(X_{n+1}|M_{n+1})$, so that $E(X_{n+1}|\mathcal{F}_{n+1}) = E(X_{n+1}|M_{n+1})$. Now note that for $1 \le k \le n+1$, the variables (X_k, M_{n+1}) have the same distribution. In particular, for any $A \in \sigma(M_{n+1})$, we find $E1_A E(X_k|M_{n+1}) = E1_A X_k = E1_A X_{n+1} = E1_A E(X_{n+1}|M_{n+1})$, allowing us to conclude $E(X_{n+1}|M_{n+1}) = E(X_k|M_{n+1})$ for all $k \le n+1$, and therefore we obtain

$$E(X_{n+1}|M_{n+1}) = \frac{1}{n+1}\sum_{k=1}^{n+1} E(X_k|M_{n+1}) = E(M_{n+1}|M_{n+1}) = M_{n+1},$$

finally showing that $E(nM_n|\mathcal{F}_{n+1}) = nM_{n+1}$, so that $E(M_n|\mathcal{F}_{n+1}) = M_n$ and so (M_n) is a backwards martingale. By the backwards Martingale Convergence Theorem A.3.6, M_n is convergent almost surely and in $\mathcal{L}^1$ to some variable M_∞.

It remains to prove that M_∞ is almost surely equal to ξ. To this end, first note that for any $m \ge 1$, we have $\lim_n M_n = \lim_n \frac{1}{n}\sum_{k=1}^n X_k = \lim_n \frac{1}{n}\sum_{k=m+1}^n X_k$, so we conclude that $M_\infty \in \sigma(X_m, X_{m+1}, \ldots)$ for all m. Thus, M_∞ is measurable with respect to the tail-σ-algebra $\cap_{n=1}^\infty \sigma(X_n, X_{n+1}, \ldots)$ of (X_n). By Kolmogorov's zero-one law, every element of this σ-algebra all have probability zero or one, and therefore $P(M_\infty \in A)$ is either zero or one for all $A \in \mathcal{B}$. We claim that this implies that M_∞ is almost surely constant. To see this, note that for any $t \in \mathbb{R}$, $P(M_\infty \le t)$ is either zero or one. Letting $\alpha = \sup\{t \ge 0 \mid P(M_\infty \le t) = 0\}$, we find that $P(M_\infty < \alpha) = 0$ and $P(M_\infty \le \alpha + \varepsilon) = 1$ for all $\varepsilon > 0$. By the continuity properties of probability measures, $P(M_\infty \le \alpha) = 1$, and we conclude that $P(M_\infty = \alpha) = 1$. Therefore, M_∞ is almost surely equal to α. Therefore, from the convergence of M_n to M_∞ in $\mathcal{L}^1$, we obtain in particular that $\alpha = EM_\infty = \lim_n EM_n = \xi$, so M_∞ is almost surely equal to ξ, as desired. This proves that $\frac{1}{n}M_n$ converges to ξ almost surely and in $\mathcal{L}^1$. □

Solution to exercise A.3. We use Lemma A.2.3. Pick $\varepsilon > 0$ and let $\delta > 0$ be such that whenever $P(F) \le \delta$, then $E1_F|X_i| \le \varepsilon$ for all $i \in I$. Let $F \in \mathcal{F}$ with $P(F) \le \delta$ be given, let $X \in \mathbb{A}$ and let (X_{i_n}) be a sequence in $(X_i)_{i\in I}$ converging in $\mathcal{L}^1$ to X. Then $E||X| - |X_{i_n}|| \le E|X - X_{i_n}|$ by the reverse triangle inequality, so $|X_{i_n}|$ converges to $|X|$ in $\mathcal{L}^1$as well, and so $1_F|X_{i_n}|$ converges to $1_F|X|$ in $\mathcal{L}^1$. In particular, the means converge. We may then pick m so that whenever $n \ge m$, $E1_F|X| \le \varepsilon + E1_F|X_{i_n}|$, and for such n, we find $E1_F|X| \le \varepsilon + E1_F|X_{i_n}| \le 2\varepsilon$. Also, as X_{i_n} converges in $\mathcal{L}^1$ to X, $E|X| \le \sup_{i\in I} E|X_i|$, so $\mathbb{A}$ is bounded in $\mathcal{L}^1$. By Lemma A.2.3, $\mathbb{A}$ is uniformly integrable. □

Solution to exercise A.4. From Lemma A.2.7, we know that the result holds in the case where (X_n) bounded in $\mathcal{L}^2$. We will prove the extended result by a truncation argument. By uniform integrability, $\lim_{\lambda\to\infty}\sup_n E|X_n - X_n 1_{(|X_n|\le\lambda)}| = \lim_{\lambda\to\infty}\sup_n E|X_n|1_{(|X_n|>\lambda)} = 0$. Therefore, there exists an increasing sequence (λ_k) tending to infinity such that by defining $X_n^k = X_n 1_{(|X_n|\le\lambda_k)}$, we have $\sup_n E|X_n - X_n^k| \le 2^{-k}$. Then $(X_n^k)_{n\ge1}$ is a sequence bounded in $\mathcal{L}^2$ for any k. We will recursively define sequences (Y_n^k) such that Y_n^k is a finite convex combination of elements in $\{X_n, X_{n+1}, \ldots\}$ and such that Y_n^k converges in $\mathcal{L}^1$ to Y^k, where (Y^k) is a sequence itself convergent in $\mathcal{L}^1$. This will allow us to prove the desired result.

By Lemma A.2.7, there is a sequence (Y_n^1) such that Y_n^1 is a finite convex combination of elements in $\{X_n^1, X_{n+1}^1, \ldots\}$ and such that Y_n^1 converges in $\mathcal{L}^2$ to some variable Y^1, in particular we also have convergence in $\mathcal{L}^1$. Assume now that the k'th sequence $(Y_n^k)_{n\ge1}$ with corresponding $\mathcal{L}^1$ limit Y^k has been defined, we construct the $k+1$'th sequence. Since Y_n^k is a finite convex combination of elements in $\{X_n^k, X_{n+1}^k, \ldots\}$, $Y_n^k = \sum_{i=n}^{K_n^k}(\lambda_n^k)_i X_i^k$ for some convex weights. Define $R_n^{k+1} = \sum_{i=n}^{K_n^k}(\lambda_n^k)_i X_i^{k+1}$, then (R_n^{k+1}) is a pointwisely bounded sequence, in particular it is bounded in $\mathcal{L}^2$. Therefore, again applying what was already proven, there exists a sequence (Y_n^{k+1}) of variables with Y_n^{k+1} being a finite convex combination of elements in $\{R_n^{k+1}, R_{n+1}^{k+1}, \ldots\}$, converging to some limit Y^{k+1}. Thus, for some convex weights, we have $Y_n^{k+1} = \sum_{j=n}^{M_n^{k+1}}(\mu_n^k)_j R_j^{k+1} = \sum_{j=n}^{M_n^{k+1}}(\mu_n^k)_j \sum_{i=j}^{K_j^k}(\lambda_j^k)_i X_i^{k+1}$, and in particular, Y_n^{k+1} is a finite convex combination of elements in $\{X_n^{k+1}, X_n^{k+1}, \ldots\}$. Now, as we have

$$E\left|Y^k - \sum_{j=n}^{M_n^{k+1}}(\mu_n^k)_j Y_j^k\right| = E\left|\sum_{j=n}^{M_n^{k+1}}(\mu_n^k)_j (Y^k - Y_j^k)\right| \le \sum_{j=n}^{M_n^{k+1}}(\mu_n^k)_j E|Y^k - Y_j^k|,$$

we find that the sequence $\sum_{j=n}^{M_n^{k+1}}(\mu_n^k)_j Y_j^k$ also converges to Y^k. Thus, summing up our results, $\sum_{j=n}^{M_n^{k+1}}(\mu_n^k)_j \sum_{i=j}^{K_j^k}(\lambda_j^k)_i X_i^{k+1}$ converges in $\mathcal{L}^1$ to Y^{k+1} and $\sum_{j=n}^{M_n^{k+1}}(\mu_n^k)_j \sum_{i=j}^{K_j^k}(\lambda_j^k)_i X_i^k$ converges in $\mathcal{L}^1$ to Y^k. Therefore,

$$\begin{aligned} E|Y^k - Y^{k+1}| &= \lim_n E\left|\sum_{j=n}^{M_n^{k+1}}(\mu_n^k)_j \sum_{i=j}^{K_j^k}(\lambda_j^k)_i (X_i^k - X_i^{k+1})\right| \\ &\le \lim_n \sum_{j=n}^{M_n^{k+1}}(\mu_n^k)_j \sum_{i=j}^{K_j^k}(\lambda_j^k)_i E|X_i^k - X_i^{k+1}|, \end{aligned}$$

and as $\sup_n E|X_n^k - X_n^{k+1}| \le \sup_n E|X_n - X_n^k| + \sup_n E|X_n - X_n^{k+1}| \le 2^{-(k+1)}$, we conclude $E|Y^k - Y^{k+1}| \le 2^{1-k}$. In particular, (Y^k) is a Cauchy sequence in $\mathcal{L}^1$, therefore convergent in $\mathcal{L}^1$ to some integrable variable Y. We have now proven the existence of sequences $(Y_n^k)_{n\ge1}$ such that $Y_n^k \in \operatorname{conv}\{X_n^k, X_{n+1}^k, \ldots\}$ and such that Y_n^k converges in $\mathcal{L}^1$ to Y^k, where Y^k

converges in $\mathcal{L}^1$ to Y. We will now use these results to obtain a sequence (Y_n) such that Y_n is a finite convex combination of elements in $\{X_n, X_{n+1}, \ldots\}$ and Y_n converges in $\mathcal{L}^1$ to Y. To this end, recall that for any m and k, we have $Y^k_m = \sum_{i=m}^{K^k_m} (\lambda^k_m)_i X^k_i$ for some convex weights $(\lambda^k_m)_i$. Define Z^k_m by putting $Z^k_m = \sum_{i=m}^{K^k_m} (\lambda^k_m)_i X_i$, we then obtain

$$\begin{aligned} E|Y - Z^k_m| &\leq E|Y - Y^k| + E|Y^k - Y^k_m| + E|Y^k_m - Z^k_m| \\ &\leq E|Y - Y^k| + E|Y^k - Y^k_m| + \sum_{i=m}^{K_m} (\lambda^k_m)_i E|X^k_i - X_i| \\ &\leq E|Y - Y^k| + E|Y^k - Y^k_m| + \sup_i E|X_i| 1_{(|X_i|>k)}. \end{aligned}$$

By our previous results, the first term tends to zero as k tends to infinity, and by uniform integrability, the last term tends to zero as k tends to infinity. Therefore, we may for each n select k_n so large that the first and last terms are less than $\frac{1}{n}$ each. Then, we may fix $m_n \geq n$ such that $E|Y^{k_n} - Y^{k_n}_{m_n}|$ is less than $\frac{1}{n}$. We then obtain $E|Y - Z^{k_n}_{m_n}| \leq \frac{3}{n}$, where $Z^{k_n}_{m_n}$ is a convex combination of elements from $\{X_n, X_{n+1}, \ldots\}$. This concludes the proof. □

Solution to exercise A.5. We first consider the case of the square bracket process. By inspection, $[M]$ is increasing and adapted. Since M is zero at zero, we obtain for $n \geq 1$,

$$\begin{aligned} M^2_n - [M]_n &= \sum_{k=1}^n M^2_k - M^2_{k-1} - \sum_{k=1}^n (M_k - M_{k-1})^2 \\ &= \sum_{k=1}^n M^2_k - M^2_{k-1} - \sum_{k=1}^n M^2_k - 2M_k M_{k-1} + M^2_{k-1} \\ &= 2\sum_{k=1}^n M_{k-1}(M_k - M_{k-1}), \end{aligned}$$

and in particular,

$$E(M^2_{n+1} - [M]_{n+1}|\mathcal{F}_n) = M^2_n - [M]_n + 2E(M_n(M_{n+1} - M_n)|\mathcal{F}_n) = M^2_n - [M]_n,$$

As we also have $E(M^2_1 - [M]_1|\mathcal{F}_0) = 0$, $M^2 - [M]$ is a martingale with initial value zero. In order to obtain uniform integrability, note that by Lemma A.3.4, M_n is convergent almost surely and in $\mathcal{L}^2$ to some variable M_∞, and EM^{*2}_∞ is finite. As $M^2_n \leq M^{*2}_\infty$ for all n, Lemma A.2.4 shows that $(M^2_n)_{n\geq 1}$ is uniformly integrable. Furthermore, as $[M]$ is increasing, $[M]$ converges almost surely to a limit $[M]_\infty$ in $[0, \infty]$. By the monotone convergence theorem, $E[M]_\infty = \lim_n E[M]_n = \lim_n EM^2_n \leq EM^{*2}_\infty$, which is finite. Thus, $[M]_\infty$ is integrable. As $[M]_n \leq [M]_\infty$ for all n, we find, again by Lemma A.2.4, that $([M]_n)_{n\geq 1}$ is uniformly integrable, and so $M^2 - [M]$ is uniformly integrable.

We next turn to the case of the angle bracket process. Clearly, $\langle M\rangle$ is increasing. Furthermore, using what we already have shown, we find that for any n,

$$\begin{aligned}
E([M]_{n+1}-\langle M\rangle_{n+1}|\mathcal{F}_n) &= E([M]_{n+1}|\mathcal{F}_n)-\langle M\rangle_{n+1}\\
&= E(M_{n+1}^2|\mathcal{F}_n)-(M_{n+1}^2-[M]_{n+1}|\mathcal{F}_n)-\langle M\rangle_{n+1}\\
&= E(M_{n+1}^2|\mathcal{F}_n)-(M_n^2-[M]_n)-\langle M\rangle_{n+1}\\
&= E(M_{n+1}^2-M_n^2|\mathcal{F}_n)+[M]_n-\langle M\rangle_{n+1}\\
&= [M]_n-\langle M\rangle_n,
\end{aligned}$$

and as $E([M]_1-\langle M\rangle_1|\mathcal{F}_0)=0$, we find that $[M]-\langle M\rangle$ is a martingale with initial value zero. Therefore, as $M^2-[M]$ is a martingale with initial value zero, $M^2-\langle M\rangle$ is a martingale with intial value zero as well. As regards uniform integrability, we find that as in the previous case, it suffices to check that $\langle M\rangle$ is uniformly integrable. As $\langle M\rangle$ is increasing, it has an almost sure limit $\langle M\rangle_\infty$, and the monotone convergence theorem along with the martingale property of $[M]-\langle M\rangle$ shows that $E\langle M\rangle_\infty=\lim_n E\langle M\rangle_n=\lim_n E[M]_n=E[M]_\infty$, which is finite. Thus, $\langle M\rangle_\infty$ is integrable, and by Lemma A.2.4, $\langle M\rangle$ is uniformly integrable, showing that $M^2-\langle M\rangle$ is uniformly integrable. □

Solution to exercise A.6. First consider the case $p=\frac{1}{2}$. We first show $P(T_a=T_{-a})=0$. Note $P(T_a=T_{-a})=P(T_a=T_{-a}=\infty)=P(\ \forall\ n:|Z_n|<a)$. For any k, it holds that $P(Z_{(n+1)k}-Z_{nk}=k)=P(Z_k=k)=P(X_1=1,\ldots,X_k=1)=2^{-k}$ and the sequence of variables $(Z_{(n+1)k}-Z_{nk})_{n\geq 1}$ are independent. Therefore, as $\sum_{n=1}^\infty P(Z_{(n+1)k}-Z_{nk}=k)$ is divergent and the events $(Z_{(n+1)k}-Z_{nk}=k)$ are independent, the Borel-Cantelli lemma shows that $Z_{(n+1)k}-Z_{nk}=k$ infinitely often with probability one. As this holds for all k, we also have that for all k, $|Z_n|\geq k$ infinitely often with probability one, in particular $P(\ \forall\ n:|Z_n|<a)=0$ and so $P(T_a=T_{-a})=0$.

We may then define the auxiliary variables $X'_n=-X_n$, $Z'_n=\sum_{k=1}^n X'_k$ and the stopping times $T'_a=\inf\{n\geq 1|Z'_n=a\}$, $T_{a-}=\inf\{n\geq 1|Z'_n=-a\}$. As $(Z_n)_{n\geq 1}$ has the same distribution as $(-Z_n)_{n\geq 1}$, we then find that (T'_a,T'_{-a}) has the same distribution as (T_a,T_{-a}). However, we also have $T'_a=T_{-a}$ and $T'_{-a}=T_a$, so

$$\begin{aligned}
P(T_a>T_{-a}) &= 1-P(T_a\leq T_{-a})=1-P(T_a<T_{a-})=1-P(T'_{-a}<T'_a)\\
&= 1-P(T_{-a}<T_a)=1-P(T_a>T_{-a}),
\end{aligned}$$

so $P(T_a>T_{-a})=\frac{1}{2}$.

Next, we consider the case $p\neq\frac{1}{2}$. Our plan is to identify a martingale related to the sequence (Z_n) and use the optional sampling theorem. Note that we always have $c>0$, so we may define a process (M_n) by putting $M_n = c^{Z_n}$. Putting $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, we claim that (M_n) is a martingale with respect to $(\mathcal{F}_n)$. As $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$ as well, M_n is $\mathcal{F}_n$ measurable. As $|Z_n| \le n$, we find $|M_n| \le c^n$ and so M_n is integrable. Furthermore, as X_{n+1} is independent of $X_1, \ldots, X_n$, $E(M_{n+1}|\mathcal{F}_n) = M_n E(c^{X_{n+1}}|\mathcal{F}_n) = M_n(cp + c^{-1}(1-p)) = M_n$, proving that (M_n) is a martingale. Furthermore, by the strong law of large numbers, $\frac{1}{n} Z_n$ converges almost surely to $2p-1$, the common mean of the elements of the sequence (X_n). Therefore, if $p > \frac{1}{2}$, $Z_n \xrightarrow{\text{a.s.}} \infty$ and as $c < 1$ in this case, $M_n \xrightarrow{\text{a.s.}} 0$. If $p < \frac{1}{2}$, $Z_n \xrightarrow{\text{a.s.}} -\infty$, and in this case $c > 1$, so $M_n \xrightarrow{\text{a.s.}} 0$ as well. In both cases, M_n converges almost surely to its limit M_∞, which is zero.

We are now ready to prove the results on the stopping times. We first argue that we never have $T_a = T_{-a}$. As noted before, if $p > \frac{1}{2}$, $Z_n \xrightarrow{\text{a.s.}} \infty$ and if $p < \frac{1}{2}$, $Z_n \xrightarrow{\text{a.s.}} -\infty$, so we always have that either T_a or T_{-a} is finite. Therefore, if $T_a = T_{-a}$, we have that both are finite and so $a = Z_{T_a} = Z_{T_{-a}} = -a$, which is impossible. Therefore, $P(T_a = T_{-a}) = 0$.

Now put $S = T_a \wedge T_{-a}$. We then have $M_S = c^a$ when $T_a < T_{-a}$ and $M_S = c^{-a}$ when $T_{-a} < T_a$. In particular, $|M_S| \le c^a$, so M_S is integrable and as $P(T_a = T_{-a}) = 0$, we find $EM_S = c^a P(T_a < T_{-a}) + c^{-a} P(T_{-a} < T_a)$. As we also have $|M_{n\wedge S}| \le c^a$, applying the dominated convergence theorem and the optional sampling theorem allows us to conclude that $EM_S = E \lim_n M_{n\wedge S} = \lim_n EM_{n\wedge S} = 1$. Using that $P(T_{-a} < T_a) = 1 - P(T_a < T_{-a})$, we thus find

$$\begin{aligned} 1 &= c^a P(T_a < T_{-a}) + c^{-a}(1 - P(T_a < T_{-a})) \\ &= (c^a - c^{-a}) P(T_a < T_{-a}) + c^{-a}, \end{aligned}$$

which shows

$$P(T_a < T_{-a}) = \frac{1 - c^{-a}}{c^a - c^{-a}} = \frac{c^a - 1}{c^{2a} - 1} = \frac{1 - c^a}{1 - c^{2a}},$$

as desired. For completeness, we check manually that the expression obtained is a number between zero and one. Note that when $p > \frac{1}{2}$, $c < 1$ and so $0 < c^{2a} < c^a < 1$, yielding $0 < 1 - c^a < 1$ and $0 < 1 - c^{2a} < 1$, so that $0 < \frac{1-c^a}{1-c^{2a}} < \frac{1-c^{2a}}{1-c^{2a}} = 1$ and when $p < \frac{1}{2}$, $c > 1$ so that $1 < c^a < c^{2a}$, and then $1 - c^{2a} < 1 - c^a < 0$, proving that $0 < \frac{1-c^a}{1-c^{2a}} < 1$ in this case as well. $\square$

Solution to exercise A.7. From Lemma A.2.5, we know that if (X_t) is uniformly integrable and converges in probability to X, then X_t converges in $\mathcal{L}^1$ to X. We need to prove the other implication.

Therefore, assume that X_t converges in $\mathcal{L}^1$ to some integrable variable X. We apply Lemma A.2.3. First fix $u \ge 0$ so large that, say, $E|X - X_t| \le 1$ when $t \ge u$. We then obtain that

$(X_t)_{t\geq u}$ is bounded in $\mathcal{L}^1$, as $E|X_t| \leq E|X| + 1$ for $t \geq u$. As X is continuous and $[0,t]$ is compact, we find that $(X_t)_{t\leq u}$ is bounded in $\mathcal{L}^1$ as well, since continuous functions are bounded over compact sets. We may now conclude that $(X_t)_{t\geq 0}$ is bounded, and so the first criterion in Lemma A.2.3 is satisfied.

Next, pick $\varepsilon > 0$. We will find a $\delta > 0$ satisfying the requirement given in Lemma A.2.3. Fix $u \geq 0$ so large that $E|X - X_t| \leq \frac{\varepsilon}{2}$ whenever $t \geq u$. As the set $[0,u]$ is compact, our continuity assumption on $t \mapsto X_t$ ensures that $t \mapsto X_t$ is uniformly continuous on $[0,u]$. Pick $\eta > 0$ such that whenever s and t are in $[0,u]$ with $|s-t| \leq \eta$, we have $E|X_t - X_s| \leq \frac{\varepsilon}{2}$. Now let m be so large that $\eta m \geq u$. From Lemma A.2.4, we know that the finite family $\{X_0, X_\eta, X_{2\eta}, \ldots, X_{m\eta}, X\}$ is uniformly integrable. Using Lemma A.2.3, we may then pick $\delta > 0$ such that whenever $F \in \mathcal{F}$ with $P(F) \leq \delta$, we have $E1_F|X| \leq \frac{\varepsilon}{2}$ and $E1_F|X_{i\eta}| \leq \frac{\varepsilon}{2}$ for $i = 0, \ldots, m$.

We claim that this $\delta > 0$ satisfies that whenever $F \in \mathcal{F}$ with $P(F) \leq \delta$, it holds for all $t \geq 0$ that $E1_F|X_t| \leq \varepsilon$. To see this, consider some $t \geq 0$. First assume that $t \geq u$. In this case, we have $E1_F|X_t| \leq E|X_t - X| + E1_F|X| \leq \varepsilon$. If we instead have $t \leq u$, there is $i \in \mathbb{N}_0$ with $i \leq m$ such that $i\eta \leq t \leq (i+1)\eta$. We then obtain $E1_F|X_t| \leq E|X_t - X_{i\eta}| + E1_F|X_{i\eta}| \leq \varepsilon$. Lemma A.2.3 now shows that $(X_t)_{t\geq 0}$ is uniformly integrable, as was to be proven. □

Solution to exercise A.8. Assume that X_n converges in probability to some variable X. As (X_n) is bounded, X is almost surely bounded. By changing X on a null set, we may assume that X is in fact bounded. Fix $p \geq 1$. We then have that $X_n - X$ converges in probability to zero, so $|X_n - X|^p$ converges in probability to zero as well. As (X_n) is bounded and X is bounded, $|X_n - X|^p$ is bounded as well, so $|X_n - X|^p$ is uniformly integrable by Lemma A.2.4 and thus converges to zero in $\mathcal{L}^1$ by Lemma A.2.5. This shows that X_n converges in $\mathcal{L}^p$ to X. □

Solution to exercise A.9. Since (X_n) is uniformly integrable, it holds that

$$0 \leq \lim_{\lambda\to\infty} \sup_n EX_n 1_{(X_n>\lambda)} \leq \lim_{\lambda\to\infty} \sup_n E|X_n|1_{(|X_n|>\lambda)} = 0$$

Let $\varepsilon > 0$ be given, we may then pick λ so large that $EX_n1_{(X_n>\lambda)} \leq \varepsilon$ for all n. Now, the sequence $(\lambda - X_n1_{(X_n\leq\lambda)})_{n\geq 1}$ is nonnegative, and Fatou's lemma therefore yields

$$\begin{aligned}
\lambda - E\limsup_n X_n 1_{(X_n\leq\lambda)} &= E\liminf_n(\lambda - X_n1_{(X_n\leq\lambda)}) \\
&\leq \liminf_n E(\lambda - X_n1_{(X_n\leq\lambda)}) \\
&= \lambda - \limsup_n EX_n1_{(X_n\leq\lambda)}.
\end{aligned}$$

The terms involving the limes superior may be infinite and are therefore a priori not amenable to arbitrary arithmetic manipulation. However, by subtracting λ and multiplying by minus one, we may still conclude that $\limsup_n EX_n 1_{(X_n \leq \lambda)} \leq E \limsup_n X_n 1_{(X_n \leq \lambda)}$. As we have ensured that $EX_n 1_{(X_n > \lambda)} \leq \varepsilon$ for all n, this yields

$$\limsup_n EX_n \leq \varepsilon + E \limsup_n X_n 1_{(X_n \leq \lambda)} \leq \varepsilon + E \limsup_n X_n,$$

and as $\varepsilon > 0$ was arbitrary, the result follows. $\square$

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