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PhD Thesis

On the Algebraic Structure of Hochschild Complexes and the Free Loop Space

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This PhD thesis has been submitted to the PhD School of The Faculty of Science, University of Copenhagen.

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Abstract

The overarching themes of this thesis are the algebraic structure of Hochschild complexes and free loop spaces. It is a presentation of three projects in progress, followed by two papers.

In the first project, which is based on work of Kontsevich–Soibelman and Wahl–Westerland, we define a three-coloured differential graded operad $\mathcal{T}$ using graph complexes, and sketch how it acts on $(C^*(A, A), C_*(A, A), A)$, the triple of Hochschild (co)chains of an $\mathcal{A}_\infty$-algebra A.

The Hochschild cochains $C^*(A, A)$ are not functorial in the algebra A. In the second project we make sense of 'natural' operations on $C^*(-, -)$ by defining a functor from multiplicative PROPs to chain complexes instead. This functor recovers the usual definition of Hochschild cochains when applied to endomorphism algebras, and its definition is based on work of McClure and Smith.

The third project discusses operations on cyclic chains. In particular, for a given operad that acts on the Hochschild chains on an algebra, we construct an operad acing on the cyclic chains of that algebra.

In the first paper we fix a gap in the paper "Cyclic homology and equivariant homology" by John D.S. Jones. To achieve this, we use the $\mathcal{E}_\infty$-structure of singular cochains to construct a homotopy coherent map between the cyclic bar construction of the differential graded algebra of cochains on a space and a model for the cochains on its free loop space.

The second paper proves an $O(2)$-equivariant version of the Jones isomorphism, relating Borel $O(2)$-equivariant cohomology of free loop spaces to negative dihedral homology, a variation of cyclic homology. After discussing a variation of the de Rham isomorphism, we apply the results to calculate the rational Borel $O(2)$-equivariant cohomology of the free loop space of the 2-sphere.

Resumé

De overordnede temaer i denne afhandling er den algebraiske struktur af Hochschild komplekser og frie løkke rum. Afhandlingen er en præsentation af tre igangværende projekter, efterfulgt af to artikler.

I det første projekt, som er baseret på arbejde af Kontsevich–Soibelman og Wahl–Westerland, definerer vi en tre-farvet d.g. operad $\mathcal{T}$ ved brug af graf komplekser, og skitserer hvordan den virker på $(C^*(A,A), C_*(A,A), A)$, triplet af Hochschild (co)kæder af en $\mathcal{A}_\infty$-algebra A.

Hochschild cokæderne $C^*(A,A)$ er ikke funktoriale i algebraen A. I det andet projekt giver vi mening til 'naturlige' operationer på $C^*(-,-)$, ved at definere en funktor fra multiplikative PROPer til kædekomplekser. Denne funktor genopretter den sædvanlige definition af Hochschild cokæder, når den anvendes til endomorfi algebraer, og definitionen er baseret på arbejde af McClure og Smith.

Det tredje projekt diskuterer operationer på cykliske kæder. Særligt konstruerer vi for en given operad der virker på Hochschild kæderne af en algebra, en operad der virker på de cykliske kæder af denne algebra.

I den første artikel fikser vi et hul i artiklen "Cyclic homology and equivariant homology" af John D.S. Jones. For at gøre det bruger vi $\mathcal{E}_\infty$-strukturen af singulære cokæderne, til at konstruere en homotopi sammenhængende afbildning mellem den cykliske barkonstruktion af d.g. algebraen af singulære cokæder, og en model for cokæderne på det frie løkke rum.

Den anden artikel beviser en $O(2)$-ækvivariant version af Jones isomorfi, som knytter Borel $O(2)$-ækvivariant cohomologi af frie løkke rum til negativ dihedral homologi, en variation af cyklisk homologi. Efter at have diskuteret en variation af de Rham isomorfien anvender vi resultaterne til at beregne den rationelle Borel $O(2)$-ækvivariante cohomologi af det frie løkke rum af 2-sfæren.

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CHAPTER 1

Introduction

The two main objects of study in this thesis are Hochschild homology and free loop spaces. We are in particular concerned with operations on Hochschild complexes and how these operations relate to free loop spaces and other areas. In this introduction we give a general motivation and introduce the main topics studied in this thesis.

1.1. Hochschild homology

Hochschild homology, and its variations like cyclic homology, topological Hochschild homology and factorization homology, have become a common face in topology and its neighbouring fields. As a homology theory for associative algebras, it can play different roles, for example:

- Deformation complexes of associative algebras
- Differential forms in non-commutative geometry
- Algebraic models for free loop spaces in string topology
- Approximation to K-theory via Chern characters and trace maps

A common theme amongst these different roles is the consideration of the operations on the complexes involved. Often, these operations are appropriately packaged into operads or PROPs, which nowadays are standard players in the landscape of algebraic topology and other fields. Examples of such operations include:

Deligne conjecture: How do the cup product and Lie algebra structure of Hochschild cohomology lift to the chain level? Or more precisely, is there a homotopy action of the chains of the little disks on the Hochschild cochains of associative algebras? Many proofs are now known of this fact, and the study of variations and extensions of the Deligne conjecture remains an active area. See [GJ94; Tam03; KS00; MS02; DTT11].

Non-commutative calculus: The calculus of differential forms and multivector fields can be formulated entirely in terms of the algebra of smooth functions $C^\infty(X)$. In fact, this can be done in such a way that $C^\infty(X)$ may be replaced by any non-commutative algebra A, where the Hochschild chains $C_*(A)$ and cochains $C^*(A)$ of A play the roles of differential forms and multivector fields respectively.

Algebraic string topology: Ever since the seminal paper by Chas and Sullivan [CS99] appeared, many have been actively looking for operations on the (co)homology $H_*(\mathcal{L}X)$ of free loop spaces of manifolds. Using the Jones isomorphism as an algebraic model for the cohomology of free loop spaces, one can produce operations in string topology. See [WW16; TZ06; Wah14; KS09; KP06].

In [TT00], Tamarkin and Tsygan proved that the pair of Hochschild cochains and chains[1] $(C^*(A), C_*(A))$ of an associative differential graded algebra A carries the structure of a 'calculus up to homotopy', mimicking the algebraic structure of multivector fields and differential forms. In [KS09], Kontsevich and Soibelman described, for A an $\mathcal{A}_\infty$-algebra, the corresponding coloured operad in terms of graph complexes, and relate this to a generalization of the Deligne conjecture. Examples of operations in these complexes are:

- Connes' B operator $B\colon C_*(A) \to C_{*+1}(A)$, the analogue of the de Rham differential.
- The cup product $C^*(A) \otimes C^*(A) \to C^*(A)$, which is the analogue of the product of multivector fields.
- The Lie bracket $C^*(A) \otimes C^*(A) \to C^{*-1}(A)$, which models the bracket of multivector fields.
- The cap product $C^*(A) \otimes C_*(A) \to C_*(A)$, which is like the insertion of a multivector field into a differential form.
- A map $C^*(A) \to C_*(A)$, which factors through the map $C^*(A) \to A$, a projection of the multivector fields onto the functions.

If one also allows to mix in factors of the algebra A, then there are more operations available. For example, in [WW16] there is the 'annulus complex', which acts on the pair $(C_*(A), A)$. In homology, $\mathcal{A}nn$ is generated by the product structure on A, the B operator and the map $A \to C_*(A)$ that models the inclusion of smooth functions into the differential forms. Additionally, there exist operations of the form $C^*(A)^{\otimes p} \otimes A^{\otimes q} \to C^*(A)$, which are studied in [DTT11].

These known operations are summarized in Figure 1.1 and are discussed further in Chapter 3.

[1] In this thesis, we use coefficients $C^*(A) = C^*(A, A)$ and $C_*(A) = C_*(A, A)$.

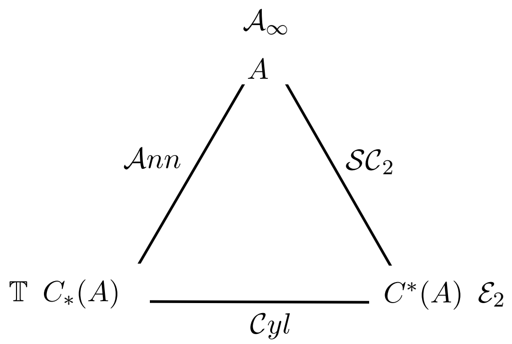


FIGURE 1.1. Summary of operations on Hochschild complexes of $\mathcal{A}_\infty$-algebras. See Chapter 3 for more details.

In this thesis, we present work in progress that aims to 'fill in' this triangle by constructing a three-coloured operad that acts on the triple $(C^*(A), C_*(A), A)$, and that includes all operations of the diagram. We use complexes of black and white graphs similar to those in [KS09] and [WW16].

When studying operations, a sensible question to ask is whether the operations under consideration are natural. For a complex like the Hochschild chains $C_*(A)$, this question has a clear answer because the complex is functorial in the algebra A. But when it comes to Hochschild cochains there is no functoriality in algebra morphisms that are not isomorphisms. All the operations on the cochains that have been discussed so far can be defined for every algebra, and are defined 'in the same way' for every isomorphism class of algebras.

Several possible angles of attack have been given over time and we discuss them in Chapter 4. In particular, we use the work of [MS04] to introduce a new way of thinking about the problem by defining a sense in which $C^*(-)$ is a functor. This has the added benefit that it produces a way of answering a more general question:

QUESTION. What are the operations on Hochschild cochains that are natural in a given class of algebras?

In the different roles Hochschild homology plays there is usually also a place for cyclic homology, meaning that one might also be interested in the operations on cyclic chains. An example of such an operation is the shuffle product $CC_*(A) \otimes CC_*(A) \to CC_*(A)$ if A is commutative. This product is also defined for Hochschild chains, and the version on cyclic chains is in fact built out of this one. A similar pattern happens in equivariant string topology, where non-equivariant operations, like the string product, induce operations on the $S^1 = SO(2)$-equivariant homology. Such equivariant string operations are related to cyclic homology by the Jones isomorphism, which we discuss in the next section.

This analogy leads to a general question, which we discuss in Chapter 5 in more detail.

QUESTION. Given operations on Hochschild chains, can one produce operations on cyclic chains?

1.2. Free loop spaces

Free loop spaces $\mathcal{L}X = \mathrm{Map}(S^1, X)$ have played a big role in geometry and physics. For example, when $X = G$ is a Lie group, $\mathcal{L}G$ forms an infinite dimensional Lie group whose subgroups are the much studied loop groups. In string theory, loops (closed strings) form the fundamental object of study, together with the surfaces they sweep out in space. In attempting to formulate a 'Theory of Everything', string theory has sparked many interesting pieces of mathematics not the least of which is Kontsevich's homological mirror symmetry. Loop spaces are also used in the study of geodesics through the use of the energy functional. This last fact is a major part of the motivation of Paper B and we discuss this further in the introduction of that paper.

A widely used tool for studying the cohomology of $\mathcal{L}X$ is the set of ideas flowing from Adams' paper on the cobar construction [Ada56]. Amongst others [Goo85; BF86], the most popular such tool in string topology is the Jones isomorphism.

THEOREM ([Jon87]). *Let $\Bbbk$ be a field and X a simply connected space with finite type homology over $\Bbbk$. Then there is an isomorphism*

$$H^*(\mathcal{L}X; \Bbbk) \cong HH_*(S^*(X; \Bbbk)).$$

Here $S^*(X; \Bbbk)$ is the differential graded algebra of (normalized) singular cochains with the cup product.

This theorem is a way of taking the free loop space, a big and usually complicated space, and study it algebraically using Hochschild homology. The isomorphism is proven in the following way.

- Using a simplicial model $S^1_\bullet$ for the circle, the free loop space can be modelled by the mapping space $\mathrm{Map}(S^1_\bullet, X) = X^{\times(\bullet+1)}$, which forms a cosimplicial space and totalizes to $\mathcal{L}X$.
- To compute the cohomology of the free loop space using this cosimplicial space, we use a map

$$\mathrm{Tot}_\oplus S^*(\mathrm{Map}(S^1_\bullet, X)) \to S^*(\mathrm{tot}\,\mathrm{Map}(S^1_\bullet, X)) \cong S^*(\mathcal{L}X).$$

This map is a quasi isomorphism under the appropriate assumptions.

- The source of the map above is the totalization of a simplicial chain complex that is very similar to the cyclic bar complex used to define Hochschild chains of $S^*(X)$: In simplicial degree n it is the chain complex $S^*(X^{\times(n+1)})$, compared to $S^*(X)^{\otimes(n+1)}$ for the cyclic bar complex. Jones compares the two simplicial objects using the Alexander–Whitney map

$$S^*(X) \otimes S^*(X) \to S^*(X \times X).$$

- Under some assumptions, the map above is a quasi isomorphism, and a spectral sequence argument shows that the comparison of simplicial chain complexes induces a quasi isomorphism on the totalizations, concluding the proof.

Upon careful inspection however, the Alexander–Whitney map does not commute with the structure maps. For example, if the structure map d_1 in simplicial degree one would commute with the Alexander–Whitney map, this would imply that the cup product on $S^*(X)$ is commutative, which is certainly not the case in general. Fortunately, the cup product is commutative up to homotopy, or more precisely: $S^*(X)$ is an $\mathcal{E}_\infty$-algebra. In Paper B we use this $\mathcal{E}_\infty$ to construct a so-called 'homotopy coherent natural transformation' between the two simplicial chain complexes. This turns out to be enough to imply that the two objects are connected by a zig-zag of quasi isomorphisms, fixing the gap in the original proof.

The orthogonal group $\mathbb{O} \subset \mathrm{Homeo}(S^1)$ acts on $\mathcal{L}X$ by rotating and flipping the loops. In addition to the isomorphism discussed earlier, Jones proved that there is an isomorphism

$$H^*(\mathcal{L}X_{h\mathbb{T}}; \Bbbk) \cong HC_*^-(S^*(X; \Bbbk)).$$

Here, the left hand side is the Borel equivariant homology with respect to the circle group $SO(2) = \mathbb{T} \subset \mathbb{O}$ and the right hand side is a variant of cyclic homology called *negative cyclic homology*. This raises the question of what needs to replace the right hand side if one wants to use more of the symmetry and calculate $H^*(\mathcal{L}X_{h\mathbb{O}}; \Bbbk)$ instead. This question is the topic of Paper B and the answer is a tool called *negative dihedral homology* HD_*^-, which originally showed up in [Lod87] in the study of Lie algebra homology of orthogonal matrices.

1.3. Contents of the thesis

We now give a more detailed description of the main results of each chapter in this thesis.

After discussing some preliminary material in Chapter 2, there are three chapters that present work in progress on some of the topics laid out in this introduction, followed by two papers. We discuss some perspectives and ideas for further research on the topics of this thesis in Chapter 6.

In Chapter 3, we use black and white graphs to construct a three-coloured differential graded operad $\mathcal{T}$. We also sketch how the triple $(C^*(A), C_*(A), A)$ is an algebra over $\mathcal{T}$ and identify familiar examples.

Chapter 4 proposes an approach to the problem that the Hochschild cochains do not form a functor $C^*\colon \mathsf{Alg} \to \mathsf{Ch}$. In order to still talk about 'natural' operations on $C^*(A)$, we change perspective and instead define functors using categories of 'multiplicative PROPs'

$$C^*, C_*\colon m\mathsf{PROP} \to \mathsf{Ch}.$$

These functors recover the usual definitions of Hochschild chains and cochains when applied to endomorphism algebras. The advantage of defining these functors is that it immediately leads to definitions of PROPs of natural operations on the pair $(C^*(A), C_*(A))$.

In Chapter 5, we provide two recipes for constructing operations on cyclic chains from operations on Hochschild chains. The first recipe uses maps $CC_*(A) \to C_*(A)$ and $C_*(A) \to CC_{*+1}(A)$, and the second uses linear extension. In particular, the second recipe leads to the following theorem, which is a reformulation of Proposition 5.2.5 and Theorem 5.2.6.

THEOREM. *Given a differential graded operad $\mathcal{D}$ that acts on $C_*(A)$ and comes with a compatible map $\mathbb{T} \to \mathcal{D}$, then there exists an operad $\mathcal{D}[u]$ that acts on $CC_*(A)$, $CC_*^-(A)$ and $CC_*^{per}(A)$ and in arity n is given by $\mathcal{D}[u](n) = \mathcal{D}(n)[u]$ as a graded module.*

In Paper A, titled 'Free loop space and the cyclic bar construction', we fix a gap in the proof of Jones' isomorphism $H^*(\mathcal{L}X; \Bbbk) \cong HH_*(S^*(X; \Bbbk))$. In particular we prove the following theorem.

THEOREM. *Let X be a space with finite type homology over a principal ideal domain $\Bbbk$. There is a natural zigzag of equivalences of cyclic chain complexes*

$$B_\bullet^{\mathrm{cyc}} S^*(X; \Bbbk) \xleftarrow{\simeq} QB_\bullet^{\mathrm{cyc}} S^*(X; \Bbbk) \xrightarrow{\simeq} S^*(\mathrm{Map}(S^1_\bullet, X); \Bbbk),$$

where $QB_\bullet^{\mathrm{cyc}} S^*(X; \Bbbk)$ is a resolution of the cyclic bar construction.

In Paper B, which is titled 'Free loop spaces and dihedral homology', we prove a variation of Jones' theorem that takes into account the orthogonal action rather than only the special orthogonal group.

Theorem. *Let* $\Bbbk$ *be a field and* X *a simply connected space with finite type homology over* $\Bbbk$*. Then there is an isomorphism*

$$H^*(\mathcal{L}X_{h\mathbb{O}};\Bbbk) \cong HD^-_*(S^*(X;\Bbbk)).$$

The right hand side of this isomorphism is a variation on cyclic homology called *negative dihedral homology* and requires additional information on the algebra $S^*(X;\Bbbk)$. More specifically it requires the data of an *involution* of $S^*(X)$. That is, a chain map $\overline{(-)}\colon S^*(X) \to S^*(X)$ that satisfies $\overline{\gamma_1 \cup \gamma_2} = (-1)^{|\gamma_1||\gamma_2|}\overline{\gamma_2} \cup \overline{\gamma_1}$ and $\overline{1} = 1$. Apart from providing such a structure for $S^*(X;\Bbbk)$, we also study the behaviour of the involution when considering other models for the cohomology ring like polynomial forms over $\mathbb{Q}$. In particular we get the following corollary.

Corollary. *Let* X *be a rationally formal space. Then there is an isomorphism*

$$HD^-_*(S^*(X;\mathbb{Q})) \cong HD^-_*(H^*(X;\mathbb{Q})).$$

This corollary is then applied to the sphere and we compute the cohomology $H^*((\mathcal{L}S^2)_{h\mathbb{T}};\mathbb{Q})$. We also compute $HD^-_*(H^*(S^2;\mathbb{F}_2))$, although it remains to prove that this is isomorphic to $H^*((\mathcal{L}S^2)_{h\mathbb{T}};\mathbb{F}_2)$. Such an isomorphism could in turn help to settle the following open problem, as described at length in Paper B.

Question. Does the 2-sphere, equipped with an arbitrary Riemannian metric, admit infinitely many distinct geodesics?

CHAPTER 2

Preliminaries

2.1. Conventions

Let $\mathsf{Ch} = \mathsf{Ch}_{\Bbbk}$ be the closed symmetric monoidal category of chain complexes over a commutative ring $\Bbbk$. The degree of a homogeneous element is denoted by $|x|$. For the tensor product we use the Koszul-sign: The tensor product of two chain complexes has differential $d(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy$ and symmetry morphism $x \otimes y \mapsto (-1)^{|x||y|}(y \otimes x)$. From this, one can deduce other identities, like $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$. The inner hom $\underline{\mathsf{Ch}}(A, B) \in \mathsf{Ch}$ has a sign convention that makes the evaluation into a chain map. The n'th graded piece consists of degree n maps

$$\underline{\mathsf{Ch}}_{\Bbbk}(X, Y)_n = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_{\Bbbk}(X_i, Y_{i+n}),$$

with differential

$$(df)(x) = d(f(x)) - (-1)^{|f|} f(dx).$$

Note that $df = 0$ is exactly the condition for being a chain map.

The Hochschild chains $C_*(A) = C_*(A; A)$ and Hochschild cochains $C^*(A) = C^*(A; A)$ are always taken with coefficients in the algebra A.

2.2. Ends and coends in chain complexes

We recall the description of an end as an equalizer. In a bicomplete category D, the end of a bifunctor $S\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{D}$, is the equalizer

$$\textstyle\int_{\mathsf{C}} S \longrightarrow \prod_{C \in \mathsf{C}} S(C, C) \overset{\prod S(1_C, f)}{\underset{\prod S(f, 1_C)}{\rightrightarrows}} \prod_{f\colon A \to B} S(A, B) .$$

Dually, the coend of S is the coequalizer

$$\textstyle\int^{\mathsf{C}} S \longleftarrow \coprod_{C \in \mathsf{C}} S(C, C) \overset{\coprod S(f, 1_A)}{\underset{\coprod S(1_B, f)}{\leftleftarrows}} \coprod_{f\colon A \to B} S(B, A) .$$

The end construction may be used to form complexes of natural operations between any two functors $F, G\colon \mathsf{C} \to \mathsf{Ch}_{\Bbbk}$. Plugging in

$$\underline{\mathrm{Nat}}(F, G) = \int_{\mathsf{C}} \underline{\mathsf{Ch}}_{\Bbbk}(F(-), G(-)),$$

the equalizer ends up giving families of maps $\alpha \in \prod_{C \in \mathsf{C}} \underline{\mathsf{Ch}}_{\Bbbk}(F(C), G(C))$, subject to the condition that for all morphisms $f\colon A \to B$ in C, the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ F(B) & \xrightarrow{\alpha_B} & G(B) \end{array}$$

commutes. As this is precisely the naturality condition, we are left with a chain complex of natural families of graded maps of any degree. Note that only the cycles are always chain maps.

Similarly, given functors $F\colon \mathsf{C} \to \mathsf{Ch}_{\Bbbk}$ and $G\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Ch}_{\Bbbk}$, we may take their tensor product as functors using a coend construction

$$F \otimes_{\mathsf{C}} G = \int^{\mathsf{C}} F(-) \otimes G(-) = \bigoplus_{A \in \mathsf{C}} F(A) \otimes G(A)/\sim .$$

Here, one divides out by the relation generated by $F(f)(\alpha) \otimes \beta \sim \alpha \otimes G(f)(\beta)$, which is similar to the definition of the tensor product of a left and a right module.

The Moore complex. We recall how the end and coend constructions can be used to define the normalized total complexes of simplicial and cosimplicial chain complexes.

DEFINITION 2.2.1. Let $\delta^{\bullet}_{\Bbbk}$ be the standard cosimplicial chain complex. That is,

$$\delta^{\bullet}_{\Bbbk}\colon \Delta \xrightarrow{\delta_{yon}} \mathsf{sSet} \xrightarrow{N_{\bullet}} \mathsf{Ch}_{\geq},$$

where δ_{yon} is the Yoneda embedding and $N_{\bullet}$ is the normalized Moore complex functor. More concretely, $\delta^{\bullet}_{\Bbbk}$ is generated by the non-degenerate elements, which are all of the form $d_{i_k} \dots d_{i_1} \iota_n$. Here $\iota_n \in \Delta^n_n = \Delta([n],[n])$ is the top dimensional generator of the standard n-simplex, which is the identity map on $[n]$. The codegeneracies are all zero due to the normalization, and the cofaces are $d^i(d_{i_k} \dots d_{i_1} \iota_n) = d_{i_k} \dots d_{i_1} d_i \iota_{n+1}$.

For a simplicial chain complex $X_{\bullet}$, the normalization functor from the Dold–Kan correspondence can be written as a coend $N_{\bullet}X = X_{\bullet} \otimes_{\Delta} \delta^{\bullet}_{\Bbbk} \in \mathsf{Ch}_{\Bbbk}$. This amounts to dividing out by degeneracies in the simplicial direction, followed by a totalization of a bicomplex. Using the coequalizer description of coends, we take a homogeneous element $x \otimes d_{i_k} \dots d_{i_1} \iota_p$ in the image of the p'th piece of $\bigoplus_p (X_p \otimes \delta^p_{\Bbbk})$. This element has (total) degree $q + p - k$, where q is the degree of $x \in X_p$. We can now calculate the differential to be

$$\begin{aligned} d(x \otimes d_{i_k} \dots d_{i_1} \iota_p) &= dx \otimes d_{i_k} \dots d_{i_1} \iota_p + (-1)^q x \otimes \sum (-1)^i d^i (d_{i_k} \dots d_{i_1} \iota_p) \\ &= dx \otimes d_{i_k} \dots d_{i_1} \iota_p + (-1)^q \sum (-1)^i d_i x \otimes d_{i_k} \dots d_{i_1} \iota_p \\ &= d_{internal} + (-1)^q d_{simplicial} \end{aligned}$$

We may view the conormalization (as in the dual Dold–Kan correspondence) of a cosimplicial chain complex $X^\bullet$ as the end construction $N^\bullet X = \underline{\mathrm{Nat}}_\Delta(\delta^\bullet_{\Bbbk}, X^\bullet) \in \mathsf{Ch}$. An element $\{\phi^n\} \in N^\bullet X \subset \prod_n \underline{\mathsf{Ch}}(\delta^n_{\Bbbk}, X^n)$ is completely determined by what it does on the ι_n, and these choices are all independent. This is because $\phi(d_i \iota_n) = \phi(d^i \iota_{n-1}) = d^i \phi(\iota_{n-1})$. Note that the ϕ is zero on codegenerate elements for the same reason. If we simply identify ϕ^n with the image $\phi^n(\iota_n) \in {X^n}_p$, we see that the degree is $p-n$. The differential comes from the inner hom complex.

$$\begin{aligned} (d\phi^n)(x) &= d(\phi^n x) - (-1)^{p-n} \phi^n(dx) \\ &= d(\phi^n x) - (-1)^{p-n} \sum (-1)^i d^i (\phi^n x) \\ &= d_{internal} - (-1)^{p-n} d_{cosimplicial} \end{aligned}$$

2.3. Operads and PROPs

This section recalls the notions and some basic examples of *operads* and *PROPs*, which are used throughout this thesis as a bookkeeping tool for spaces of operations and to characterize classes of algebras.

Operads. Although PROPs appeared in the literature first, operads are more widely known in algebraic topology and its neighbouring fields. The notion originates from work of May, and Boardman and Vogt on iterated (based) loop spaces in the 70's, but has since also flourished in many other areas like non-commutative geometry and computer science.

DEFINITION 2.3.1. A (symmetric) *operad* $\mathcal{P}$ in a closed symmetric monoidal category $(\mathsf{M}, \otimes, 1_{\mathsf{M}})$ consists of the following data.

- A sequence of objects $\mathcal{P}(n) \in \mathrm{Ob}\,\mathsf{M}$ for each $n = 0, 1, 2, \dots$ that represent spaces of operations.
- Right actions of the symmetric groups $\mathcal{P}(n) \curvearrowleft \Sigma_n$ representing permutations of the inputs.
- The *unit* map $1_{\mathsf{M}} \to \mathcal{P}(1)$ representing the identity operation.
- A map $\circ_i \colon \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(m+n-1)$ for each m, n and $i \leq m$. These map are known as the *partial composition maps* and represent piping the output of the operation in $\mathcal{P}(n)$ into the i'th slot of the operation in $\mathcal{P}(m)$.

The maps are assumed to satisfy associativity, unitality and equivariance axioms. See Figure 2.1. This definition is also known as the 'partial' definition, referring to the fact that one only specifies insertion/composition in one input at a time. A morphism of operads in M is a collection of maps between the spaces of operations that preserve all the structure maps. For more details on the theory of operads, see for example [Fre16; Mar08].

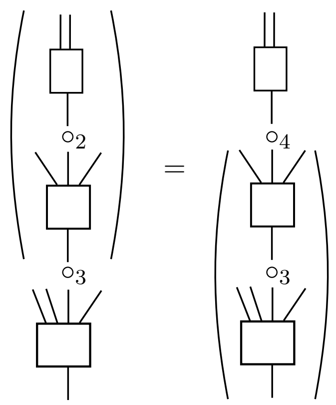


FIGURE 2.1. Associativity of partial composition

EXAMPLE 2.3.2. The prototypical example of an operad is the *endomorphism operad* $\mathrm{End}(V)$ associated to an object $V \in \mathrm{Ob}\,\mathsf{M}$. This operad is given by $\mathrm{End}(V)(n) = \underline{\mathsf{M}}(V^{\otimes n}, V)$, where $\underline{\mathsf{M}}$ denotes the inner hom of M. The symmetric groups act by permutations of the input factors $V^{\otimes n}$, and the unit is given by the identity on V. The partial compositions are given by the compositions of maps.

DEFINITION 2.3.3. An algebra V over an operad $\mathcal{P}$ is an object $V \in \mathrm{Ob}\,\mathsf{M}$ and map of operads $\mathcal{P} \to \mathrm{End}(V)$. Equivalently, it is a sequence of maps

$$\mathcal{P}(n) \otimes V^{\otimes n} \to V,$$

satisfying associativity, unitality and equivariance axioms.

EXAMPLE 2.3.4. The commutative and associative operads are given as $\mathcal{C}om(n) = 1_{\mathsf{M}}$ and $\mathcal{A}ss(n) = \coprod_{\Sigma_n} 1_{\mathsf{M}}$ for $n \geq 1$ respectively. Algebras over these operads are the commutative and the associative monoids in M. The operad describing unital associative monoids is denoted $u\mathcal{A}ss$.

Things get more interesting when the category M is also equipped with homotopy theoretic information, for example when $\mathsf{M} = \mathsf{Top}$ or Ch. In that case, one can often study the homotopy theory of such operads, which in particular encompasses the study of 'algebras up to homotopy', see e.g [BM03;

Val14]. In this thesis we do not need the homotopy theory of operads except briefly in Paper B. What we do use, however, is the notion of $\mathcal{A}_\infty$-algebra, which is the 'algebra up to homotopy' for $\mathcal{A}ss$. We discuss the corresponding operad in Definition 2.5.3.

Coloured Operads. There are many variations on the notion of operad. Most of these variations can be modelled on graphs that are more general than the planar trees that operads are modelled on. The main variation we use is that of coloured operads, which are modelled on trees with coloured leaves.

DEFINITION 2.3.5. A *coloured operad* is the data of a set of colours C, and objects $\mathcal{P}(c_1, \ldots, c_n; c) \in \operatorname{Ob} \mathsf{M}$ for all combinations of colours $c, c_i \in C$. These spaces are then equipped with units, actions and composition maps when the inputs and outputs match up appropriately.

EXAMPLE 2.3.6. An algebra over a C-coloured operad $\mathcal{P}$ consists of objects V_c, one for each $c \in C$ and a map of C-coloured operads $\mathcal{P} \to \operatorname{End}(V)$. Here $\operatorname{End}(V)(c_1, \ldots, c_n; c) = \underline{\mathsf{M}}(V_{c_1} \otimes \ldots \otimes V_{c_n}, V_c)$ is a coloured generalization of the endomorphism operad.

EXAMPLE 2.3.7. An example of a two-coloured topological operad is the *Swiss cheese operad* of Voronov [Vor99], which mixes the little intervals and the little disks operads.

PROPs. For some classes of algebras, like Hopf algebras, the notion of operad is too restrictive as it does not allow for operations of the form $V \to V \otimes V$. One way to handle such algebras is to use PROPs instead, which in fact appeared in the literature earlier than operads did. A coloured version also exists and is studied for example in [HR15].

DEFINITION 2.3.8. A *PROP* $\mathcal{P}$ in M is a symmetric monoidal category $\mathcal{P}$, enriched over M, with objects the natural numbers $(\mathbb{N}, +, 0)$. A PROP is determined by the following data.

- Spaces of operations $\mathcal{P}(m, n) \in \operatorname{Ob} \mathsf{M}$.
- Actions $\Sigma_n \curvearrowright \mathcal{P}(m, n) \curvearrowleft \Sigma_m$
- A unit $1_{\mathsf{M}} \to \mathcal{P}(1, 1)$
- Horizontal and vertical composition maps.

$$\circ_h \colon \mathcal{P}(m_1, n_1) \otimes \mathcal{P}(m_2, n_2) \to \mathcal{P}(m_1 + m_2, n_1 + m_2)$$
$$\circ_v \colon \mathcal{P}(m, n) \otimes \mathcal{P}(n, l) \to \mathcal{P}(m, l)$$

The map $\circ_h$ corresponds to taking a tensor product of operations and $\circ_v$ composes operations of matching arities.

EXAMPLE 2.3.9. Apart from the endomorphism PROP, which is defined by $\mathrm{End}(V)(m,n) = \underline{\mathsf{M}}(V^{\otimes m}, V^{\otimes n})$, another interesting example is $\mathcal{C}ob$, which has $\mathcal{C}ob(m,n)$ the set of equivalence classes of cobordisms from m to n circles.

REMARK 2.3.10. Associated to a PROP $\mathcal{P}$, there is an operad given by restricting attention to the spaces $\mathcal{P}(n,1)$. Similarly, every operad induces a PROP with morphism spaces

$$\mathcal{P}(m,n) = \bigoplus_{m_1+\ldots+m_n=m} (\mathcal{P}(m_1) \otimes \ldots \otimes \mathcal{P}(m_n)) \otimes_{\Sigma_{m_1}\times\ldots\times\Sigma_{m_n}} \Sigma_m.$$

2.4. Hochschild and cyclic homology

This section recalls some of the basics of Hochschild and cyclic homology. The presentation is based on the canonical reference [Lod98]. Every definition implicitly assumes a base ring $\Bbbk$. In particular, all tensor products are over $\Bbbk$.

Hochschild chains. Many define the Hochschild homology of a $\Bbbk$-algebra A to be $HH_*(A) = Tor_*^{A\otimes A^{\mathrm{op}}}(A,A)$. When A is flat, a useful chain complex to compute these Tor-groups is the following.

DEFINITION 2.4.1. Given a unital differential graded algebra (A, d_A), the *cyclic bar construction* $B_\bullet^{\mathrm{cyc}}A$ is the simplicial (cyclic) chain complex that on level $n \in \mathrm{Ob}\,\Delta$ is the chain complex $B_n^{\mathrm{cyc}}A = A \otimes A^{\otimes n}$ with the following structure maps.

$$\begin{aligned} d_{i<n}(a) &= a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \ldots a_n \\ d_n(a) &= (-1)^{|a_n|(|a_0|+\cdots+|a_{n-1}|)} a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \\ s_i(a) &= a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n \\ t_n(a) &= (-1)^{|a_n|(|a_0|+\cdots+|a_{n-1}|)} a_n \otimes a_0 \otimes \cdots \otimes a_{n-1} \end{aligned}$$

The *Hochschild chains* are defined as the totalization of the simplicial object. In other words, $C_*(A) = \bigoplus_n A^{\otimes n+1}[n]$ with differential $d_A + (-1)^{\mathrm{int}} \sum_i (-1)^i d_i$. Here "int" means the internal degree in A. See also Definition 2.2.1 or [Lod98, §5.3].

In this thesis, we do not concern ourselves with flatness issues and always write $C_*(A)$ to mean the totalization of the cyclic bar complex. We also do not use coefficients, so $HH_*(A)$ and $C_*(A)$ always mean $HH_*(A;A)$ and $C_*(A;A)$.

REMARK 2.4.2. Note that the definition above uses only half of the structure maps. The degeneracies s_i are used to form the *normalized Hochschild complex* by taking the Moore complex of $B_\bullet^{\mathrm{cyc}}A$ instead. The normalized chains are canonically quasi isomorphic to the regular chains. We use the normalized chains by default, as it makes the definitions of the B operator and cyclic homology simpler. See also Definition 2.2.1 and [Lod98, §1.1.14]

Example 2.4.3. Let G be a discrete group. Then the Hochschild homology of its group algebra is $HH_*(\Bbbk G) \cong H_*(G; \Bbbk G)$, where the right hand side is the group homology of G and $\Bbbk G$ is a right G-module using conjugation. See Exercise 1.1.4 in [Lod98].

Example 2.4.4 (Morita Invariance). Let A be any algebra. Then there is an isomorphism $HH_*(M_n(A)) \cong HH_*(A)$, where $M_n(A)$ is the matrix algebra associated to A. See [Lod98, §1.2].

Example 2.4.5. Let $A = T(V)$ be a tensor algebra on a free $\Bbbk$-module V. Then the Hochschild homology is concentrated in degrees 0 and 1, and is given by

$$\begin{aligned} HH_0(A) &\cong \bigoplus_{m\geq 0} (V^{\otimes m})_\tau \quad \text{coinvariants} \\ HH_1(A) &\cong \bigoplus_{m\geq 0} (V^{\otimes m})^\tau \quad \text{invariants,} \end{aligned}$$

where $\tau\colon V^{\otimes m} \to V^{\otimes m}$ is a generator of cyclic permutations. This is Theorem 3.1.4 in [Lod98].

Example 2.4.6. Two algebras of particular interest in this thesis are the cochains dga $S^*(X)$, and the chains on based loop space $S_*(\Omega^{Moore}X)$. Under appropriate assumptions these two algebras have Hochschild homology $HH_*(S^*(X)) \cong H^{-*}(\mathcal{L}X)$ and $HH_*(S_*(\Omega^{Moore}X)) \cong H_{-*}(\mathcal{L}X)$ respectively. These isomorphisms are the topic of Paper B.

Cyclic chains. In order to define cyclic homology, we first define the Connes B operator on the Hochschild chains. See also Chapter 2 of [Lod98] and Section 4 of Paper B.

Definition 2.4.7. On the elements of simplicial degree n, define

$$\begin{aligned} T_n &= (-1)^n t_n && \text{The cyclic generator} \\ N_n &= \mathrm{id} + T + T^2 + \dots T^n && \text{The norm operator} \\ B_n &= (-1)^{\mathrm{int}}(1-T)t_{n+1}s_n N && \text{The Connes } B \text{ operator.} \end{aligned}$$

We usually suppress the n in B_n. The B operator satisfies $B^2 = 0$ and anti-commutes with the differential, making B a chain map of degree one in our conventions. Concretely, the operator takes the following form on the normalized chains

$$B(a_0 \otimes \dots a_n) = \sum_{i=0}^{n} (-1)^{\epsilon_i} 1 \otimes a_{n-i+1} \otimes \dots \otimes a_n \otimes a_0 \otimes \dots \otimes a_{n-i},$$

where the sign is given by

$$\epsilon_i = ni + (1 + |a_0| + \dots + |a_{n-i}|)(1 + |a_{n-i+1}| + \dots + |a_n|).$$

With the B operator in place, we are ready to define the three variants of cyclic homology.

Definition 2.4.8. Let u denote a formal variable of degree -2. We define the negative, ordinary and periodic cyclic chains of A respectively as

$$CC_*^-(A) = C_*(A)[\![u]\!]$$
$$CC_*(A) = C_*(A)[u^{-1}]$$
$$CC_*^{per}(A) = C_*(A)[u^{-1}][\![u]\!].$$

All three variants have the differential $d + uB$ where d is the differential of $C_*(A)$.

It is important to keep in mind the differences between the ways the three variants are totalized. For example, when totalized incorrectly, CC^{per} is always acyclic. See also [Lod98, §5.1.2].

There are several other, mostly equivalent, definitions of cyclic homology. In particular, when $\mathbb{Q} \subset \Bbbk$, some define the cyclic chains using the coinvariants of $C_{n+1} \curvearrowright A^{\otimes n+1}$. Another popular definition uses resolutions of all the different C_{n+1} when $\mathbb{Q} \not\subseteq \Bbbk$. From this definition it is then more clear that $HC_*(A) \cong Tor_*^{\Delta C}(\Bbbk^\dagger, B_\bullet^{\mathrm{cyc}} A)$, where ΔC is the cyclic category, an enlargement of Δ, and $\Bbbk^\dagger$ is the constant functor. All these variations are discussed and compared in [Lod98].

Hochschild cochains. A not quite dual version of the Hochschild chains is the *Hochschild cochains*. Analogously to how one might define $HH_*(A) = Tor_*^{A \otimes A^{\mathrm{op}}}(A, A)$, Hochschild cohomology is commonly defined as $HH^*(A) = Ext^*_{A \otimes A^{\mathrm{op}}}(A, A)$. The analogue of the cyclic bar construction is the cosimplicial chain complex that on level n is given by $\underline{\mathbf{Ch}}(A^{\otimes n}, A)$, and has the following structure maps.

$$\delta_0(\gamma)(a_1, \dots, a_n) = (-1)^{|a_1||\gamma|} a_1 \gamma(a_2, \dots, a_n)$$
$$\delta_i(\gamma)(a_1, \dots, a_n) = \gamma(a_1, \dots, a_i a_{i+1}, \dots, a_n)$$
$$\delta_n(\gamma)(a_1, \dots, a_n) = \gamma(a_1, \dots, a_{n-1}) a_n$$
$$\sigma_i(\gamma)(a_1, \dots, a_n) = \gamma(a_1, \dots, a_{i-1}, 1, a_i, \dots, a_n)$$

Here $\gamma \in \underline{\mathsf{Ch}}_{|\gamma|}(A^{\otimes n-1}, A)$. The Hochschild cochains $C^*(A)$ are defined as the (product) totalization of this cosimplicial object. See Definition 2.2.1 and [Lod98, §1.5].

REMARK 2.4.9. Note that this construction is not functorial in A. This problem is discussed at length in Chapter 4.

2.5. Graph complexes

A graph complex is a chain complex that is generated by certain graphs, and where the differential can be described concretely in terms of the graphs. Usually this is in the form of blowing up vertices or collapsing edges. Fat graphs were used by Penner, Kontsevich, Igusa, Godin and others to model spaces of surfaces with decorations. See Chapter 1 of [Ega14] for an overview.

In this section we recall the definitions of the complexes of black and white graphs that are needed for later chapters. These graphs were used as a model for moduli spaces of open/closed cobordisms by Costello in [Cos07a; Cos07b] and later by Wahl and Westerland in [WW16]. Our presentation closely follows that in [WW16] and [Kla13a].

Graphs are modelled as quadruples (V, H, s, i), where V and H are the sets of *vertices* and *half edges* respectively. The *source map* $s\colon H \to V$ encodes the vertex to which a half edge is attached and $i\colon H \to H$ encodes what other half edge a given half edge is attached to, and is required to satisfy $i^2 = i$. If a half edge $h \in H$ satisfies $i(h) = h$ we call it a leaf, otherwise we call the unordered pair $(h, i(h))$ and edge. Note that this allows for vertices of all valences. We also explicitly allow for the empty graph and a single leaf. See also [WW16, §2.1].

Roughly speaking, a *fat graph* is a graph plus extra information that allows one to thicken the edge to strips and obtain a surface with boundary. More precisely, it is a graph plus a cyclic ordering of all the half edges at each vertex. See Figure 2.2.

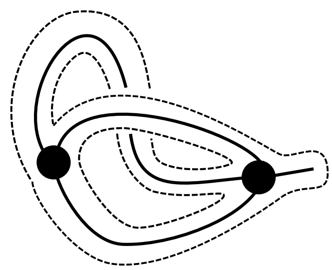
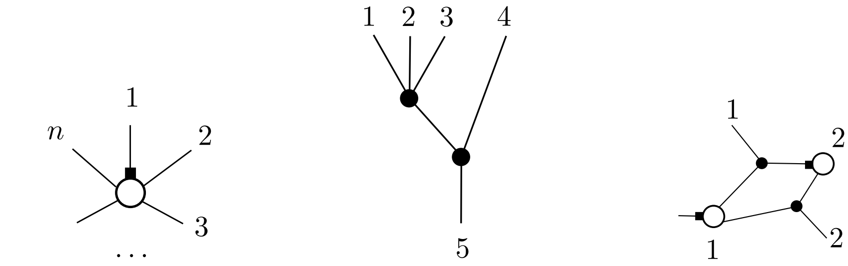


FIGURE 2.2. A thickened fat graph



(A) The $\genfrac{[}{]}{0pt}{}{n}{1}$-graph l_n (B) A $\genfrac{[}{]}{0pt}{}{5}{0}$-graph (C) A $\genfrac{[}{]}{0pt}{}{2}{2}$-graph

FIGURE 2.3. Examples of $\genfrac{[}{]}{0pt}{}{p}{m}$-graphs

An additional piece of data we need in order to define the black and white graphs is an *orientation* on a graph, which is a unit vector in $\det(\mathbb{R}(V \sqcup H))$. Such orientations are written as expressions like $v_1 \wedge h_1^1 \wedge h_2^2 \wedge \ldots \wedge v_n$. Note that odd-valent fat graphs have canonical orientations. See also [WW16, §2.2].

DEFINITION 2.5.1. A *black and white graph* is a an oriented fat graph plus a labelling of the vertices as either black or white. Black vertices are required to have valence three or higher, and white vertices can have any non-zero valence. The white vertices are ordered and come with a distinguished half edge called the *start half edge*, which we thicken in drawings.

DEFINITION 2.5.2. A $\genfrac{[}{]}{0pt}{}{p}{m}$-graph is a black and white graph with p white vertices and m labelled leaves. There may be additional unlabelled leaves if they are the start half edge of a white vertex. This later corresponds to how the unit behaves with respect to the $\mathcal{A}_\infty$-structure and normalized Hochschild (co)chains. For examples of $\genfrac{[}{]}{0pt}{}{p}{m}$-graphs, see Figure 2.3.

In order to make the $\genfrac{[}{]}{0pt}{}{p}{m}$-graphs into a chain complex, we discuss the degree, edge collapse and blowup operations on graphs and construct the differential.

Let G be a $\genfrac{[}{]}{0pt}{}{p}{m}$-graph. Its degree is given by summing over the vertices, counting black vertices as $|v| - 3$ and white vertices as $|v| - 1$, where $|v|$ denotes the valence of a vertex. In other words, every half edge contributes with a degree of one and each vertex as -3 or -1. Because all of the vertices have valence at least one and three, the degree always ends up non-negative.

The differential of G is defined as a blowup, or more precisely as

$$d(G) = \sum_{(\tilde{G}, e)} \lfloor \tilde{G} \rfloor,$$

where the sum ranges over graphs $\tilde{G}$ and edges $e \in \tilde{G}$ such that if you collapse e in $\tilde{G}$, you get G. The $\lfloor - \rfloor$ is an operation that ensures that unlabelled leaves are handled correctly. For examples, see Figures 2.4 and 2.5. For a more precise definition, see [WW16, §2.4].

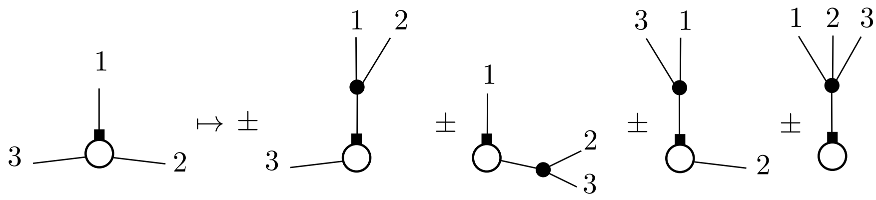


FIGURE 2.4. The differential of $l_3 \in \left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]$ - Graphs

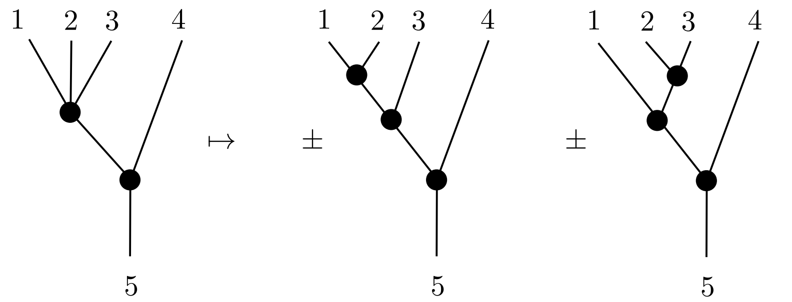


FIGURE 2.5. Example of a differential in $\mathcal{A}_\infty(4) \subset \left[\begin{smallmatrix}5\\0\end{smallmatrix}\right]$ - Graphs

The operad $\mathcal{A}_\infty$. Just as in [WW16], we define the differential graded operad $\mathcal{A}_\infty$ using graph complexes, as this suits our needs in Chapter 3. For an introduction to $\mathcal{A}_\infty$-algebras, see [Val14]. For a discussion on how the definition in terms of graph complexes compares to other definitions, see [WW16, §3.1].

DEFINITION 2.5.3 ([WW16, §2.7]). Let

$$\mathcal{A}_\infty(n) \subset \left[\begin{smallmatrix}n+1\\0\end{smallmatrix}\right] \text{ - Graphs}$$

be generated by the trees where all leaves are labelled. The first n leaves are the inputs and the last leaf is marked as an output. We explicitly allow the doubly labelled leaf as the generator $id \in \mathcal{A}_\infty(1)$ and the singly labelled leaf $1 \in \mathcal{A}_\infty(0)$. The graph in Figure 2.3b is an example.

Composition is defined by first grafting the trees and then orienting the resulting tree by juxtaposition. Finally, one needs to apply the procedure $\lfloor - \rfloor$ to take care of the unit.

REMARK 2.5.4. A black vertex of valency $n+1$ denotes a higher multiplication map $A^{\otimes n} \to A$ of degree $n-2$.

REMARK 2.5.5. This is in fact the definition of $\mathcal{A}_\infty^+$ in [WW16], the operad describing *unital* $\mathcal{A}_\infty$*-algebras*.

REMARK 2.5.6. The operad $\mathcal{A}_\infty$ can be obtained from $\mathcal{T}$ by restricting to the third colour, the algebra colour.

2.6. Formal operations

Let $\mathcal{E}$ be a differential graded PROP, and assume we are given a morphism of PROPs $\mathcal{A}_\infty \xrightarrow{i} \mathcal{E}$, inducing a functor $i^*\colon \mathcal{E}\text{-}\mathsf{Alg} \to \mathcal{A}_\infty\text{-}\mathsf{Alg}$. We outline the construction of the PROP of natural operations on Hochschild chains.

- Consider the functor $C_*\colon \mathcal{A}_\infty\text{-}\mathsf{Alg} \to \mathsf{Ch}$ that takes Hochschild chains.
- Take pointwise tensor products to obtain functors ${C_*}^{\otimes r}$.
- Apply the end construction to these functors to get complexes of operations, natural in $\mathcal{E}$-algebras $\underline{Nat}_{\mathcal{E}}({C_*}^{\otimes r}, {C_*}^{\otimes s})$.
- These complexes assemble into a dg PROP, similar to the endomorphism PROP of a chain complex.

The paper [Wah14] discusses a strategy to approximate or compute this PROP of natural operations. In particular, it defines a PROP of *formal operations* that is closely related to the PROP of natural operations. In this section we recall the relevant parts of that framework.

The category of $\mathcal{E}$-algebras is equivalent to the category of enriched symmetric monoidal functors $\mathrm{Fun}^{\otimes}(\mathcal{E}, \mathsf{Ch})$: The underlying chain complex of such a functor $\Phi \in \mathrm{Fun}^{\otimes}(\mathcal{E}, \mathsf{Ch})$ is $\Phi(1)$, and $\Phi(n)$ is canonically isomorphic to $\Phi(1)^{\otimes n}$. All of the $\mathcal{E}$-operations act using the enriched nature of the functor.

To define the Hochschild chains of an $\mathcal{E}$-algebra, we normally take $\bigoplus_n \Phi(1)^{\otimes n+1}$ and use the $\mathcal{A}_\infty$-structure to construct the differential. But the Hochschild chains may be defined for any functor, not necessarily symmetric monoidal, by taking $\bigoplus_{n\geq 0} \Phi(n+1)$ and adding the Hochschild differential. Definition 2.6.3 makes this precise.

REMARK 2.6.1. It is important to note that a symmetric monoidal functor involves additional data, such as the isomorphisms $\Phi(n) \cong \Phi(1)^{\otimes n}$.

DEFINITION 2.6.2 ([Wah14, Section 1]). In order to define the Hochschild complex, we first define an enriched functor $\mathcal{L}\colon \mathcal{A}_\infty^{\mathrm{op}} \to \mathsf{Ch}$. As a graded module,

$$\mathcal{L}(m) = \bigoplus_{n\geq 1} \mathcal{A}_\infty(m,n) \otimes L_n.$$

Here L_n stands for the free module on a single generator l_n of degree $n-1$. The enriched functoriality $\mathcal{A}_\infty^{\mathrm{op}}(m_1, m_2) \otimes \mathcal{L}(m_1) \to \mathcal{L}(m_2)$ is defined using precomposition. The differential is a sum of the differential $d_{\mathcal{A}_\infty}$ of the first factor plus d_L. By considering l_n to be the graph in Figure 2.3a, we get a map $L_n \to \bigoplus_{k\geq 1} \mathcal{A}_\infty(m,k) \otimes L_k$ from the differential in the graph complex. See also Figure 2 in [Wah14] and Figure 2.4.

DEFINITION 2.6.3. For functors $\Phi \in \mathrm{Fun}(\mathcal{E}, \mathsf{Ch})$ and $\Psi \in \mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathsf{Ch})$ we use the notation

$$C(\Phi)(m) = (i^*\Phi(- + m)) \otimes_{\mathcal{A}_\infty} \mathcal{L}$$
$$D(\Psi)(m) = \mathrm{Nat}_{\mathcal{A}_\infty^{\mathrm{op}}}(\mathcal{L}, i^*\Psi(- + m)).$$

These constructions $C(-)$ and $D(-)$ define endofunctors on the functor categories. See also Definition 1.2 in [Wah14].

By iterating the functor C, we obtain the endofunctors C^r. When applied to $\Phi \in \mathcal{E}\text{-}\mathsf{Alg}$, we recover the pointwise tensor product $C^r(\Phi)(0) = C_*(\Phi)^{\otimes r}$. Using the iterations and the end construction we define the complex of *formal operations*

$$\underline{\mathrm{Nat}}_{\mathcal{E}}^{\mathrm{formal}}(r,s) = \underline{\mathrm{Nat}}_{\mathrm{Fun}(\mathcal{E},\mathsf{Ch})}(C^r(0), C^s(0)),$$

which again forms a differential graded PROP.

THEOREM 2.6.4 ([Wah14] Theorem A). *For any PROP $\mathcal{E}$ and a morphism of PROPs $\mathcal{A}_\infty \to \mathcal{E}$, there is an isomorphism of chain complexes*

$$\underline{\mathrm{Nat}}_{\mathcal{E}}^{\mathrm{formal}}(n_1,n_2) \cong \prod_{j_1,\ldots,j_{n_1}\geq 1} \bigoplus_{k_1,\ldots,k_{n_2}\geq 1} \mathcal{E}(j,k)[k - j + n_1 - n_2],$$

where $j = j_1 + \ldots j_{n_1}$, $k = k_1 + \ldots + k_{n_2}$, and where the differential on the right hand side is the sum of the differential of $\mathcal{E}$, a multi-Hochschild and multi-coHochschild differential. The square brackets indicate a shift in grading.

By forgetting the symmetric monoidal structure, we get a functor

$$\mathcal{E}\text{-}\mathsf{Alg} = \mathrm{Fun}^{\otimes}(\mathcal{E}, \mathsf{Ch}) \to \mathrm{Fun}(\mathcal{E}, \mathsf{Ch}),$$

inducing a map of PROPS

$$\underline{\mathrm{Nat}}_{\mathcal{E}}^{\mathrm{formal}} \to \underline{\mathrm{Nat}}_{\mathcal{E}}.$$

In [Wah14], there are various statements about the properties of this map. For example, if $\mathcal{E}$ is a PROP associated to an operad, then the comparison map is injective.

CHAPTER 3

Operations on Hochschild complexes

In this chapter we present the current state of work in progress that aims to use graph complexes to define complexes of operations of the forms

$$
\begin{aligned}
&C^*(A)^{\otimes m} \otimes C_*(A)^{\otimes n} \otimes A^{\otimes p} \to C^*(A) \\
&C^*(A)^{\otimes m} \otimes C_*(A)^{\otimes n} \otimes A^{\otimes p} \to C_*(A) \\
&C^*(A)^{\otimes m} \otimes C_*(A)^{\otimes n} \otimes A^{\otimes p} \to A,
\end{aligned} \tag{1}
$$

where A is an $\mathcal{A}_\infty$-algebra. The natural way to encode these is in a three coloured differential graded operad $\mathcal{T}$ that acts on the triple $(C^*(A), C_*(A), A)$. Operads acting on two of the colours, rather than all three colours have appeared in the literature:

- An operad $\mathcal{A}nn$ of operations on $(C_*(A), A)$ is discussed in [WW16]. The operad has the homology of a topological operad of certain open closed cobordisms of disks and annuli. The homology $H_*(\mathcal{A}nn)$ is generated by the unit and multiplication of $H_*(A)$, the Connes B operator on $HH_*(A)$, and a map $H_*(A) \to HH_*(A)$.
- A generalization of the Deligne conjecture for the pair $(C^*(A), A)$ for an associative algebra A is discussed in [DTT11]. The authors construct a coloured operad that acts on $(C^*(A), A)$ and that is quasi isomorphic to the Swiss cheese operad, a topological operad. The homology of this Swiss cheese operad is the operad describing Gerstenhaber algebras and modules.
- In [TT00], Tamarkin and Tsygan proved that the pair of Hochschild cochains and chains $(C^*(A), C_*(A))$ of an associative differential graded algebras A, carries the structure of a 'calculus up to homotopy'. In [KS09], Kontsevich and Soibelman described, for A an $\mathcal{A}_\infty$-algebra, the corresponding coloured operad in terms of graph complexes and relate this to a generalization of the Deligne conjecture: The two coloured operad that acts on $(C^*(A), C_*(A))$ is quasi isomorphic to the chains on a topological operad $\mathcal{C}yl$. This operad $\mathcal{C}yl$ restricts to the little disk operad $\mathcal{E}_2$ on the cochains, and is the space of little disks on a cylinder for the operations landing in the

second colour. See also Figure 3.1. This topological operad has also been studied in [Hor13].

These results are summarized in Figure 3.2.

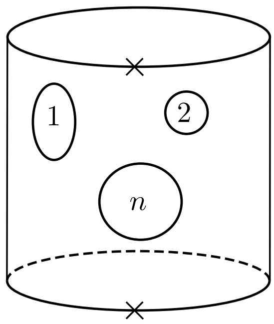


FIGURE 3.1. Elements of the topological operad $\mathcal{C}yl$ are configurations of disks on a cylinder, modulo rotations of the boundary.

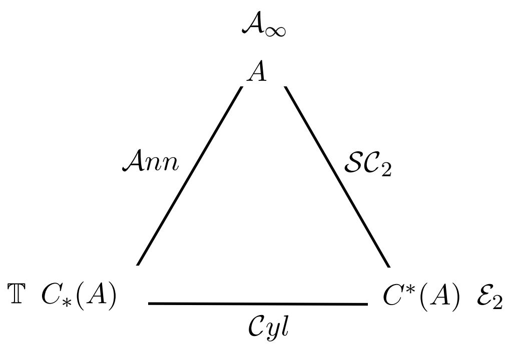


FIGURE 3.2. Summary of operations on Hochschild complexes of $\mathcal{A}_\infty$-algebras

The complexes of the forms described in Equation 1 are denoted

$$\mathcal{T}\left(\begin{bmatrix} m \\ n \\ p \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), \mathcal{T}\left(\begin{bmatrix} m \\ n \\ p \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \quad \text{and} \quad \mathcal{T}\left(\begin{bmatrix} m \\ n \\ p \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

respectively. Defining the operad $\mathcal{T}$ is the subject of Section 3.1. For an $\mathcal{A}_\infty$-algebra A, we outline how the triple $(C^*(A), C_*(A), A)$ is an algebra over $\mathcal{T}$ in Section 3.2 and give examples in Section 3.3.

All of the ideas and proofs are heavily influenced by [WW16] and [KS09] and it is advised to read about the PROP $\mathcal{OC}$ in [WW16] when confusion arises.

Dictionary. To ease into the definition of the operad $\mathcal{T}$, we provide a dictionary that helps to interpret the graph complexes. The definition of $\mathcal{T}$ is in terms of the $\left[\begin{smallmatrix}p\\m\end{smallmatrix}\right]$-graphs discussed in Section 2.5. Compared to how the graph complexes are used to define $\mathcal{OC}$ in [WW16], the only additions are the last two entries.

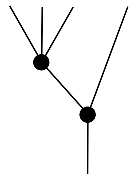
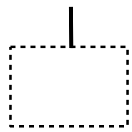
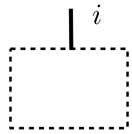
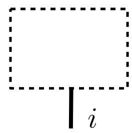
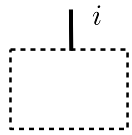
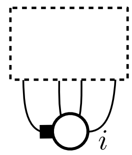
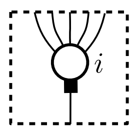
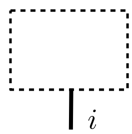
Multiplication in $\mathcal{A}_\infty$ is defined using black trees as in Section 2.5. A black vertex of valency n is the higher multiplication $\mu_{n-1}\colon A^{\otimes n-1} \to A$.	
The *unit* of the multiplication is denoted using an unlabelled leaf. These may only occur as the start half edge of a white vertex.	
An *incoming algebra* factor is a labelled leaf.	
An *outgoing algebra* factor is a labelled leaf.	
An *incoming chain* is a labelled leaf.	
An *outgoing chain* is a white vertex. The start half edge denotes the 0'th factor in $A \otimes A^{\otimes n}$.	
An *incoming cochain* is a white vertex. The start half edge denotes the output of the cochain.	
An *outgoing cochain* is a labelled leaf.	

FIGURE 3.3. Dictionary for the graph complexes. The dashed boxes denote other parts of the graph.

3.1. The operad

In this section we define the graphs that span $\mathcal{T}$ and prove they form a differential graded operad. In order to digest the definition, it is advised to keep the dictionary and the category $\mathcal{OC}$ of [WW16] in mind.

Definition 3.1.1. Let

$$\mathcal{T}\left(\begin{bmatrix} m_1 \\ n_1 \\ p_1 \end{bmatrix}, \begin{bmatrix} m_2 \\ n_2 \\ p_2 \end{bmatrix}\right) \subset \begin{bmatrix} m_1+n_2 \\ n_1+m_2+p_1+p_2 \end{bmatrix}\text{- Graphs}$$

be generated as a graded $\mathbb{Z}$-module by the isomorphism classes of *$\mathcal{T}$-type graphs*. That is, graphs that satisfy the following three properties.

(1) The graph is a forest, and the trees are rooted by the m_2+p_2 labelled leaves and the n_2 white vertices that denote the outputs.
(2) The first n_1+m_2 labelled leaves are the only labelled leaves in their boundary cycle. Note that every tree has only a single boundary component.
(3) The m_1 white vertices that denote incoming cochains have their start half edge in the direction of the root.

Lemma 3.1.2. *The $\mathcal{T}$-type graphs form a subcomplexes of the $\left[\begin{smallmatrix} p \\ m \end{smallmatrix}\right]$-graphs.*

Proof. As $\mathcal{T}$ is defined using a subset of graphs, it suffices to show that $\mathcal{T}$ is closed under the differential. That is, we show that for a $\mathcal{T}$-type graph G, the differential $dG = \lfloor \hat{d}G \rfloor$ is a linear combination of $\mathcal{T}$-type graphs. Here, the $\lfloor - \rfloor$ is the underlying graph (as in [WW16, Section 2.5]), and $\hat{d}$ is the differential of black and white graphs. This differential is a sum over all possible blowups of the vertices of the graph, taking into account the orientation and labellings. We show that these blowups preserve the properties that define $\mathcal{T}$-type graphs.

First of all, by collapsing an edge one can never get rid of a cycle. This means that the differential of a tree or forest is a sum of trees or forests. No labelled leaves are created when blowing up, so the second condition is satisfied. The condition on the start half edge is seen to be satisfied on all of the terms where a black vertex is blown up as no start half edges are involved. However, when a white vertex is blown up, it splits into a black and a white vertex (by definition, an edge between to white vertices can not be collapsed). So we have terms of two types (see Figure 3.4). In both cases there is only one possible choice of marking the start half edge as this is the only one collapsing to G. $\square$

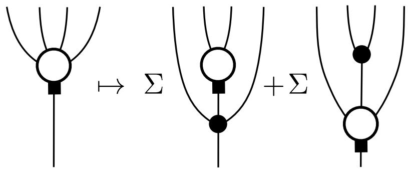


FIGURE 3.4. The two possible types of blowups

We define composition of graphs as a slight adaptation of the recipe given in Section 2.8 of [WW16].

DEFINITION 3.1.3. Let

$$G_1 \in \mathcal{T}\left(\begin{bmatrix} m_1 \\ n_1 \\ p_1 \end{bmatrix}, \begin{bmatrix} m_2 \\ n_2 \\ p_2 \end{bmatrix}\right) \quad \text{and} \quad G_2 \in \mathcal{T}\left(\begin{bmatrix} m_2 \\ n_2 \\ p_2 \end{bmatrix}, \begin{bmatrix} m_3 \\ n_3 \\ p_3 \end{bmatrix}\right)$$

be graphs. We define their composition as the sum over all possible black and white graphs $G_2 \circ G_1 = \sum \lfloor G \rfloor$ that can be obtained from G_1 and G_2 by:

1a Removing the m_2 white vertices in G_2 that represent the incoming cochains.
1b Removing the n_2 white vertices in G_1 that represent the outgoing chains.
2a Identifying the start half edge of the jth white vertex u_j with the jth leaf μ_j of G_1.
2b Identifying the start half edge of the ith white vertex v_i with the ith leaf λ_i of G_2.
3a Attach the remaining leaves $s^{-1}(u_j)$ to vertices of the boundary cycle of G_1 containing μ_j, respecting the cyclic ordering of the leaves.
3b Attach the remaining leaves $s^{-1}(v_i)$ to vertices of the boundary cycle of G_2 containing λ_i, respecting the cyclic ordering of the leaves.
4 Attach the p_2 leaves of G_1 to those in G_2 in the corresponding order. This represents the composition of the algebra-colour.

The orientation of each such graph is obtained in exactly the same way as in [WW16]. First remove all white vertices together with their start half edge in pairs $(v \wedge h)$ from the original orientations to get $[\tilde{G_1}]$ and $[\tilde{G_2}]$. Then simply wedge (juxtapose) the results $[\tilde{G_1}] \wedge [\tilde{G_2}]$. Finally one should remember to take into account the $\lfloor - \rfloor$ procedure, getting rid of redundant unlabelled leaves.

REMARK 3.1.4. Step 3 makes sense because the $n_1 + m_2$ leaves that represent incoming chains and outgoing cochains are the only labelled leaves in their boundary cycles and these boundary cycles are not affected by step 1. This last half goes wrong if we do not assume the graphs to be rooted trees.

Remark 3.1.5. When using the above recipe to extract the operadic partial compositions, only one of the two situation a or b happens at a time. As the PROP in consideration is operadic, the partial composition determines all of the composition structure.

Remark 3.1.6. The partial compositions can also be obtained directly from [WW16] by reinterpreting their recipe as a general way of glueing graphs at a single spot. In order to do a partial composition along a chain or algebra factor, one can simply use the recipe and the result agrees with our definition. However, if one uses the recipe to glue along a cochain, the orientation differs by $(-1)^{|G_1||G_2|}$.

Proposition 3.1.7. *The composition recipe determines associative chain maps*

$$\mathcal{T}\left(\begin{bmatrix} m_1 \\ n_1 \\ p_1 \end{bmatrix}, \begin{bmatrix} m_2 \\ n_2 \\ p_2 \end{bmatrix}\right) \otimes \mathcal{T}\left(\begin{bmatrix} m_2 \\ n_2 \\ p_2 \end{bmatrix}, \begin{bmatrix} m_3 \\ n_3 \\ p_3 \end{bmatrix}\right) \to \mathcal{T}\left(\begin{bmatrix} m_1 \\ n_1 \\ p_1 \end{bmatrix}, \begin{bmatrix} m_3 \\ n_3 \\ p_3 \end{bmatrix}\right),$$

and $\mathcal{T}$ forms a three-coloured differential graded operad.

Proof. The symmetric groups act by permuting labels. The only non-trivial task is to prove that the partial composition maps are associative chain maps. To prove this, we use the perspective of the previous remark and Lemma 2.5 of [WW16]. First of all, it should be stressed that the composition procedure lands in $\mathcal{T}$ as the three determining properties are preserved under partial composition: During the procedure one grafts trees onto trees, which can never produce cycles. For example, by removing a white vertex from a tree a forest is created, which is then grafted onto another tree. The labelling property is shown to be preserved in the proof of [WW16][Lemma 2.5]. As the white vertices either stay the same or a single white vertex gets removed, the last property is not an issue either.

The degree works out correctly for the same reason as in [WW16]: When attaching a leaf to a leaf (an algebra factor) the total degree is the same as the degree of the disjoint union which is the sum of the two degrees. When a white vertex gets removed the start half edge is identified with an existing leaf and the rest of the leaves stay around. So in effect we loose a white vertex of valence one, which does not contribute to the degree.

To see that a partial composition $\circ_i$ along a factor i satisfies

$$d(G_2 \circ_i G_1) = G_2 \circ_i dG_1 + (-1)^{|G_1|} dG_2 \circ_i G_1,$$

we may use Remark 3.1.6, and see directly that the glueing satisfies this relation as long as i is not of the cochain type. In that case, the only thing to check is that the sign works out correctly.

Say we have three composable graphs G_1, G_2 and G_3 with specified in and outputs along which we glue/compose. To see that the partial composition is associative, we proceed case by case depending on the in- and outputs of G_2 along which we compose.

- If either the input or the output is a leaf that is labelled as an algebra factor, the associativity is automatic, just as in [WW16].
- The case where both in- and output are chains is covered by [WW16] directly.
- The case where both in- and output are cochains is covered by applying the associativity proved in [WW16] in reversed order.
- The case where the input is a chain and the output a cochain can not happen in a $\mathcal{T}$ graph due to the condition that both leaves have to be the unique labelled leaves in their boundary cycle.
- The case where the input is a cochain and the output a chain can also be deducted from the proof in [WW16], but is a bit more tricky. Instead of thinking of the composition as first applying G_1, then G_2 and then G_3, we can also think of the glueing as considering G_2 and attaching $G_1 \sqcup \mathrm{id}$ and $\mathrm{id} \sqcup G_3$. The fact that $\mathcal{OC}$ forms a PROP can then be seen to imply that it does not matter whether we first attached $G_1 \sqcup \mathrm{id}$ and then $\mathrm{id} \sqcup G_3$, or the other way around. This then implies the associativity.

□

3.2. Algebra structure

In this section we sketch the definition of the algebra structure of the triple $(C^*(A), C_*(A), A)$ over the coloured operad $\mathcal{T}$. To really prove that this is an algebra would require many elementary but tedious checks, which at the time of writing the author has not performed.

To find the operation corresponding to a graph G in $\mathcal{T}$ we distinguish two cases A and B, which do not correspond to the a and b of Definition 3.1.3.

Case A: The output is an algebra factor or a chain.

Step 1A There can be at most one incoming chain (represented by a labelled leaf). Say this is the case and we try to map out of $C_i(A) = A^{i+1}$. Then we attach l_{i+1} to the leaf that labels the input, exactly as is done in [WW16, §6.2], which in turn is modelled on [KS09, pp.58–62].

Step 2A The previous step produces black and white graphs of the same kind as G, but with i more labelled leaves. These i leaves, plus the original, are interpreted as the $i+1$ incoming tensor factors of A that form the incoming Hochschild chain. The outgoing part consists of either a leaf labelled as an algebra factor, or a white vertex l_{r+1} representing the outgoing chain in $C_r(A)$. The rest of the graph is a combination of black and white forests representing operations in $\mathcal{A}_\infty$, and white vertices with a downward pointing start half edge. To evaluate the operation, we now take the graph, cochains, chain and algebra factors and read the graphs top to bottom. When encountering a cochain represented by a white vertex of valency $j+1$, project the corresponding cochain $C^*(A) \twoheadrightarrow \mathrm{Hom}(A^{\otimes j}, A)$. Unlabelled leaves denote the unit of the algebra.

Case B: The output is a cochain. Note that there can not be any incoming chains due to the second condition in the definition of $\mathcal{T}$-type graphs.

Step 1B This is similar to Step 1A, but instead we apply the procedure to the outgoing cochain. That is, compose $G \circ l_{r+1}$ to determine the component that lands in $\mathrm{Hom}(A^{\otimes r}, A)$.

Step 2B This is the same as Step 2A, except for that the extra labelled leaves are interpreted as the r incoming tensor factors of A needed by the equivalence

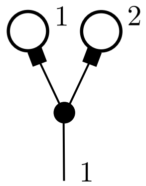
$$C^*(A)^{\otimes m} \otimes A^{\otimes p} \to C^r(A) \Leftrightarrow C^*(A)^{\otimes m} \otimes A^{\otimes p} \otimes A^{\otimes r} \to A.$$

One of the steps in checking that this recipe really does give an action of $\mathcal{T}$ on the triple, consists of identifying every possible term that shows up in the differential of the graph G with a corresponding term in the differential of the operation it produces. For example, when blowing up at a white vertex that represents an incoming cochain, one recovers the differential of that incoming Hochschild cochain. See also Figure 2.4.

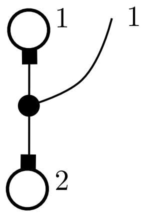
3.3. Examples

The operations from the calculus discussed in Chapter 1 are all represented by graphs in $\mathcal{T}$ and are depicted in Figure 3.5.

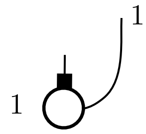
Example evaluation: The cap product. In order to illustrate the procedure of the previous section, we evaluate the cap product of (B), which falls in Case A as it is an operation $C^*(A) \otimes C_*(A) \to C_*(A)$. Say we evaluate the graph on a cochain $\gamma \in C^*(A)$ and a chain $a = a_0 \otimes \ldots \otimes a_n \in C_*(A)$. Following Steps 1A and 2A, we have to attach the graph l_{n+1} at the labelled leaf and distribute the remaining n leaves. This results in graphs of the type



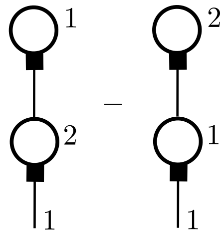
(A) The cup product



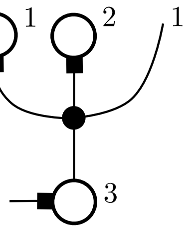
(B) The cap product



(C) The B operator



(D) The Lie bracket



(E) A more exotic operation

FIGURE 3.5. Examples of graphs representing operations. The graph of (E) can be interpreted either as an operation in $\mathcal{T}(\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix})$ or in $\mathcal{T}(\begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix})$.

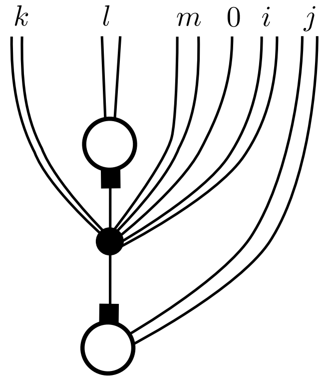


FIGURE 3.6. Term in the evaluation of the cap product

depicted in Figure 3.6. Going top to bottom we can now read off the result of applying the operation to $\gamma \otimes a$ as sums over $i+j+k+l+m=n$ of terms

$$m_{i+k+m+2}(a_{i+j+1}, \dots, a_{i+j+k}, \gamma_l(a_{i+j+k+1}, \dots, a_{i+j+k+l}),$$
$$a_{n-m}, \dots a_n, a_0, \dots, a_i) \otimes a_{i+1} \otimes \dots \otimes a_{i+j},$$

where γ_l is the component of γ in $\mathrm{Hom}(A^{\otimes l}, A)$ and $m_{i+k+m+2}$ is a higher multiplication map on A that lives in $\mathcal{A}_\infty(i+k+m+2)$.

CHAPTER 4

Naturality of Hochschild cochains

The material in this chapter is a presentation of work in progress. First, we make definition of a functor $C^*(-)\colon um\mathsf{PROP} \to \mathsf{Ch}$ that is only a slight variation of a definition made by McClure and Smith in [MS04]. The novel contribution is applying this definition to make sense of natural operations on Hochschild cochains. We now proceed by outlining the problem that this chapter aims to solve.

4.1. The problem

In contrast to Hochschild homology, Hochschild *co*homology of an algebra $HH^*(A) = HH^*(A; A)$ is not functorial in the algebra A (see also [Lod98, §1.5.5]). This is already evident from the fact that $HH^0(A) = Z(A)$ is the centre, which is not functorial either. This failure to be functorial raises the question in what right operations like the bracket, cup and cap product may be called natural, as they certainly seem to behave as such. In general we would like to know, given a multiplicative PROP $\mathcal{F}$, what would be the differential graded coloured PROP $\mathrm{Nat}_{\mathcal{F}}$ of operations, natural in $A \in \mathcal{F}\text{-}\mathsf{Alg}$ of the form

$$(C_*(A))^{\otimes p_0} \otimes (C^*(A))^{\otimes q_0} \otimes A^{\otimes r_0} \to (C_*(A))^{\otimes p_1} \otimes (C^*(A))^{\otimes q_1} \otimes A^{\otimes r_1}.$$

Any reasonable such definition should at least include known operations for various classes of algebras $\mathcal{F}$ and the restriction of colours to $p_0 = p_1 = 0$ should be related to some previously studied complex of operations, see also [Wah14; WW16] and Section 6.1.

For the case of strictly associative algebras $\mathcal{F} = u\mathcal{A}ss$, an ad hoc solution to this problem is given in [BBM13]. They define a non-coloured operad of operations on Hochschild cochains as the complex of all finite sums of compositions of some elementary operations. A similar approach is used in [DTT11].

It is worth mentioning that a similar question was investigated by Boris Tsygan [Tsy13], who also takes into consideration that the coefficients M may vary for $HH^*(A; M)$. See also [DKR15].

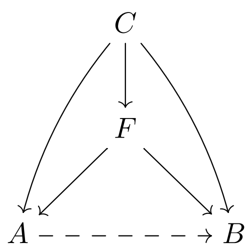
The aim of this chapter is to propose a solution to this problem by changing perspective: Rather than trying to be a functor from $\mathcal{F}$-**Alg**, we consider a different category out of which Hochschild cochains is a functor. For simplicity, we only treat strictly associative (differential graded) algebras, although the $\mathcal{A}_\infty$ case should be similar.

4.2. Intermezzo - Facts about under categories

Let $\mathbf{C}$ be any category and $C \in \operatorname{Ob}\mathbf{C}$. We denote the category of objects under C by $C \downarrow \mathbf{C}$.

PROPOSITION 4.2.1. *Given an object $(C \to F) \in C \downarrow \mathbf{C}$, there is a canonical isomorphism of categories $(C \to F) \downarrow (C \downarrow \mathbf{C}) \cong F \downarrow \mathbf{C}$.*

PROOF. This is seen from the commuting diagram



which characterizes morphisms on both sides. □

PROPOSITION 4.2.2. *The projection functor $C \downarrow \mathbf{C} \to \mathbf{C}$ creates all limits.*

PROOF. This is [Mac98, Exercise V.1.1]. The proof consists of showing there is a unique C-structure on the limiting cone in $\mathbf{C}$ and observing that it is also a limiting cone in $C \downarrow \mathbf{C}$. □

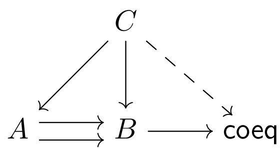
COROLLARY 4.2.3. *If $\mathbf{C}$ has limits of shape $\mathbf{J}$, then so does $C \downarrow \mathbf{C}$. If $\mathbf{C}$ is complete, then so is $C \downarrow \mathbf{C}$.*

For colimits in under categories, it suffices to identify the small coproducts and coequalizers. To see what the coproducts are, consider for example the following pushout diagram.

$$\begin{array}{ccc} C & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \sqcup_{\mathbf{C}} B \end{array}$$

The pushout comes with a map from C, and morphisms from A and B that commute with the structure maps in the under category. As it also has the correct universal property, we see that this $C \to A \sqcup_{\mathbf{C}} B$ is the coproduct of A and B in the under category. By the same argument, we see that all coproducts in $C \downarrow \mathsf{C}$ are fibered coproducts over C.

The coequalizers come for free: Simply take the coequalizer in C and one gets the map from C by composition.



4.3. Categories of multiplicative PROPs

We wish to consider props with multiplicative structure. That is, ones in which we can identify a multiplication that behaves like the one in $\mathcal{A}ss$. To do this, we may do the following construction.

Definition 4.3.1. The category of multiplicative PROPs is the under category $m\mathsf{PROP} = \mathcal{A}ss \downarrow \mathsf{PROP}$. The unital multiplicative PROPs form a category $um\mathsf{PROP} = u\mathcal{A}ss \downarrow \mathsf{PROP}$.

Remark 4.3.2. The construction that assigns the multiplicative endomorphism PROP to an associative algebra is not functorial: It is not clear what one should do to the morphisms.

Concretely, the datum of a (unital) multiplicative PROP consists of a PROP $\mathcal{P}$, and a structure map $u\mathcal{A}ss \xrightarrow{\mu} \mathcal{P}$. In particular, one can use this map to obtain the multiplication in $\mathcal{P}$ as the composition

$$\mu_{\mathcal{P}} \colon I \to u\mathcal{A}ss(2,1) \xrightarrow{\mu} \mathcal{P}(2,1).$$

And its unit as

$$1_{\mathcal{P}} \colon I = u\mathcal{A}ss(0,1) \xrightarrow{\mu} \mathcal{P}(0,1).$$

The rest of the structure map μ encodes the relations like associativity and equivariance with respect to permutation of the inputs. Morphisms of multiplicative PROPs are morphism of PROPs that preserve the multiplicative structure.

It also makes sense to further consider PROPs with more structure than just associative multiplication, e.g., Frobenius PROPs. To be able to do this, we do the same construction again.

Definition 4.3.3. Let $\mathcal{F}$ be a PROP with unital multiplication. We define the category of $\mathcal{F}$-PROPs as $\mathcal{F}\text{-}\mathsf{PROP} = \mathcal{F} \downarrow um\mathsf{PROP}$.

Applying the facts about under categories from Section 4.2 to PROPs over a bicomplete symmetric monoidal category, we get the following.

COROLLARY 4.3.4. *For every (unital) multiplicative PROP $\mathcal{F}$, we have $\mathcal{F}$-**PROP** $\cong \mathcal{F} \downarrow$ **PROP** and it is bicomplete.*

PROOF. This follows from the fact that categories of PROPs are bicomplete. See the proof of Theorem 5.5 in [Fre10]. □

4.4. Hochschild simplicial and cosimplicial objects

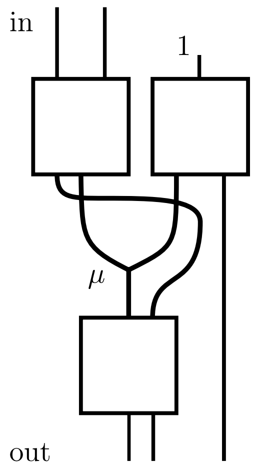
The following material is a variation on a construction in [MS04, §10], which associates a cosimplicial space to every non-symmetric topological operad with multiplication. We construct functors

$$(-)^\bullet : um\mathsf{PROP} \to c\mathsf{M} \quad \text{and} \quad (-)_\bullet : um\mathsf{PROP} \to s\mathsf{M},$$

where $c\mathsf{M}$ and $s\mathsf{M}$ are the categories of cosimplicial and simplicial objects in M respectively. These two functors model the Hochschild (co)simplicial objects when acting on the endomorphism PROPs of algebras.

REMARK 4.4.1. The unitality of the algebra is only used for the construction of the degeneracy and codegeneracy maps. Hence the same construction would result in functors to the semi-(co)simplicial objects when considered as a construction on $m\mathsf{PROP}$, instead of on $um\mathsf{PROP}$.

In order to make the definitions, it is useful to first introduce a pictorial language for PROPs. Let $\mathcal{P} \in um\mathsf{PROP}$, be a multiplicative PROP.



- An element in $\mathcal{P}(k,l)$ is denoted by a box with k strands coming in from the top and l strands going out from the bottom.
- Horizontal composition is done by putting two boxes next to each other.
- Vertical composition is done by putting boxes on top of each other and connection the strands.
- The unit of the multiplication is denoted by a short strand coming into a box.
- The symmetric group action is represented pictorially by 'braids' on the strand.
- When two or more strands come together, apply the associative multiplication.

DEFINITION 4.4.2. Let $\mathcal{P} \in um\mathsf{PROP}$, the associated cosimplicial object $\mathcal{P}^\bullet \in c\mathsf{M}$ has k'th level $\mathcal{P}^k = \mathcal{P}(k,1) \in \mathsf{M}$. The coface maps

$$d^i : \mathcal{P}^k \to \mathcal{P}^{k+1}, 0 \leq i \leq k$$

are defined using the multiplicative structure and the codegeneracy maps s^i are defined by inserting a unit at the i'th place. See Figure 4.1.

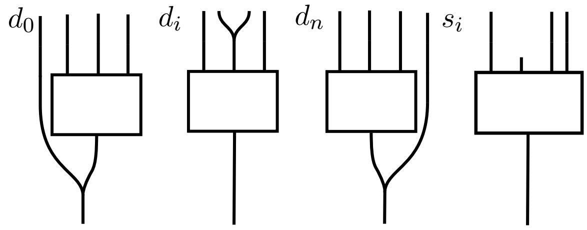


FIGURE 4.1. Definition of the cosimplicial structure maps

DEFINITION 4.4.3. The simplicial object $\mathcal{P}_\bullet \in s\mathsf{M}$ associated to a PROP $\mathcal{P} \in um\mathsf{PROP}$ has k'th level $\mathcal{P}_k = \mathcal{P}(0, k+1) \in \mathsf{M}$. The faces are defined using multiplication of the outputs, the degeneracies again come from inserting units. See Figure 4.2.

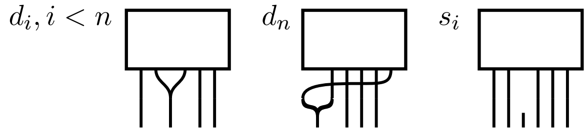


FIGURE 4.2. Definition of the simplicial structure maps

It is easy to see that the cosimplicial and simplicial identities are satisfied by drawing the corresponding pictures. As maps in $um\mathsf{PROP}$ are defined to commute with all structure maps, we see that these constructions are functors from $um\mathsf{PROP}$ to (co)simplicial objects in M.

DEFINITION 4.4.4. Let $\mathsf{M} = \mathsf{Ch}_{\Bbbk}$. The (normalized) Hochschild chains $C_*(\mathcal{P})$ and cochains $C^*(\mathcal{P})$ of a multiplicative PROP $\mathcal{P}$ are defined to be the (normalized) totalizations of $\mathcal{P}_\bullet$ and $\mathcal{P}^\bullet$ respectively.

It is now elementary to check the following.

PROPOSITION 4.4.5. *The Hochschild chains and cochains of multiplicative PROPs form functors*

$$C^* \colon um\mathsf{PROP} \xrightarrow{(-)^\bullet} c\mathsf{Ch}_{\Bbbk} \xrightarrow{N^*} \mathsf{Ch}_{\Bbbk}$$
$$C_* \colon um\mathsf{PROP} \xrightarrow{(-)_\bullet} c\mathsf{Ch}_{\Bbbk} \xrightarrow{N_*} \mathsf{Ch}_{\Bbbk}.$$

Taking $\mathsf{M} = \mathsf{Ch}_{\Bbbk}$ as the symmetric monoidal category and considering the multiplicative PROPs, we have the endomorphism PROP $\mathrm{End}(A)$ for every dga A. By considering the constructions above, we potentially have two different definitions for the Hochschild (co)chains of A.

Proposition 4.4.6. *Let A be a differential graded algebra, then we have the following isomorphisms.*

$$C_*(A) \cong C_*(\mathrm{End}(A))$$
$$C^*(A) \cong C^*(\mathrm{End}(A))$$

Proof. This can easily be seen by comparing the definitions as simplicial and cosimplicial objects. For example, $C_*(\mathrm{End}(A))$ is defined as the total complex of $\mathrm{End}(A)_\bullet$, which in simplicial degree n is given by

$$\mathrm{End}(A)_n = (\mathrm{End}(A))(0, n+1) = \underline{\mathsf{Ch}}(\Bbbk, A^{\otimes n+1}) \cong A^{\otimes n+1}.$$

□

Remark 4.4.7. The construction that assigns the endomorphism PROP to a chain complex or dga is not functorial. Hence, the above does not violate our previous claim that the Hochschild cochains are not functorial in the algebra. See also Remark 4.3.2.

4.5. Natural operations

Given that the constructions $C_*(-)$ and $C^*(-)$ are functorial in $um\mathsf{PROP}$ by Proposition 4.4.5, we may tensor the functors together to get functors

$$(C_*)^{\otimes p} \otimes (C^*)^{\otimes q} \colon um\mathsf{PROP} \to \mathsf{Ch}_\Bbbk,$$

for any $p, q \geq 0$. Thus, it now makes sense to use the end construction on such functors and obtain chain complexes of natural operations

$$\underline{\mathrm{Nat}}((C_*)^{\otimes p_0} \otimes (C^*)^{\otimes q_0}, (C_*)^{\otimes p_1} \otimes (C^*)^{\otimes q_1}).$$

These chain complexes automatically form a differential graded two-coloured PROP. Also, given a multiplicative PROP $\mathcal{F}$, we get another PROP $\underline{\mathrm{Nat}}_{\mathcal{F}}$ of operations that are natural in $\mathcal{F}$-PROP.

Remark 4.5.1. The construction of Hochschild cochains works equally well when restricting attention to operads rather than PROPs. For example, for a given multiplicative operad $\mathcal{F}$, one may construct the coloured PROP or operad of operations that are natural in $\mathcal{F}$-Operad instead. The functor $\mathcal{F}$-$\mathsf{Operad} \to \mathcal{F}$-$\mathsf{PROP}$ induces a map of PROPs from $\underline{\mathrm{Nat}}_{\mathcal{F}}$ to the version natural in operads.

4.6. Examples

All operations that may be defined aritywise by summing, composing, permuting, multiplying and inserting units are examples of operations in $\underline{\mathrm{Nat}}_{\mathcal{A}ss}$. For example, the cup and cap are described pictorially in Figures 4.3 and 4.4. Other examples include the Lie bracket and the Connes B operator.

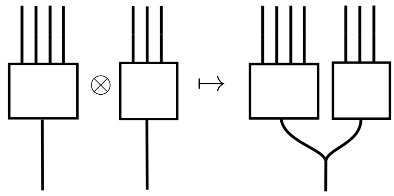


FIGURE 4.3. Pictorial description of the cup product

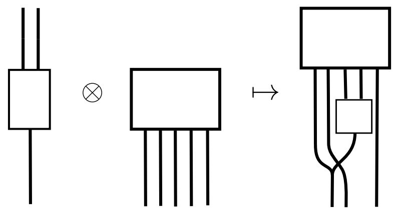


FIGURE 4.4. Pictorial description of the cap product

If one also allows compositions with fixed elements of a chosen multiplicative PROP $\mathcal{F}$, one obtains the operations in $\underline{\mathrm{Nat}}_{\mathcal{F}}$. For example, take $\mathcal{F}$ to be the PROP associated to the operad $u\mathcal{A}ss \rtimes C_2$ that models algebras with involutions (see Paper B). Then associated to the involution is a C_2 action on Hochschild chains, described in Figure 4.5.

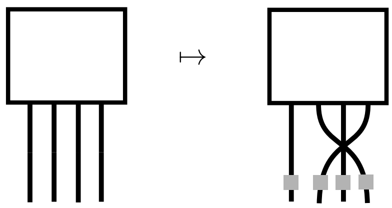


FIGURE 4.5. Pictorial description of the C_2-action on Hochschild chains of involutive algebras. The shaded boxes denote the involution operator.

In Section 6.1 we discuss plans on how to compare the formal definition of the last section to other, more concrete complexes.

CHAPTER 5

Operations on cyclic homology

This chapter discusses the current state of work in progress within the context outlined below and in Section 6.3. The material in Sections 5.1–5.3 has been checked reasonably rigorously, but Section 5.4 is to be read with some caution.

Similarly to how operations on Hochschild chains show up in different places, so do operations on cyclic chains of algebras. For example, Loday studied λ-operations on the cyclic homology of commutative algebras in [Lod89], and shuffle type products are discussed in [Lod98] and [HJ87].

An important place where operations play a role is *equivariant string topology.* In [CS99; CS04], Chas and Sullivan introduce the string bracket, a graded Lie bracket on $H_*(\mathcal{L}X_{h\mathbb{T}})$ of degree $2-d$ where d is the dimension of the manifold X. In general, one wants to study operations on Borel equivariant (co)homology and one way to attack this is by using the Jones isomorphism $HC_*^-(S^*(X)) \cong H^*(\mathcal{L}X_{h\mathbb{T}})$ from [Jon87]. From this perspective, one needs to capture the algebraic structure of the singular cochains $S^*(X)$ on a manifold X, and study operations on the negative cyclic chains of such algebras. Such a program has been undertaken for the Hochschild chains in [Cos07b] and [WW16; Wah14]. Two of the statements that comes out of this study are that there is a dg PROP $\mathcal{O}$ that (roughly) describes the algebraic structure of cochains on a compact oriented manifold and that the homology of the space of (formal) $\mathcal{O}$-natural operations from r Hochschild chains to s Hochschild chains is $\bigoplus_{[\Sigma]} H_*(B\operatorname{Diff}^\partial \Sigma)$, where Σ ranges over topological cobordisms with r inputs and s outputs.

This raises the question what the analogous statement is for (negative) cyclic chains:

Question. Are there moduli spaces of certain types of cobordisms that model operations on the (negative) cyclic homology of $\mathcal{O}$-algebras?

This question has been studied by Richard Hepworth who conjectured in the lecture [Hep13] that the answer would be

$$\bigoplus_{[\Sigma]} \pi_*(H\mathbb{Z} \wedge (B\operatorname{Diff}^\partial \Sigma)_{h\mathbb{T}^s})^{h\mathbb{T}^r}.$$

In order to study this question and generally gain a better understanding of operations of cyclic chains, we outline two recipes that can be used to produce such operations from operations on Hochschild chains.

5.1. Recipe I - Transfers

Associated to the fibration

$$S^1 \to E\mathbb{T} \times \mathcal{L}X \xrightarrow{p} E\mathbb{T} \times_{\mathbb{T}} \mathcal{L}X = \mathcal{L}X_{h\mathbb{T}}$$

is the Gysin sequence

$$\dots \to H^{i-2}(\mathcal{L}X_{h\mathbb{T}}) \xrightarrow{-\cup e} H^i(\mathcal{L}X_{h\mathbb{T}}) \xrightarrow{p^*} H^i(\mathcal{L}X) \xrightarrow{\tau} H^{i-1}(\mathcal{L}X) \to \dots$$

The connecting homomorphism τ is also known as the *transfer*, and can be used to construct equivariant string operations. If one starts with an operation on $H^*(\mathcal{L}X)$, precomposition by p^* and postcomposition by τ results in an operation (of a different degree) on $H^*(\mathcal{L}X_{h\mathbb{T}})$.

Example 5.1.1. Dual to the Chas–Sullivan product

$$\mu\colon H_p(\mathcal{L}X) \otimes H_q(\mathcal{L}X) \to H_{p+q-d}(\mathcal{L}X)$$

is the coproduct $\mu^\vee$ on $H^*(\mathcal{L}X)$. From this we construct the cobracket of degree $-(2+d)$ on $H^*(\mathcal{L}X_{h\mathbb{T}})$ as $(\tau \otimes \tau) \circ \mu^\vee \circ p^*$.

This way of building equivariant operations has an analogue in cyclic homology, and one can use the Jones isomorphism to compare the two. Consider the short exact sequence

$$CC^-_{*+2}(A) \xhookrightarrow{\mathrm{S}} CC^-_*(A) \xrightarrow{h} CC^-_*(A)/CC^-_{*+2}(A) \simeq C_*(A).$$

The long exact sequence associated to this short exact sequence is the analogue of the Gysin sequence for negative cyclic homology

$$\dots \to HC^-_{j+2}(A) \xrightarrow{\mathrm{S}} HC^-_j(A) \xrightarrow{h} HH_j(A) \xrightarrow{B} HC^-_{j+1}(A) \to \dots$$

Here $\mathrm{S}(\sum a^p u^p) = \sum a^p u^{p+1}$ is the periodicity operator, $h(\sum a^p u^p) = a^0$ and $B(a) = Ba \in CC^-_{j+1}(A)$.

Remark 5.1.2. The procedure of pre- and postcomposing by h and B gives maps $\mathrm{Nat}((C_*)^{\otimes r}, (C_*)^{\otimes s}) \to \mathrm{Nat}((CC^-_*)^{\otimes r}, (CC^-_*)^{\otimes s})$ of degree s. However, this is not a map of PROPs or even of operads as it does not behave well with respect to composition.

5.2. Recipe II - Linear extension

In order to state the second recipe, we use the following language. Let (C_*, b) be a chain complex and B be an operator of degree 1 on C_*, satisfying $B^2 = Bb + bB = 0$. This is also called a differential graded $\mathbb{T}$-module in Paper B. For such a $\mathbb{T}$-module, we may define the following 'cyclic theories' in analogy to the definition of cyclic homology in Section 2.4.8.

$$\begin{aligned} CC_*^- &= C_*[\![u]\!] \\ CC_*^{\oplus} &= C_*[u] \\ CC_* &= C_*[u^{-1}] \\ CC_*^{per} &= C_*[u^{-1}][\![u]\!] \end{aligned}$$

Here u is a formal variable of degree -2, and all four complexes come with the differential $\partial = b + uB$.

Example 5.2.1. For an algebra A, the Hochschild chains $(C_*, b) = (C_*(A), d^{\mathrm{int}} + d^{\mathrm{Hoch}})$ together with Connes' B operator are an example. The resulting cyclic theories are the variations of cyclic homology discussed in Section 2.4.

Given a chain map $\phi\colon C_* \to C_*$, we can extend this map u-linearly to all four of the complexes if it commutes with B. This is for example done when defining the C_2 action on cyclic chains of involutive algebras later in Paper B. More generally, if one does not assume ϕ to be a chain map but still insists that $\phi \circ B = (-1)^{|\phi|} B \circ \phi$ we have that $\delta\widehat{\phi} = \widehat{\delta\phi}$, where the hat denotes u-linear extension and δ denotes the differential as a map, which is discussed in the conventions section. One may further generalize this construction to include higher arity operations and the possibility that ϕ does not commute with B.

Proposition 5.2.2. *Let $\phi\colon (C_*)^{\otimes n} \to C_*$ be a map of degree $|\phi|$. The u-linear extension $\widehat{\phi}$ defines n-ary operations on the cyclic theories and satisfy the identity*

$$\delta_\partial \widehat{\phi} = \widehat{\delta_b \phi} + \mathrm{S} \circ \widehat{\delta_B \phi}.$$

Here δ_b denotes the differential as an operation with respect to the b differential and similarly for δ_∂ and δ_B.

Proof. We illustrate this elementary check in the binary case.

$$\begin{aligned} (\delta_\partial \widehat{\phi})(xu^i, yu^j) =& (b+uB)\widehat{\phi}(xu^i, yu^j) - (-1)^{|\phi|}(\widehat{\phi}((b+uB)xu^i, yu^j) \\ &+ (-1)^{|x|}\widehat{\phi}(xu^i, (b+uB)xu^j)) \end{aligned}$$

Expanding and regrouping, we recognize the two needed terms

$$b(\phi(x,y))u^{i+j} - (-1)^{|\phi|}(\phi(bx,y) + (-1)^{|a|}\phi(x,by))u^{i+j} = \widehat{\delta_b\phi}(xu^i, yu^j)$$

and

$$\begin{aligned} -(-1)^{|\phi|}(\phi(Bx,y) + (-1)^{|a|}\phi(x,By))u^{i+j+1} \\ +B(\phi(x,y))u^{i+j+1} = \mathrm{S}(\widehat{\delta_B\phi}(xu^i, yu^j)) \end{aligned}$$

□

REMARK 5.2.3. Note that the u-linear extension would not make sense for multilinear operations on $C_*[\![u^{-1}]\!][\![u]\!]$, as direct products do not commute with tensor products. The reason we can get away with linear extension for the other complexes is that there is a finite number of ways to write a specific power of u as $u^{i_0+\ldots+i_n}$.

We can further formalize this procedure by applying it to an entire operad of operations on C_*. To do this, we first make a definition.

DEFINITION 5.2.4. Let $\mathbb{T}$ be the operad associated to the monoid $\mathbb{T} = \Bbbk[B]$ where $|B| = 1$ (and $B^2 = 0$ for degree reasons). Given a differential graded operad $\mathcal{D}$ and a map of operads $\mathbb{T} \to \mathcal{D}$, we define the differential graded operad $\mathcal{D}[u]$ as follows:

- As a graded module, $(\mathcal{D}[u])(n) = \mathcal{D}(n)_*[u]$.
- The differential is given by $\widehat{\delta} + u\widehat{\delta_B}$. Here the hat stands for u-linear extension, δ denotes the differential of $\mathcal{D}$ and δ_B denotes the differential on $\mathcal{D}(n)$ that uses the $\mathbb{T}$-module structure.
- Composition of operations in $\mathcal{D}[u]$ is defined as the linear extension of the composition of $\mathcal{D}$.
- The symmetric groups act u-linearly.
- The unit is given by $1_{\mathcal{D}} \in \mathcal{D}(1)_0 \subset \mathcal{D}[u](1)_0$.

PROPOSITION 5.2.5. *The $\mathcal{D}(n)_*[u]$ are chain complexes and form a differential graded operad $\mathcal{D}[u]$.*

PROOF. The map $\mathbb{T} \to \mathcal{D}$ picks out an element $B \in \mathcal{D}(1)_1$, which is then used to build the differential δ_B, using pre- and postcomposition in analogy to the differential on $\mathrm{End}(C_*, B)$:

$$\delta_B\phi = B \circ \phi - (-1)^{|\phi|} \sum_{i=1}^{n} \phi \circ (I^{\otimes i-1} \otimes B \otimes I^{\otimes n-i}).$$

The fact that $\widehat{\delta} + u\widehat{\delta_B}$ squares to zero follows from elementary checks that imply $\delta^2 = \delta_B^2 = \delta_B\delta + \delta\delta_B = 0$. For example, $\delta(B \circ \phi) = (\delta B) \circ \phi - B \circ \delta\phi$

means that the first term in δ_B is a chain map of degree one and hence $\delta_B\delta + \delta\delta_B = 0$.

To see that $\mathcal{D}[u]$ forms an operad, define the composition by u-linear extension

$$(\phi u^{i_0}) \circ (\phi_1 u^{i_1} \otimes \ldots \otimes \phi_n u^{i_n}) := \phi \circ (\phi_1 \otimes \ldots \otimes \phi_n) u^{i_0 + \ldots + i_n}.$$

The fact that this satisfies all the axioms of being an operad follows easily from the fact that $\mathcal{D}$ does. The fact that $\widehat{\delta} + u\widehat{\delta_B}$ is a derivation for the operadic composition on $\mathcal{D}[u]$ is an elementary check that boils down to checking that δ_B is a derivation for the operad composition on $\mathcal{D}$. □

THEOREM 5.2.6. *Let C_* be a $\mathcal{D}$-algebra. In particular, C_* is a $\mathbb{T}$-module using $\mathbb{T} \to \mathcal{D} \to \mathrm{End}(C_*)$. Then the operad $\mathcal{D}[u]$ acts on the cyclic theories and the actions are compatible with the action of $\mathcal{D}$ on C_*.*

PROOF. The action of $\mathcal{D}[u]$ on the cyclic theories is defined by the formula

$$(\phi u^p)(a^1 u^{i_1}, \ldots, a^n u^{i_n}) = \phi(a^1, \ldots, a^n) u^{p + i_1 + \ldots i_n}.$$

To see that this formula defines an action, see the proof of Proposition 5.2.2. □

REMARK 5.2.7. At first sight it might seem that a similar statement should hold for PROPs, but the author was not able to find or prove such a statement.

COROLLARY 5.2.8. *If $\mathcal{F}$ is a multiplicative differential graded PROP, and $\mathcal{D}$ is an operad of natural operations on Hochschild chains of $\mathcal{F}$-algebras, then $\mathcal{D}[u]$ is an operad of natural operations on (negative) cyclic chains.*

5.3. The shuffle product as a combination of the recipes

The two recipes may also be combined. To see this, we review the definition of the shuffle product on cyclic chains as treated in [Lod98]. For any two algebras A and A', the Eilenberg–Zilber Theorem gives a map

$$- \times -\colon C_p(A) \otimes C_q(A') \to C_{p+q}(A \otimes A').$$

When $A = A'$ is commutative, the multiplication map $\mu\colon A \otimes A \to A$ is a map of algebras, so we get a map called the *shuffle product*

$$- * -\colon C_*(A) \otimes C_*(A) \xrightarrow{-\times-} C_*(A \otimes A) \xrightarrow{C_*(\mu)} C_*(A).$$

This operation lifts to the cyclic chains as

$$(\sum x^i u^{-i}) * (\sum y^i u^{-i}) = \sum ((Bx^0) * y^i) u^{-i}.$$

The reason this works, is that b is a derivation for $*$, and that B is almost a derivation for $*$. That is, $*$ and B satisfy $B(Bx * y) = (-1)^{|x|} Bx * By$, making B a derivation for $x \otimes y \mapsto (Bx) * y$. So essentially one uses Recipe I on the first input and Recipe II on the second. The interaction property then ensures that the resulting operation is a chain map.

5.4. Formal operations on cyclic chains

The goal of this section is to apply the framework of formal operations as recalled in Section 2.6, to cyclic chains rather than Hochschild chains. The material of this section is work in progress and not all statements have been checked with complete rigour.

DEFINITION 5.4.1. In analogy to the enriched functor $\mathcal{L} : \mathcal{A}_\infty^{\mathrm{op}} \to \mathsf{Ch}$ of Definition 2.6.2, let $\mathcal{L}^\lambda : \mathcal{A}_\infty^{\mathrm{op}} \to \mathsf{Ch}$ be the functor

$$m \mapsto (\bigoplus_{n\geq 1} \mathcal{A}_\infty(m,n) \otimes L_n \otimes \Bbbk[u^{-1}], d_{\mathcal{A}_\infty} + d_L + ud_B).$$

The morphisms act by precomposition in the PROP $\mathcal{A}_\infty$, and the $d_{\mathcal{A}_\infty}$ and d_L are the u-linear extensions of those that show up in $\mathcal{L}$. The last term of the differential, ud_B, is defined by sending

$$l_n \mapsto uBl_n \colon L_n \otimes \Bbbk[u^{-1}] \to \mathcal{A}_\infty(n+1,n) \otimes L_{n+1} \otimes \Bbbk[u^{-1}].$$

Here B is interpreted as an element of a graph complex as in Figure 3.5c, which may then be applied to l_n to obtain a black and white graph that can be split up as an $\mathcal{A}_\infty(n+1,n)$-part attached to l_{n+1}. This is very similar to the description of d_L in Definition 2.6.2.

PROPOSITION 5.4.2. *Let $\Phi \in \mathcal{A}_\infty$-**Alg**, then $\Phi \otimes_{\mathcal{A}_\infty} \mathcal{L}^\lambda = CC_*(\Phi)$.*

PROOF. By inspecting the definition of the coend construction, and because $\Phi \otimes_{\mathcal{A}_\infty} \mathcal{L} = C_*(\Phi)$, we see that $\Phi \otimes_{\mathcal{A}_\infty} \mathcal{L}^\lambda = C_*(\Phi)[u^{-1}]$ as a graded module. A calculation then shows that the differential is exactly the desired $b + uB$. □

REMARK 5.4.3. One may similarly define $\mathcal{L}^{-\lambda}$ using $\Bbbk[u]$ rather than $\Bbbk[u^{-1}]$ in order to get $CC^{-}_{\oplus *}(\Phi) = \Phi \otimes_{\mathcal{A}_\infty} \mathcal{L}^{-\lambda}$. It seems, however, that the variations of cyclic homology that are defined as product totalizations may not be obtained in a similar fashion.

DEFINITION 5.4.4. For functors $\Phi \in \mathrm{Fun}(\mathcal{E}, \mathsf{Ch})$ and $\Psi \in \mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathsf{Ch})$ we use the notation

$$C_{\mathcal{L}^\lambda}(\Phi)(m) = (i^*\Phi(-+m)) \otimes_{\mathcal{A}_\infty} \mathcal{L}^{-\lambda}$$
$$D_{\mathcal{L}^\lambda}(\Psi)(m) = \mathrm{Nat}_{\mathcal{A}_\infty^{\mathrm{op}}}(\mathcal{L}^\lambda, i^*\Psi(-+m)).$$

These constructions $C(-)_{\mathcal{L}^\lambda}$ and $D(-)_{\mathcal{L}^\lambda}$ define endofunctors on the functor categories similarly to Definition 1.2 in [Wah14].

An argument similar to the proof of Theorem A in [Wah14] or an application of Theorem 4.15 in [Kla13c, Paper C] gives the following theorem.

THEOREM 5.4.5 (In progress). *For a multiplicative PROP $\mathcal{E}$, we can rewrite the PROP of formal operations as*

$$\mathrm{Nat}_{\mathcal{E}}^{\mathrm{formal}}(n_1, n_2) \cong D_{\mathcal{L}^\lambda}^{n_1} C_{\mathcal{L}^\lambda}^{n_2} \mathcal{E}(-,-).$$

This expression allows us to further identify the complex of formal operations in more concrete terms as in Theorem 2.6.4.

$$\prod_{j_1,\ldots,j_{n_1}} \bigoplus_{k_1,\ldots,k_{n_2}} \mathcal{E}(j,k)[k-j+n_1-n_2][\![u_1,\ldots,u_{n_1}]\!][u_1^{-1},\ldots,u_{n_2}^{-1}]$$

Here, all the indices run from one to infinity, $j = j_1 + \ldots + j_{n_1}$ and similarly for k. The first set of brackets denotes a degree shift. The differential is similar to the one in the statement for Hochschild chains, except that there are extra terms for the pre- and postcomposition by the B operator in a fashion very similar to Recipe II.

CHAPTER 6

Perspectives for further research

As the research presented in Chapters 3–5 is work in progress, many loose ends remain to be tied up. Apart from that, many other opportunities for further research arise from the work. We outline some of those opportunities.

6.1. Operations on Hochschild complexes

The first step that needs to be taken in order to complete the work of Chapter 3, is to rigorously prove that the operad $\mathcal{T}$ acts on the triple $(C^*(A), C_*(A), A)$ for every $\mathcal{A}_\infty$-algebra A. A sensible next step would be to compare the operations in $\mathcal{T}$ to the known existing operations described in the introduction of Chapter 3.

The three two-coloured operads obtained by omitting one of the three colours from $\mathcal{T}$ are thought to be equivalent to the singular chains on a topological operad. For example, the operad from [DTT11] which encodes the operations on $(C^*(A), A)$ is equivalent to chains on Voronov's Swiss cheese operad $\mathcal{SC}_2$. This is a generalization of the Deligne conjecture, also referred to as the Swiss cheese conjecture, see [DTT11; Gin15; Tho16]. This begs the question whether there is also a three-coloured topological operad, of which the chains are equivalent to $\mathcal{T}$, giving another generalization of the Deligne conjecture. A solution to this Deligne conjecture would in particular reveal the homology of $\mathcal{T}$, although this is likely easier to calculate directly.

As mentioned in Remark 1.1 of [DTT11], Kontsevich has conjectured in [Kon99] that the pair $(C^*(A), A)$, where A is an $H_*(\mathcal{E}_d)$-algebra, is an algebra over the chains of the d-dimensional Swiss cheese operad, and that the Hochschild cochains are final in an appropriate category of Swiss cheese algebras. It would be interesting to investigate whether such a statement holds in the three coloured case also. Some evidence for this is provided in [Cos07b], where a similar homotopy universality property is proven for the Hochschild chains of an $\mathcal{O}$ algebra.

Variations of the graph complexes considered might also be useful for studying operations on the triple $(C^*(A), C_*(A), A)$ for other classes of algebras, e.g., for $\mathcal{C}om$, $\mathcal{E}_2$, or even $\mathcal{O}$-algebras.

6.2. Naturality of Hochschild cochains

The definitions made in Chapter 4 should be compared to other parts of the literature. For example, in [BBM13] the operad of operations on $C^*(A)$ for associative A, is as sums of compositions of elementary operations. As our natural operations are closed under composition, it is most important to show that the elementary operations are in our complex and to check that the differential agrees.

As a sanity check, it is also wise to compare the natural operations on chains to those in the literature.

QUESTION. When we restrict the colours of $\underline{\text{Nat}}_{\mathcal{F}}$ to only consider Hochschild chains, how does our construction compare to other complexes of operations, e.g., the formal operations from [Wah14]?

We have found a comparison map between our PROP of natural operations and the PROP of formal operations, but so far it is not known to have any good properties.

Two other possible directions are: Can the definition be extended to include cyclic and dihedral cohomology and are the operations from Chapter 3 examples of natural operations?

6.3. Operations on Cyclic homology

After thoroughly checking Theorem 5.4.5, one could try to compute the complex of formal operations for the commutative operad, similarly to what Klamt did in [Kla13b]. The expected answer is that the operations are generated by shuffles and the periodicity operator, given that the operations on the Hochschild chains are generated by shuffles and the B operator.

R. Cohen and S. Ganatra are currently establishing an equivalence between the symplectic field theory structure on the symplectic homology $SH_*(T^*M)$ and the string topology field theory on $H_*(\mathcal{L}M)$, extending Viterbo's isomorphism $SH_*(T^*M) \cong H_*(\mathcal{L}M)$ [CG15]. Both sides have $\mathbb{T}$-equivariant analogues which have the structure of an 'Involutive Lie Bialgebra' (IBL) [Sei08; CS04]. In the very recent work [CFL15], Cieliebak, Fukaya and Latschev describe a chain level version of IBL-algebras called IBL_∞-algebras. Establishing such IBL_∞-structures on $SH^*_{\mathbb{T}}(T^*M) \cong H^*(\mathcal{L}M_{h\mathbb{T}})$ and their equivalence remains open. An interesting problem would be to describe the algebraic structure of equivariant string topology on the chain level using negative cyclic homology. To achieve this, one could follow a specific path:

(1) First, construct a graph complex that forms a generalisation of Costello's open-closed conformal field theory [Cos07b]. More precisely, find a differential graded PROP that acts naturally the negative cyclic chains of $\mathcal{O}$-algebras, or Calabi–Yau A_∞ categories, generalisations of dga's with Poincaré duality.
(2) Show a homotopy universality property and extend the works [Wah14; WW16] to justify that all operations have been found.
(3) Finally, analyse the structure obtained, comparing it in particular to the IBL_∞-algebras of equivariant symplectic and string topology field theory in [CFL15; CS04].

In [Cos07b], the complexes of operations on such Hochschild chains are isomorphic to chains on moduli spaces of open-closed cobordisms (hence the name topological conformal field theory). This leads to the following interesting question.

Question. Are there moduli spaces of certain types of cobordisms that model operations on negative cyclic homology of $\mathcal{O}$-algebras?

This question has been studied by Richard Hepworth, who conjectured in the lecture [Hep13] that the answer would be

$$\bigoplus_{[\Sigma]} \pi_*(H\mathbb{Z} \wedge (B\,\mathrm{Diff}^\partial\,\Sigma)_{h\mathbb{T}^s})^{h\mathbb{T}^r}.$$

6.4. Dihedral homology

It is not entirely clear to the author which parts of the literature are affected by the gap that is fixed in Paper A. A particular issue that might need attention is for example: Let X be a compact oriented simply connected manifold of finite type over a field $\Bbbk$, how are the product and coproduct on $HH_*(S^*(X;\Bbbk))$ related to the cup product and Chas–Sullivan coproduct on $H^*(\mathcal{L}X;\Bbbk)$?

Other topics that need investigation are:

- Paper B is an instance of the Universal Exercise in [Lod98, p. 5.2.1]: "Take any result in this book about cyclic homology and try to find and prove an analog for quaternionic or dihedral homology.". Another instance would be to construct a shuffle product on (negative) dihedral chains and relate this to the string bracket on $H^*(\mathcal{L}X_{h\mathbb{O}};\Bbbk)$.
- Can the ideas from Chapter 5 also be applied to dihedral homology and not just to cyclic homology?

- Can the simply connectedness assumptions of the Main Theorem of Paper B be weakened to a nilpotence condition? See also Remark 8.2 in Paper B and Proposition 5.3 in [AF15].
- Can the results of Paper B be applied to learn something about Hermitian K-Theory? See [Cor93; KrS86; Ldd96].
- The proof of Proposition 6.4 seems somewhat suboptimal. It seems like the existence of an appropriate operad should follow more directly from the $\mathcal{E}_\infty$-structure of $S^*(X)$ and some facts from the homotopy theory of operads.
- What is the role of dihedral homology in the Hochschild–Kostant–Rosenberg Theorem?
- Given a simplicial set $X_\bullet$, the singular cochain algebra $S^*(|X_\bullet|)$ is an involutive dga and it is quasi isomorphic as a dga to the simplicial cochains of $X_\bullet$. Is there a quasi isomorphic involutive dga that is smaller than $S^*(|X_\bullet|)$? One approach would be to replace the simplicial set $X_\bullet$ by a reflexive set, for example by using one of the two Kan extensions along $\Delta \hookrightarrow \Delta R$. The author pursued this direction, but did not manage to find a positive answer. A solution to this problem would provide smaller models that are probably easier to compute with.

As suggested in the introduction of Paper B, perhaps the most important thing to prove is that spheres are involutively formal over $\mathbb{F}_2$. That is, is $S^*(S^n; \mathbb{F}_2)$ quasi isomorphic to its cohomology as an involutive dga? If so, the calculations in the last section show that the Betti numbers of $H^*(\mathcal{L}S^2_{h\mathbb{O}}; \mathbb{F}_2)$ are unbounded. As explained in the Introduction of Paper B, this might have consequences for the following open question.

Question. Does the 2-sphere, equipped with an arbitrary Riemannian metric, admit infinitely many distinct geodesics?

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Paper A

FREE LOOP SPACE AND THE CYCLIC BAR CONSTRUCTION

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Abstract. Using the E_∞-structure on singular cochains, we construct a homotopy coherent map from the cyclic bar construction of the differential graded algebra of cochains on a space to a model for the cochains on its free loop space. This fills a gap in the paper "Cyclic homology and equivariant homology" by John D.S. Jones.

1. Introduction

Hochschild homology has been widely used to provide an algebraic model for the cohomology of free loop spaces. In particular, there is an isomorphism $HH_*(S^*(X)) \cong H^*(LX)$ for the singular cochain algebra $S^*(X)$ of a simply connected space X. This was proved by John D.S. Jones in [Jon87], together with its $SO(2)$-equivariant version.

One step in the proof of this isomorphism requires one to establish the equivalence of two diagrams of chain complexes, $B_\bullet^{\mathrm{cyc}} S^*(X)$ and $S^*(\mathrm{Map}(S^1_\bullet, X))$. The first is the cyclic bar construction of cochains on a space X and the second is given by the cochains on $\mathrm{Map}(S^1_\bullet, X)$, a cocyclic space modelling the free loop space of X. Jones uses the Alexander–Whitney map to compare these two cyclic objects, which gives a map on every simplicial level; however, it does not form a map of cyclic objects, as it does not commute with the structure maps of the cyclic category. This fact can already be seen in simplicial level one, where the Alexander–Whitney map should be symmetric on the cochain level in order to commute with the cyclic operator t and the first boundary map d_1. This cannot be the case, as it would imply that the cup product is commutative on the cochain level. The cup product is, however, commutative up to coherent homotopy and it is this natural E_∞-structure that will be used to construct a homotopy coherent isomorphism instead, filling the gap in the proof.

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Main Theorem. *Let X be a space with finite type homology over a principal ideal domain $\Bbbk$. There is a natural zigzag of equivalences of cyclic chain complexes*

$$B^{\mathrm{cyc}}_{\bullet} S^*(X;\Bbbk) \xleftarrow{\simeq} QB^{\mathrm{cyc}}_{\bullet} S^*(X;\Bbbk) \xrightarrow{\simeq} S^*(\mathrm{Map}(S^1_{\bullet}, X);\Bbbk),$$

where $QB^{\mathrm{cyc}}_{\bullet} S^*(X;\Bbbk)$ is a resolution of the cyclic bar construction.

Remark 1. The finiteness assumption is not needed when working with chains rather than cochains: the cyclic cobar construction of the coalgebra of chains $\Omega^{\mathrm{cyc}}_{\bullet} S_*(X)$ is equivalent to $S_*(\mathrm{Map}(S^1_{\bullet}, X))$. This statement works over the integers and uses the same proof combined with the observation that the E_∞-structure on cochains described in [MS03] is the linear dual of an operad coaction on chains.

If X is simply connected and of finite type over a field $\Bbbk$, Jones' proof implies the isomorphisms

$$\begin{aligned} H^*(LX;\Bbbk) &\cong HH_*(S^*(X;\Bbbk)) \\ H^*(LX \times_{\mathrm{SO}(2)} E\mathrm{SO}(2);\Bbbk) &\cong HC^-_*(S^*(X;\Bbbk)). \end{aligned}$$

The assumption that X is of finite type over a field $\Bbbk$ is not explicitly stated in [Jon87], but it is used in a cited paper: That $\Bbbk$ is a field is assumed in [And72] to establish the "convergence" of the cosimplicial mapping space $\mathrm{Map}(S^1_{\bullet}, X)$ over $\Bbbk$. The finite type assumption ensures that the Alexander–Whitney map $S^*(X) \otimes S^*(X) \to S^*(X \times X)$ is a quasi isomorphism.

In Proposition 5.3 of [AF15], the authors use factorization homology over S^1 to show that the assumptions can be weakened to allow $\Bbbk$ to be any commutative ring, X a nilpotent space equivalent to a finite type CW complex and π_1 finite.

From the algebraic theorem in [JM92], one can reprove both isomorphisms using the same conditions. For this one needs to start with the isomorphism $H_*(LX) \cong HH_*(S_*(\Omega X))$ from [Goo85] and combine this with Adams' cobar equivalence.

A failure to commute with the last boundary map also appears in the papers [PS16] and [Wah04], where methods similar to ours are used.

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1.2. **Conventions.** We use the closed symmetric monoidal structure of **Ch**, the category of homologically graded chain complexes of abelian groups. The tensor product of two chain complexes carries a differential $d(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy$. The internal hom is a chain complex $\underline{\mathsf{Ch}}(X, Y)$ that in degree n consists of linear maps of degree n and has differential $(d\psi)(x) = d \circ \psi(x) - (-1)^n \psi(dx)$. This means that the chain maps are the 0-cycles in this chain complex. Any cosimplicial object in the category of chain complexes $A^\bullet$ gives a double complex using $\Sigma(-1)^i \delta^i$. Its (product) totalization is written as A, omitting the bullet. We use similar notation for simplicial chain complexes $A_\bullet$.

1.3. **Cyclic objects.** We briefly recall some definitions of cyclic objects and refer to [Jon87, Lod98] for more details. The morphisms of the category of finite ordered sets Δ are generated by δ^i, σ^i, which satisfy the simplicial relations. By appropriately adding cyclic permutations $\langle \tau \rangle = C_{n+1}$ as the automorphisms of $[n]$, one obtains Connes' cyclic category Λ. Functors out of this category are called (co)cyclic objects.

Example 1. There is a cyclic set $[n] \mapsto S^1_n = \mathbb{Z}/(n+1)\mathbb{Z}$ that realizes to the circle. From this, one obtains for each space X a cocyclic space $[n] \mapsto \mathrm{Map}(S^1_n, X) = X^{n+1}$ which totalizes to the free loop space LX. The coboundaries are given by the diagonal maps, the codegeneracies by forgetting factors and the cyclic maps by cyclically permuting the factors. For example,

$$\delta^{n+1}(x_0, \dots, x_n) = (x_0, x_1, \dots, x_n, x_0).$$

By functoriality of S^*(-), $S^*(\mathrm{Map}(S^1_\bullet, X))$ is a cyclic chain complex.

Example 2. For any unital differential graded algebra A, we have the *cyclic bar complex* $(B^{\mathrm{cyc}} A)[n] = A^{\otimes n+1}$, that can be used to compute Hochschild and cyclic homology. The structure maps are given by multiplication, insertions of the unit and cyclic permutations of the tensor factors.

1.4. **Homotopy commutative structure of cochains.** The main ingredient for the proof of our main theorem is the natural E_∞-structure on the normalized singular cochains $S^*(X)$. Such operad actions are given, for example, in [BF04, MS03] and are the integral analogue of Sullivan and Quillen models [Man06]. In our proofs, we only use the fact that there exists a symmetric differential graded operad $\mathcal{S}$ which has the homology of a point in every arity and which comes with a map from the unital associative operad and a map to the natural operations on $S^*(X)$ which specify the cup product and its unit.

Remark 2. An inductive argument for the contractibility of an operad $\mathcal{S}$ is given on p. 689 of [MS03]. However, there is a minor mistake that may be spotted by applying the formula $\partial s + s\partial = id + \iota r$ to the example $\langle 3123 \rangle$. To fix this, it is enough to change the map r to only be 0 unless the sequence contains exactly a single 1.

2. Homotopy coherent natural transformations

In this section we adapt the treatment of homotopy coherent natural transformation in [Dug08, §8] from spaces to chain complexes.

Definition 1. Let I be a small category and $F, G \colon I \to \mathsf{Ch}$ two diagrams of chain complexes. Define the cosimplicial chain complex of *homotopy coherent natural transformations* $\mathrm{hc}(F,G)^\bullet \colon \Delta \to \mathsf{Ch}$ as

$$\mathrm{hc}(F,G)^n = \prod_{\underline{\phi} \in N_n I} \underline{\mathsf{Ch}}(F(i_0), G(i_n)),$$

where the product runs over simplices of the nerve $N_\bullet I$ of I. The structure maps on such families $A \in \mathrm{hc}(F,G)^n$ are given by

$$(\sigma^i A)_{\underline{\phi}} = A_{s_i \underline{\phi}}$$

$$(\delta^i A)_{\underline{\phi}} = \begin{cases} F(i_0) \xrightarrow{F(i_0 \to i_1)} F(i_1) \xrightarrow{A_{d_0 \underline{\phi}}} G(i_{n+1}) & \text{if } i = 0, \\ F(i_0) \xrightarrow{A_{d_i \underline{\phi}}} G(i_{n+1}) & \text{if } 0 < i < n+1, \\ F(i_0) \xrightarrow{A_{d_{n+1} \underline{\phi}}} G(i_n) \xrightarrow{G(i_n \to i_{n+1})} G(i_{n+1}) & \text{if } i = n+1. \end{cases}$$

A single *homotopy coherent natural transformation* is defined to be a 0-cycle in the totalization $\mathrm{hc}(F,G)$.

Example 3. Finding single homotopy coherent natural transformation means finding a family of elements $A^n \in \mathrm{hc}(F,G)^n$ of degree n such that $dA^0 = 0$ and $\Sigma_i (-1)^i \delta^i A^n = (-1)^n dA^{n+1}$. These A^n are themselves families indexed by $\underline{\phi} \in N_n I$, which we write as $A_{\underline{\phi}} \in \underline{\mathsf{Ch}}(F(i_0), G(i_n))$. For $n = 0$ this means that we have $A_i \colon F(i) \to G(i)$ a chain map of degree zero for each object $i \in I$. In the case when $n = 1$, we have for each morphism $\phi \colon i_0 \to i_1$ in I a map $A_\phi \colon F(i_0) \to G(i_1)$ of degree one. These maps are not required to be chain maps but instead satisfy

$$d_{G(i_1)} \circ A_\phi + A_\phi \circ d_{F(i_0)} = (\delta^0 A)_\phi - (\delta^1 A)_\phi = A_{i_1} \circ F(\phi) - G(\phi) \circ A_{i_0}.$$

That is, the maps A_ϕ provide chain homotopies implementing the failure of the naturality squares to commute on the nose.

$$\begin{array}{ccc} F(i_0) & \xrightarrow{F(\phi)} & F(i_1) \\ {\scriptstyle A_{i_0}}\downarrow & & \downarrow{\scriptstyle A_{i_1}} \\ G(i_0) & \xrightarrow{G(\phi)} & G(i_1) \end{array}$$

For $n \geq 1$, we have homotopies relating all the different ways of composing a string of morphisms and the previous homotopies.

Definition 2. For a small diagram $F\colon I \to \mathsf{Ch}$, we define the resolution $QF_\bullet\colon I \times \Delta^{\mathrm{op}} \to \mathsf{Ch}$ as the two-sided bar construction $QF_\bullet = B_\bullet(I, I, F)$, where the first I is shorthand for the bifunctor $\mathbb{Z}I(\text{-},\text{-})$. Concretely, this gives a simplicial I-diagram, which at the object $i \in \operatorname{Ob} I$ in simplicial degree n is a sum over n-simplices in $N_\bullet(I/i)$.

$$QF_n(i) = \bigoplus_{i_0 \to \ldots \to i_n \to i} F(i_0)$$

For a morphism $\alpha\colon i \to j$, we have a map $QF_\bullet(i) \to QF_\bullet(j)$ induced by $\alpha_*\colon N_\bullet(I/i) \to N_\bullet(I/j)$. The simplicial structure maps all act on the indexing sets $N_\bullet(I/i)$, where a composition with $F(i_0 \to i_1)$ is needed in the definition of d_0.

Proposition 1. *For a small diagram $F\colon I \to \mathsf{Ch}$, the resolution $QF_\bullet\colon I \times \Delta^{\mathrm{op}} \to \mathsf{Ch}$ has the following properties:*

(1) *There is a canonical isomorphism of cosimplicial chain complexes*

$$\alpha^\bullet\colon \underline{\mathrm{Nat}}_I(QF_\bullet, G) \xrightarrow{\cong} \mathrm{hc}(F,G)^\bullet.$$

(2) *There is a natural object-wise quasi isomorphism $QF \xrightarrow{\simeq} F$.*

(3) *Under the identification of total complexes*

$$\alpha\colon \underline{\mathrm{Nat}}_I(QF, G) \xrightarrow{\cong} \mathrm{hc}(F,G),$$

the quasi isomorphisms on the left hand side correspond on the right hand side to the 0*-cycles that in cosimplicial degree 0 give quasi isomorphisms $F(i) \xrightarrow{\simeq} G(i)$.*

Proof.

(1) On cosimplicial level n, the left hand side is a subset

$$\underline{\mathrm{Nat}}_I(QF_n, G) \subset \prod_{i \in \operatorname{Ob} I} \underline{\mathsf{Ch}}\Big(\bigoplus_{i_0 \to \ldots \to i_n \to i} F(i_0), G(i)\Big) = \prod_{i_0 \to \ldots \to i_n \to i} \underline{\mathsf{Ch}}(F(i_0), G(i)).$$

It is exactly the subset determined by a naturality condition, comparing $\alpha_*\colon N_\bullet(I/i) \to N_\bullet(I/j)$ with $G(\alpha)$ for morphisms $\alpha\colon i \to j$.

One sees that this amounts exactly to the data being determined by the simplices of the form $i_0 \to \ldots \to i_n \xrightarrow{id} i_n$. To obtain the value at $i_0 \to \ldots \to i_n \to i$, post-compose by $G(i_n \to i)$.

It remains to compare the cosimplicial structure maps. The δ^j for $j \neq 0, n$ are clear as they only affect the index. Since d_0 used $F(i_0 \to i_1)$, we see that δ^0 precomposes by this map. For the last coboundary, we need to use the naturality to see the postcomposition by $G(i_{n-1} \to i_n)$.

(2) There is a standard augmentation $QF \to F$ defined as

$$\sum F(i_0 \to i) \colon \bigoplus_{i_0 \to \ldots \to i_n \to i} F(i_0) \to F(i),$$

with contracting homotopy s_{n+1} given by appending the identity at the end of the indexing simplex.

(3) First, observe that the chain maps are the 0-cycles. The behaviour of a map $QF(i) \to G(i)$ in homology is determined by what it does in cosimplicial degree 0. This can be seen by inspecting the augmentation and $H_*(F) \cong H_*(QF) \to H_*(G)$.

□

3. Comparing the cyclic chain complexes

In this section we construct a homotopy coherent natural transformation for $I = \Lambda^{\mathrm{op}}, F = B_\bullet^{\mathrm{cyc}} S^*(X), G = S^*(\mathrm{Map}(S^1_\bullet, X)) \colon \Lambda^{\mathrm{op}} \to \mathsf{Ch}$.

Proposition 2. *There exists a homotopy coherent natural transformation A from $B_\bullet^{\mathrm{cyc}} S^*(X)$ to $S^*(\mathrm{Map}(S^1_\bullet, X))$ that at each object i is given by the Alexander–Whitney maps*

$$B^{\mathrm{cyc}} S^*(X)(i) = S^*(X)^{\otimes i+1} \to S^*(X^{i+1}) = S^*(\mathrm{Map}(S^1_\bullet, X))(i)$$

and is natural in X.

Before giving the proof of this proposition, we introduce some notation and prove a lemma. For any $\underline{\phi} = (i_0 \xrightarrow{\phi_1} i_1 \ldots \xrightarrow{\phi_m} i_m) \in N_m \Lambda^{\mathrm{op}}$, we write ϕ for the composition $\phi_m \circ \ldots \circ \phi_1$. Associated to $\phi \in \Lambda^{\mathrm{op}}([i],[j])$ are three structure maps of (co)cyclic objects:

$$\begin{aligned}
&\phi_* \colon S^*(X)^{\otimes i+1} \to S^*(X)^{\otimes j+1} \\
&\phi \colon X^{j+1} \to X^{i+1} \\
&\phi^* \colon S^*(X^{i+1}) \to S^*(X^{j+1}).
\end{aligned}$$

Using the projections $\pi_k \colon X^{i+1} \to X, k = 0, \dots, i$, we moreover associate to each $\phi \in \Lambda^{\mathrm{op}}([i],[j])$ a map

$$\pi_\phi = (\pi_0 \circ \phi)^* \otimes \dots \otimes (\pi_i \circ \phi)^* \colon S^*(X)^{\otimes i+1} \to S^*(X^{j+1})^{\otimes i+1}.$$

Lemma 1. *The following square commutes for any $\phi \in \Lambda^{\mathrm{op}}([i],[j])$.*

$$\begin{array}{ccc}
S^*(X)^{\otimes i+1} & \xrightarrow{\phi_*} & S^*(X)^{\otimes j+1} \\
{\scriptstyle \pi_\phi}\downarrow & & \downarrow{\scriptstyle \pi_{id}} \\
S^*(X^{j+1})^{\otimes i+1} & \xrightarrow{\phi_*} & S^*(X^{j+1})^{\otimes j+1}
\end{array}$$

Proof. This is an elementary check for the boundaries, degeneracies and cyclic operators, which together generate all morphisms in Λ^{op}. If we have two composable morphisms $\psi \in \Lambda^{\mathrm{op}}([i],[j]), \phi \in \Lambda^{\mathrm{op}}([j],[k])$ that both satisfy the condition, then their composition satisfies the condition.

$$\begin{array}{ccccc}
S^*(X)^{\otimes i+1} & \xrightarrow{\psi_*} & S^*(X)^{\otimes j+1} & \xrightarrow{\phi_*} & S^*(X)^{\otimes k+1} \\
{\scriptstyle \pi_\psi}\downarrow & & \downarrow{\scriptstyle \pi_{id}} & & \\
S^*(X^{j+1})^{\otimes i+1} & \xrightarrow{\psi_*} & S^*(X^{j+1})^{\otimes j+1} & & \downarrow{\scriptstyle \pi_{id}} \\
{\scriptstyle (\phi^*)^{\otimes i+1}}\downarrow & & \downarrow{\scriptstyle (\phi^*)^{\otimes j+1}} & & \\
S^*(X^{k+1})^{\otimes i+1} & \xrightarrow{\psi_*} & S^*(X^{k+1})^{\otimes j+1} & \xrightarrow{\phi_*} & S^*(X^{k+1})^{\otimes k+1}
\end{array}$$

Note that $\pi_\phi = (\phi^*)^{\otimes i+1} \circ \pi_{id}$ and $\pi_{\psi\circ\phi} = (\phi^*)^{\otimes j+1} \circ \pi_\psi$. The top left and right hand squares commute by assumption on ψ and ϕ respectively. The bottom left square commutes by naturality of the cyclic bar construction with respect to the map ϕ^* of differential graded algebras. $\square$

Proof of Proposition 2. Fix a contractible operad $\mathcal{S}$ acting naturally on $S^*(X)$ as described in Section 1.4. We will prove the existence of the homotopy coherent map A by induction on the cosimplicial degree m. For $m = 0$ we have the Alexander–Whitney maps, which are given as

$$A_i^0 \colon S^*(X)^{\otimes i+1} \xrightarrow{\pi_{id}} S^*(X^{i+1})^{\otimes i+1} \xrightarrow{S_i} S^*(X^{i+1}).$$

Here S_i is the $i+1$ fold cup product, which lives in the operad as $S_i \in \mathcal{S}(i+1)_0$. The fact that this map can be factored as such will be the essential idea of the proof. Assume that we have defined A^m for $m < n$ satisfying the boundary condition $\sum(-1)^j(\delta^j A^{m-1})_{\underline{\phi}} = (-1)^m dA^m_{\underline{\phi}}$. Assume furthermore that the components are of the form $A_{\underline{\phi}} = S_{\underline{\phi}} \circ \pi_\phi$ for $\underline{\phi} \in N_m\Lambda^{\mathrm{op}}$ with composition

ϕ and the $S_{\underline{\phi}} \in \mathcal{S}(i_0+1)_m$ satisfying

$$(\star) \qquad (-1)^m dS_{\underline{\phi}} = S_{d_0\underline{\phi}} \circ \phi_{1*} + \sum_{j=1}^{m} (-1)^j S_{d_j\underline{\phi}}.$$

To show that we can extend this construction to level n, we need to find $A_{\underline{\phi}}$ of this form for all $\underline{\phi} \in N_n\Lambda^{\mathrm{op}}$ in such a way that the boundary condition holds and $S_{\underline{\phi}}$ satisfies $(\star)$. To do this, we describe the cosimplicial differential of $\mathrm{hc}(B^{\mathrm{cyc}}S^*(X), S^*(\mathrm{Map}(S^1_\bullet, X)))$ to see that $(\star)$ implies the boundary condition.

The first coboundary can be written as $(\delta^0 A)_{\underline{\phi}} = A_{d_0\underline{\phi}} \circ \phi_{1*} = S_{d_0\underline{\phi}} \circ \pi_{d_0\phi} \circ \phi_{1*} = S_{d_0\underline{\phi}} \circ \phi_{1*} \circ \pi_\phi$. The last equality can be seen using the commuting diagram

$$\begin{array}{ccccc}
S^*(X)^{\otimes i_0+1} & \xrightarrow{\pi_{\phi_1}} & S^*(X^{i_1+1})^{\otimes i_0+1} & \xrightarrow{(\hat{\phi}^*)^{\otimes i_0+1}} & S^*(X^{i_m+1})^{\otimes i_0+1} \\
\downarrow{\scriptstyle \phi_{1*}} & & \downarrow{\scriptstyle \phi_{1*}} & & \downarrow{\scriptstyle \phi_{1*}} \\
S^*(X)^{\otimes i_1+1} & \xrightarrow{\pi_{id}} & S^*(X^{i_1+1})^{\otimes i_1+1} & \xrightarrow{(\hat{\phi}^*)^{\otimes i_1+1}} & S^*(X^{i_m+1})^{\otimes i_m+1},
\end{array}$$

where $\hat{\phi} = \phi_m \circ \ldots \circ \phi_2$ is associated to $d_0\underline{\phi}$. The first square commutes by Lemma 1 and the second by naturality of the cyclic bar construction.

The last coboundary can be factored as

$$(\delta^m A)_{\underline{\phi}} = {\phi_m}^* \circ A_{d_m\underline{\phi}} = {\phi_m}^* \circ S_{d_m\underline{\phi}} \circ \pi_{\tilde{\phi}} = S_{d_m\underline{\phi}} \circ \pi_\phi.$$

Associated to the $d_m\underline{\phi}$ is the composition $\tilde{\phi} = \phi_{m-1} \circ \ldots \circ \phi_1$ and the last equality is a consequence of the commutativity of the diagram.

$$\begin{array}{ccccc}
S^*(X)^{\otimes i_0+1} & \xrightarrow{\pi_{\tilde{\phi}}} & S^*(X^{i_{m-1}+1})^{\otimes i_0+1} & \xrightarrow{S_{d_m\underline{\phi}}} & S^*(X^{i_{m-1}+1}) \\
& \searrow^{\pi_\phi} & \downarrow{\scriptstyle ({\phi_m}^*)^{\otimes i_0+1}} & & \downarrow{\scriptstyle {\phi_m}^*} \\
& & S^*(X^{i_m+1})^{\otimes i_0+1} & \xrightarrow{S_{d_m\underline{\phi}}} & S^*(X^{i_m+1})
\end{array}$$

The square commutes by the naturality of the operation $S_{d_m\underline{\phi}}$.

All the intermediate coboundaries $(\delta^j A)_{\underline{\phi}}$ for $j \neq 0, m$ are already of the form $(\delta^j A)_{\underline{\phi}} = A_{d_j\underline{\phi}} = S_{d_j\underline{\phi}} \circ \pi_\phi$. Observe that $\pi_{d_j\underline{\phi}} = \pi_\phi$.

This shows that $(\star)$ implies the boundary condition. Also, the expression $(\star)$ lives entirely inside $\mathcal{S}$ since ϕ_* is a composition of cup products, insertions of identities and permutations of arguments. One can see that the right hand side is in fact a cycle in $\mathcal{S}$ by applying the differential termwise and using the inductive hypotheses. This produces terms like $S_{d_0d_0\underline{\phi}} \circ \phi_{2*} \circ \phi_{1*}$ and $S_{d_ld_j\underline{\phi}}$ which all cancel out by the simplicial identities. As $\mathcal{S}$ has the homology of a point in every arity, this implies that such $S_{\underline{\phi}}$ exist. $\square$

Proof of Main Theorem. The augmentation of Proposition 1.2 provides the first quasi isomorphism $QB^{\mathrm{cyc}}_{\bullet} S^*(X;\Bbbk) \xrightarrow{\simeq} B^{\mathrm{cyc}}_{\bullet} S^*(X;\Bbbk)$. The second map $QB^{\mathrm{cyc}}_{\bullet} S^*(X;\Bbbk) \to S^*(\mathrm{Map}(S^1_{\bullet}, X);\Bbbk)$ is provided by Propositions 1.1 and 2. The finiteness assumptions imply that the Alexander–Whitney maps are quasi isomorphisms, meaning we can apply Propositions 1.3 to see that the map is a quasi isomorphism. □

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Paper B

FREE LOOP SPACES AND DIHEDRAL HOMOLOGY

MASSIMILIANO UNGHERETTI

Abstract. We prove an $O(2)$-equivariant version of the Jones isomorphism relating the Borel $O(2)$-equivariant cohomology of the free loop space to the dihedral homology of the cochain algebra. We discuss polynomial forms and a variation of the de Rham isomorphism and use these to do a computation for the 2-sphere.

1. Introduction

For any space X, one may form the (unbased) mapping space $\mathcal{L}X = \mathrm{Map}(S^1, X)$. These *free loop spaces* have played a big role in geometry, topology and physics; in particular in string theory, string topology, loop groups and the study of geodesics through the use of the energy functional on free loop space. This is exemplified by the celebrated Gromoll–Meyer Theorem [GM69], which states that a simply connected closed Riemannian manifold admits infinitely many distinct closed geodesics if the sequence of Betti numbers $\{\mathrm{rk}\, H^k(\mathcal{L}X; \Bbbk)\}_{k\geq 0}$ is unbounded for a field $\Bbbk$. Although many manifolds are covered by this theorem, it remains an open question whether the assumption on the Betti numbers can be dropped.

The Gromoll–Meyer Theorem is proven by studying the infinite dimensional Morse theory of the energy functional $E(\gamma) = \int_{S^1} ||\dot{\gamma}(t)||^2 dt$ on $\mathcal{L}X$ as the critical points of E correspond to closed geodesics. In [Bot82, p. 350] Bott proposes that one has to take into account the invariance of the energy functional under rotations and reflections of the circle. And indeed, Rademacher and Hingston have shown that some other classes of Riemannian manifolds also admit infinitely many distinct geodesics by using Borel equivariant homology with respect to the rotations in $\mathbb{T} = SO(2) \subset O(2)$. A survey of such results is given in [Oan15]. Although Lusternik and Schnirelmann proved that the 2-sphere with arbitrary Riemannian metric carries at least three distinct closed geodesics, it is not known if there are always infinitely many of them.

Taking into account the full $O(2)$-symmetry could bring a full answer even closer. This is pointed out in Remark 6.4 of [Oan15] in the following way. An

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example of Katok [Kat73] shows that there is a non-symmetric Finsler metric on the n-sphere that admits only finitely many distinct closed geodesics. The notion of a non-symmetric Finsler metric is one that generalizes that of a Riemannian metric in a way that breaks the time reversal symmetry, signifying that the full $O(2)$-symmetry is really needed for admitting infinitely many geodesics.

A common tool for computing the (co)homology of $\mathcal{L}X$ is the homology theory for algebras, Hochschild homology HH_*, and its variations like cyclic homology HC_* and negative cyclic homology HC_*^-. In particular, we have the following two theorems available.

Theorem 1.1 ([Goo85], [BF86]). *Let X be a connected space and $\Bbbk$ any ring.*

$$H_*(\mathcal{L}X;\Bbbk) \cong HH_*(S_*(\Omega^{Moore}X;\Bbbk))$$

$$H_*(\mathcal{L}X_{h\mathbb{T}};\Bbbk) \cong HC_*(S_*(\Omega^{Moore}X;\Bbbk))$$

Here $(-)_{h\mathbb{T}}$ denotes the Borel construction with respect to the circle group $\mathbb{T} = SO(2)$ and $S_*(\Omega^{Moore}X;\Bbbk)$ is the differential graded algebra of singular chains on the associative monoid of Moore loops on X with Pontryagin product.

Theorem 1.2 ([Jon87]). *Let $\Bbbk$ be a field and X a simply connected space with finite type homology over $\Bbbk$.*

$$H^*(\mathcal{L}X;\Bbbk) \cong HH_*(S^*(X;\Bbbk))$$

$$H^*(\mathcal{L}X_{h\mathbb{T}};\Bbbk) \cong HC_*^-(S^*(X;\Bbbk))$$

Here $S^*(X;\Bbbk)$ is the differential graded algebra of (normalized) singular cochains with cup product.

Although the second theorem is somewhat harder to prove, it is often preferable for computational purposes. For instance, the algebra of cochains $S^*(X)$ is smaller than the Moore loops $S_*(\Omega^{Moore}X)$ and rational homotopy theory can be used to give even smaller models for $S^*(X;\mathbb{Q})$.

As all free loop spaces come with the slightly bigger symmetry group $\mathbb{O} = O(2) \subset \text{Homeo}(S^1)$, it is natural to ask what the analogous algebraic descriptions of (co)homology of $\mathcal{L}X_{h\mathbb{O}}$ are. For the case of homology, Dunn gave the following analogue of Theorem 1.1.

Theorem 1.3 ([Dun89]). *Let X be a connected space and $\Bbbk$ any ring.*

$$H_*(\mathcal{L}X_{h\mathbb{O}};\Bbbk) \cong HD_*(S_*(\Omega^{Moore}X;\Bbbk))$$

Here HD_* is a variation of cyclic homology called *dihedral homology* [Lod87] that allows one to take into account the $\mathbb{O}$-action rather than just the $\mathbb{T}$-action. Although Hochschild homology and cyclic homology take as input (differential graded) associative algebras, dihedral homology additionally requires the data of an involution on that algebra. In the case above, this data comes from reversing the loops in $\Omega^{Moore}X$.

The aim of this article is to extend Jones' theorem to take into account the $O(2)$-symmetry of $\mathcal{L}X$.

Main Theorem. *Let $\Bbbk$ be a field and X a simply connected space with finite type homology over $\Bbbk$. Then there is an isomorphism*

$$H^*(\mathcal{L}X_{h\mathbb{O}};\Bbbk) \cong HD_*^-(S^*(X;\Bbbk)).$$

Here HD_*^- denotes a variation of dihedral homology called *negative dihedral homology* and the cochain algebra $S^*(X;\Bbbk)$ carries a homotopically trivial involution coming from changing the orientation of simplices.

Outline of the proof.

(1) To model the left hand side we start with the codihedral space $\operatorname{Map}(S^1_\bullet, X)$. A codihedral space is a cosimplicial space with extra structure that allows for an $\mathbb{O}$-action on its totalization. This codihedral space is used as a model for free loop space because $\operatorname{tot}\operatorname{Map}(S^1_\bullet, X) \cong_{\mathbb{O}} \mathcal{L}X$ and hence

$$S^*(\mathcal{L}X_{h\mathbb{O}}) \cong S^*((\operatorname{tot}\operatorname{Map}(S^1_\bullet, X))_{h\mathbb{O}}).$$

(2) The chains on the homotopy orbit space are then compared to an algebraic version of homotopy orbits

$$S^*((\operatorname{tot}\operatorname{Map}(S^1_\bullet, X))_{h\mathbb{O}}) \simeq S_*(\operatorname{tot}\operatorname{Map}(S^1_\bullet, X))_{h\mathbb{O}}{}^\vee.$$

(3) The tensor-hom adjunction relates the result of the last step to the algebraic homotopy fixed points of the dual

$$S_*(\operatorname{tot}\operatorname{Map}(S^1_\bullet, X))_{h\mathbb{O}}{}^\vee \cong S^*(\operatorname{tot}\operatorname{Map}(S^1_\bullet, X))^{h\mathbb{O}}.$$

(4) With the appropriate assumptions, comparing the two ways of totalizing the cochains on $\operatorname{Map}(S^1_\bullet, X)$ yields an equivalence

$$(S^*(\operatorname{tot}\operatorname{Map}(S^1_\bullet, X)))^{h\mathbb{O}} \simeq (\operatorname{Tot}_\oplus S^*(\operatorname{Map}(S^1_\bullet, X)))^{h\mathbb{O}}.$$

(5) After proving an equivalence of $S^*(\operatorname{Map}(S^1_\bullet, X))$ with the cyclic bar construction as a dihedral chain complex, it follows that

$$(\operatorname{Tot}_\oplus S^*(\operatorname{Map}(S^1_\bullet, X)))^{h\mathbb{O}} \simeq (\operatorname{Tot}_\oplus B^{\mathrm{cyc}} S^*(X))^{h\mathbb{O}}.$$

As the homology of the last term is our definition of $HD^-_*(S^*(X))$, the result follows after taking homology.

Since the cochain algebra $S^*(X;\Bbbk)$ is generally too big to compute with, we prove that one may instead use the polynomial forms $A^*_{PL}(X)$ when $\Bbbk$ is of characteristic 0 as $S^*(X;\Bbbk) \simeq A^*_{PL}(X)$ as involutive algebras. Similarly, we prove that the de Rham isomorphism is compatible with the involutions. The following corollary is particularly useful.

Corollary. *Let X be a rationally formal space. Then there is an isomorphism*

$$HD^-_*(S^*(X;\mathbb{Q})) \cong HD^-_*(H^*(X;\mathbb{Q})).$$

This corollary is used in the last section to compute $H^*((\mathcal{L}S^2)_{h\mathbb{T}};\mathbb{Q})$, which turns out to be one-dimensional in every dimension $* \equiv 0,3$ modulo 4, and zero otherwise. In characteristic two, the answer is more interesting. In that case the dimensions of $HD^-_*(H^*(S^2;\mathbb{F}_2))$ have been computed in low degrees to be the unbounded sequence $\lfloor (*+2)^2/4 \rfloor$. Unfortunately, the author was not able to show that the sphere is involutively formal over $\mathbb{F}_2$ meaning that the calculation does not necessarily apply to $H^*((\mathcal{L}S^2)_{h\mathbb{T}};\mathbb{F}_2)$. With the Gromoll–Meyer Theorem in mind, it does however give another hopeful indicator that the C_2-symmetry could help proving that S^2 admits infinitely many distinct closed geodesics for any Riemannian metric.

Organization of the paper. Sections 2–4 are dedicated to the definitions of involutions, dihedral objects and (negative) dihedral homology in a way suited to our application and Section 5 is an outline of the proof of Dunn's result. In Section 6 we prove Step 5 of the outline, followed by Step 2 in Section 7 and Step 4 in Section 8. The full proof of the Main Theorem is then given in Section 9. Section 10 is dedicated to polynomial forms and a version of the de Rham isomorphism that may aid in computations, which is then used in Section 11 to calculate $H^*((\mathcal{L}S^2)_{h\mathbb{O}};\mathbb{Q})$.

Conventions. All algebras are unital over a base ring $\Bbbk$. We use the closed monoidal structure of Ch, the category of unbounded homologically graded chain complexes over $\Bbbk$. For example, the tensor product of two chain complexes has differential $d(x\otimes y) = dx\otimes y + (-1)^{|x|}x\otimes dy$ and $f \in \underline{\mathsf{Ch}}(X,Y)_n$ is a map of degree n with differential $(\delta f)(x) = d(f(x)) - (-1)^n f(dx)$. The differential on the (sum) totalization of a simplicial chain complex is defined as $d_{int} + (-1)^{int}\Sigma_i(-1)^i d_i$. Similarly, the product totalization $\operatorname{Tot}_\Pi X^\bullet$ of a cosimplicial chain complex $X^\bullet$ has a differential that is $dx - (-1)^{p-n}\Sigma(-1)^i\delta^i x$ on $x \in {X^n}_p$.

If $Y^\bullet$ is a cosimplicial space, we define $\operatorname{tot} Y^\bullet = \operatorname{Nat}_\Delta(\Delta^\bullet, Y^\bullet) \in \textsf{Top}$ to be the totalization.

Let $\mathbb{T}$ be the circle group, considered as a subset of the complex numbers. We denote the orthogonal group $O(2) = \mathbb{T} \rtimes C_2$ by $\mathbb{O}$. In this notation the multiplication on $\mathbb{O}$ is $(\tilde{z}, \tilde{\alpha}) \cdot (z, \alpha) = (\tilde{z}z^{\tilde{\alpha}}, \tilde{\alpha}\alpha)$ where we consider $\alpha, \tilde{\alpha} = \pm 1 \in C_2$.

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2. Involutive algebras

Definition 2.1. Let A be a differential graded algebra; that is, a monoid in $\textsf{Ch}$. A chain map $\overline{(-)}\colon A \to A$ of degree zero is called an *involution* if $\overline{\overline{a}} = a$, $\overline{1} = 1$ and $\overline{ab} = (-1)^{|a||b|}\overline{b}\,\overline{a}$ for all homogeneous $a, b \in A$. Such a map is called an anti-involution by some due to the flipping of the order. The data of a differential graded algebra together with an involution is called an *involutive algebra.*

Example 2.2. If A is graded commutative, then the identity map is an involution for A. In fact, every algebra endomorphism that squares to the identity is an involution.

Example 2.3. Complex conjugation is an involution for $\mathbb{C}$ as an algebra over the reals.

Example 2.4. We repeatedly use the differential graded algebra of (normalized) singular cochains $S^*(X)$. Because it is defined as the linear dual of singular chains $S_*(X) = \Bbbk \otimes \operatorname{Sing}_*(X)$, it carries the differential

$$(\delta\gamma)(\sigma) = (-1)^{|\gamma+1|}\Sigma_{i=0}^{n+1}\gamma(d_i\sigma).$$

The cup product is defined as

$$(\gamma_1 \cup \gamma_2)(\sigma) = (-1)^{pq}\gamma_1(d_{p+1}\dots d_{p+q}\sigma)\gamma_2((d_0)^p\sigma),$$

where γ_1 and γ_2 are cochains of degree p and q respectively. This dga carries a natural involution given by $\overline{\gamma}(\sigma) = (-1)^{|\gamma|(|\gamma|+1)/2}\gamma(\overline{\sigma})$ where $\overline{\sigma}$ is the flipped simplex $\overline{\sigma}(t_0, \dots, t_n) = \sigma(t_n, \dots, t_0)$. See also Proposition 10.13.

Example 2.5. For a group G, the map $g \mapsto g^{-1}$ is an involution on the group algebra $\Bbbk G$.

Example 2.6. The singular chains of a topological monoid with involution form an involutive dga. An example of this is $S_*(\Omega^{Moore}X; \Bbbk)$.

3. Cyclic and dihedral objects

We recall some definitions of cyclic and dihedral objects and refer to [Jon87; Lod98; FL91; Ung16] for more details. The morphisms of the category of finite ordered sets Δ are generated by δ^i, σ^i, which satisfy the dual simplicial relations. By appropriately adding cyclic permutations $\langle \tau_n \rangle = C_{n+1}$ as the automorphisms of $[n]$, one obtains Connes' *cyclic category* ΔC. One obtains the *dihedral category* ΔD if one also adds automorphisms ρ_n at each $[n]$ such that the automorphisms become D_{n+1}^{op}. The morphisms of this category may also be described as maps of unoriented necklaces. The subcategory of ΔD generated only by the maps δ_n^i, σ_n^i and ρ_n for each n, i is called the *reflexive category*, denoted ΔR. In analogy to the definitions of simplicial and cosimplicial objects, we call a contravariant functor from ΔD^{op} to a category C a *dihedral object* in C and a covariant such functor is called a *codihedral object*.

The morphisms in ΔD will be denoted by Greek letters $\delta_n^i, \sigma_n^i, \tau_n, \rho_n$ whereas we use the Roman alphabet for morphisms in the opposite category.

Example 3.1. A dihedral set is a simplicial set $X_\bullet$ with extra structure maps $t_n, r_n \colon X_n \to X_n$ for all n such that the following identities are satisfied:

$$\begin{aligned} d_n &= d_0 t_n & t_n^{n+1} &= r_n^2 = \mathrm{id}_n \\ d_i t_n &= t_{n-1} d_{i-1} & rt &= t^{-1} r \\ s_i t_n &= t_{n+1} s_{i-1} & d_i r_n &= r_{n-1} d_{n-i} \\ s_0 t_n &= t_{n+1}^2 s_n & s_i r_n &= r_{n+1} s_{n-i} \end{aligned}$$

Example 3.2. The singular set $\mathrm{Sing}_\bullet X$ of a topological space X is a reflexive set using $r_n(\sigma)(t_0, \ldots, t_n) = \overline{\sigma}(t_0, \ldots, t_n) = \sigma(t_n, \ldots, t_0)$ on an n-simplex σ. It is in fact also dihedral, but we do not use this fact.

Example 3.3. If A is an algebra with involution, then its bar construction is a reflexive chain complex using $r_n(a_1 \otimes \ldots \otimes a_n) = \pm(\overline{a_n} \otimes \ldots \otimes \overline{a_1})$. This construction works for arbitrary monoids with involutions in symmetric monoidal categories.

Example 3.4. The simplicial model for the circle $[n] \mapsto S_n^1 = \mathbb{Z}/(n+1)\mathbb{Z}$ is not only a cyclic set, it is also dihedral by using $r_n(i) = n - i + 1$, which corresponds to reversing the orientation of the circle. From this dihedral set, one obtains for each space X a codihedral space $[n] \mapsto \mathrm{Map}(S^1_n, X) = X^{n+1}$ that totalizes to the free loop space $\mathcal{L}X$. The coboundaries are given by the diagonal maps, the codegeneracies by forgetting factors, cyclic maps by cyclically permuting the factors and the reflection by flipping the coordinates. For example,

$$\delta^{n+1}(x_0, \dots, x_n) = (x_0, x_1, \dots, x_n, x_0).$$

By functoriality of $S^*(\text{-})$, $S^*(\mathrm{Map}(S^1_\bullet, X))$ is a dihedral chain complex.

Example 3.5. For any differential graded algebra A, we have the *cyclic bar construction* $(B^{\mathrm{cyc}}A)[n] = A^{\otimes n+1}$, which is used to compute the Hochschild homology of A. The structure maps are given by multiplication, insertions of the unit and cyclic permutations of the tensor factors. If the algebra came with an involution, then $B^{\mathrm{cyc}}A$ is also a dihedral chain complex with $r_n(a_0 \otimes \ldots \otimes a_n) = (-1)^{|a_n|(|a_1|+\ldots+|a_{n-1}|)}\overline{a_0} \otimes \overline{a_n} \otimes \ldots \otimes \overline{a_1}$.

Example 3.6. [FL91; Jon87] Composing the Yoneda embedding $\Delta \to \mathbf{sSet}$ with the realization functor $\mathbf{sSet} \to \mathbf{Top}$ we obtain the *standard cosimplicial space* $\delta^\bullet$, which is the geometric standard simplex Δ^n in simplicial degree n. Using this object, one can rewrite the geometric realization of a simplicial space $X_\bullet$ as the coend construction $|X_\bullet| = X_\bullet \otimes_\Delta \delta^\bullet$ and the totalization of a cosimplicial space as $\mathrm{tot}\, Y^\bullet = \mathrm{Nat}_\Delta(\delta^\bullet, Y^\bullet)$. The same can be done for the dihedral category, obtaining the *standard codihedral space* δ_D. Concretely, $\delta^n_D \cong \mathbb{O} \times \Delta^n$ with the following structure maps.

$$\delta^i = \mathrm{id} \times \delta^i$$
$$\sigma^i = \mathrm{id} \times \sigma^i$$
$$\rho_n(z, \alpha, t_0, \dots, t_n) = (z, -\alpha, t_n, \dots, t_0)$$
$$\tau_n(z_\alpha, t_0, \dots, t_n) = (z \exp(-\alpha 2\pi i t_0), \alpha, t_1, \dots, t_n, t_0)$$

It is an elementary check that all of the structure maps are $\mathbb{O}$-equivariant if one uses left multiplication. This means that in fact δ_D is a functor $\Delta D \to \mathbb{O}\text{-}\mathbf{Top}$. The same construction can be done for the cyclic and the reflexive category and the resulting functors are all compatible.

Proposition 3.7. *The realization of a dihedral space has a natural $\mathbb{O}$-action. The same is true for the totalization of a codihedral space.*

Proof. See also Theorem 5.3 of [FL91] and §3 of [Jon87]. We have that

$$|X_\bullet| = X_\bullet \otimes_\Delta \delta^\bullet \cong X_\bullet \otimes_{\Delta D} \delta^\bullet_D$$
$$\mathrm{tot}\, Y^\bullet = \mathrm{Nat}_\Delta(\delta^\bullet, Y^\bullet) \cong \mathrm{Nat}_\Delta\, D(\delta^\bullet_D, Y^\bullet).$$

In both cases, the $\mathbb{O}$-action is now given by acting on $\delta_D^\bullet \cong \mathbb{O} \times \Delta^\bullet$. Because the action is natural in the structure maps of $\delta_D^\bullet$, these actions are well defined and natural. $\square$

Remark 3.8. Note that every dihedral space is reflexive by forgetting along the inclusion $\Delta R \hookrightarrow \Delta D$. The resulting C_2-action is surprisingly simple, given that describing the $\mathbb{T}$-action on the realization of a cyclic space is not really explicit in the same way. If $x \in X_n$ and $\underline{t} = (t_0, \ldots, t_n) \in \Delta^n$, the action of the generator of C_2 on the point $[x, \underline{t}] = [x, +1, \underline{t}]$ is $[x, -1, \underline{t}] = [x, \rho_n(+1, t_n, \ldots, t_0)] = [r_n(x), +1, t_n, \ldots, t_0]$.

Example 3.9. The dihedral set $S^1_\bullet$ from Example 3.4 realizes to the circle. The C_2-action is the map $z \mapsto z^{-1}$. This can be checked explicitly using the identification $S^1 \cong |S^1_\bullet| : z = e^{2\pi i\theta} \mapsto [1, (\theta, 1-\theta)]$ where 1 is the fundamental simplex $1 \in \mathbb{Z}/2\mathbb{Z} = S^1_1$. We also identify the totalization of the codihedral mapping space as $\mathrm{tot}(\mathrm{Map}(S^1_\bullet, X)) \cong_{\mathbb{O}} \mathcal{L}X$.

4. Cyclic and dihedral homology

Although the use of cyclic homology is widespread, Loday's dihedral homology is less commonly known. The material presented in this section is based on the various treatments in the literature, especially on [KLS88; FL91; Ldd93; Ldd96; Lod98] and of course the original source [Lod87]. Although our definitions of dihedral homology and cohomology turn out to coincide with those in the literature, the presentation is somewhat different.

Because our aim is to produce an algebraic model for homotopy orbits of an $\mathbb{O}$-space, our definitions of cyclic and dihedral homology will be in analogy to constructions in **Top**. In particular, we will abuse notation by writing $\mathbb{T} = H_*(\mathbb{T}; \Bbbk)$ and $\mathbb{O} = H_*(\mathbb{O}; \Bbbk)$ for the graded algebras obtained by applying singular homology to the two topological groups $\mathbb{T}$ and $\mathbb{O}$. We see that $\mathbb{T}$ is generated by B, the fundamental class of the circle, which is of degree one and satisfies $B^2 = 0$. The algebra $\mathbb{O} = \mathbb{T} \rtimes C_2$ has an additional generator R of degree zero, which satisfies $R^2 = 0$ and $RB = -BR$. We are especially interested in differential graded modules over these algebras.

Example 4.1. Let $E\mathbb{T}_*$ be the normalized total complex of the two sided bar construction $B_\bullet(\Bbbk, \mathbb{T}, \mathbb{T})$. This is a free contractible right differential graded $\mathbb{T}$-module of the form $\Bbbk[u^{-1}] \otimes \mathbb{T}$ where $|u| = -2$ and the differential is $u^{-p} \otimes 1 \mapsto u^{-p+1} \otimes 1, u^{-p} \otimes B \mapsto 0$. The group C_2 acts on $\mathbb{T}$ by $B \mapsto -B$, so the corresponding simplicial action on $E\mathbb{T}_*$ is $u^{-p} \otimes 1 \mapsto (-1)^p u^{-p} \otimes 1$ and $u^{-p} \otimes B \mapsto (-1)^{p+1} u^{-p} \otimes B$. That gives us a free contractible right differential graded $\mathbb{O}$-module $E\mathbb{O}_* \coloneqq E\mathbb{T}_* \otimes (EC_2)_*$. Here EC_2 denotes the periodic resolution of the trivial C_2-module that in every non-negative degree is given by $\Bbbk C_2$. The differential on an element of degree p is multiplication by $g+(-1)^p 1$, where g is the generator of C_2. We can write EC_2 as $\Bbbk C_2 \otimes \Bbbk[v^{-1}]$ with $|v| = -1$ with a non-trivial differential.

Example 4.2. (See also §4 of [Jon87]) Let W be a $\mathbb{T}$-space with action map $\mu\colon \mathbb{T} \times W \to W$. The formula $B(\sigma) = \mu_*(z \times \sigma)$ defines a left differential graded $\mathbb{T}$-module structure on the singular chains $S_*(W)$. Here $[\mathbb{T}]$ is the fundamental cycle of $\mathbb{T}$. If W was an $\mathbb{O}$-space, the chains also form an $\mathbb{O}$-module.

Example 4.3. The totalization of a cyclic chain complex is a $\mathbb{T}$-module and the totalization of a dihedral chain complex is an $\mathbb{O}$-module. Using the structure maps we may define the following operations in simplicial degree n: The simplicial boundary map $b_n = \Sigma_{i=0}^n (-1)^i d_i$, the cyclic generator $T_n = (-1)^n t_n$, the generator of the C_2-action $R_n = (-1)^{n(n+1)/2} r_n$ and the norm operator $N_n = \mathrm{id} + T + T^2 + \dots T^n$. These in turn allow us to define Connes' B operator $B_n = (-1)^{int}(1-T)t_{n+1}s_n N$. The operations T and R form chain maps with respect to both the internal differential and b whereas B anticommutes with both. The operations also satisfy the relations $(T_n)^{n+1} = R^2 = \mathrm{id}, RTR = T^{-1}, B^2 = b^2 = 0, BR = -RB$. The proofs of most of these properties and identities are found in Chapter 2 of [Lod98].

Definition 4.4. Let M_* be a (differential graded) left $\mathbb{T}$-module. We define $M_{h\mathbb{T}} := E\mathbb{T}_* \otimes_{\mathbb{T}} M_*$ and $M^{h\mathbb{T}} := \underline{\mathsf{Ch}}_{\mathbb{T}}(E\mathbb{T}_*, M_*) \subset \underline{\mathsf{Ch}}(E\mathbb{T}_*, M_*)$. If M_* is moreover an $\mathbb{O}$-module, we similarly define $M_{h\mathbb{O}} := E\mathbb{O}_* \otimes_{\mathbb{O}} M_*$ and $M^{h\mathbb{O}} := \underline{\mathsf{Ch}}_{\mathbb{O}}(E\mathbb{O}_*, M_*)$.

Proposition 4.5. *Let M_* be a differential graded left $\mathbb{O}$-module, then*

$$M_{h\mathbb{O}} = (M_{h\mathbb{T}})_{hC_2} \quad \textit{and} \quad M^{h\mathbb{O}} = (M^{h\mathbb{T}})^{hC_2}.$$

Proof. We can convert left into right modules and visa versa using the Hopf algebra structures of $\mathbb{O}$ and $\mathbb{T}$. Also, we can use the semi direct product structure $\mathbb{O} = \mathbb{T} \rtimes C_2$ to break down the tensor product over $\mathbb{O}$ into two steps $\Bbbk \otimes_{\mathbb{O}} (-) = \Bbbk \otimes_{C_2} (\Bbbk \otimes_{\mathbb{T}} (-))$. We see that

$$M_{h\mathbb{O}} = E\mathbb{O} \otimes_{\mathbb{O}} M = \Bbbk \otimes_{\mathbb{O}} (E\mathbb{T} \otimes EC_2 \otimes M) = \Bbbk \otimes_{C_2} (\Bbbk \otimes_{\mathbb{T}} (E\mathbb{T} \otimes EC_2 \otimes M)).$$

Because $\mathbb{T}$ acts trivially on the EC_2 factor we conclude that

$$M_{h\mathbb{O}} = EC_2 \otimes_{C_2} (E\mathbb{T} \otimes_{\mathbb{T}} M) = (M_{h\mathbb{T}})_{hC_2}.$$

A similar argument shows that $M^{h\mathbb{O}} = (M^{h\mathbb{T}})^{hC_2}$. $\square$

Combining the proposition with the concrete expressions we see that $M_{h\mathbb{T}} = M[u^{-1}]$ with the differential

$$u^{-q}m \mapsto u^{-q}d_M m + u^{-q+1}Bm,$$

and $M_{h\mathbb{O}} = M[v^{-1}, u^{-1}]$ with the differential

$$\begin{aligned} v^{-p}u^{-q}m \mapsto &(-1)^p(v^{-p}u^{-q}d_M m + v^{-p}u^{-q+1}Bm) \\ &+ v^{-p+1}u^{-q}((-1)^q R + (-1)^p)m. \end{aligned}$$

Note that the degree of v^{-1} is 1 and the degree of u^{-1} is 2. The signs in the last term are the differential $g + (-1)^p 1$ coming from the periodic resolution of the constant C_2-module, applied to $M_{h\mathbb{T}}$.

Definition 4.6. Let A be an involutive dga. We define the *Hochschild complex* $C_*(A)$ to be the normalized total complex of the dihedral chain complex $B^{\mathrm{cyc}}A$ of Example 3.5. We then define the *cyclic chains* and the *negative cyclic chains* to be $CC_*(A) = C_*(A)_{h\mathbb{T}}$ and $CC_*^-(A) = C_*(A)^{h\mathbb{T}}$. Similarly we define the *dihedral chains* and *negative dihedral chains* to be $DC_*(A) = C_*(A)_{h\mathbb{O}}$ and $DC_*^-(A) = C_*(A)^{h\mathbb{O}}$. The corresponding homologies are denoted $HH_*(A), CH_*(A), CH_*^-(A), DH_*(A)$ and $DH_*^-(A)$.

Proposition 4.7. *Let $\phi\colon M_* \to N_*$ be a map of differential graded $\mathbb{O}$-modules. If ϕ is a quasi isomorphism, then the associated maps $\psi_{h\mathbb{O}}\colon M_{h\mathbb{O}} \to N_{h\mathbb{O}}$ and $\psi^{h\mathbb{O}}\colon M^{h\mathbb{O}} \to N^{h\mathbb{O}}$ are also quasi isomorphisms.*

Proof. The double complex arguments in the proofs of parts ii and iii of Lemma 2.1 in [Jon87] imply that the maps $\psi_{h\mathbb{T}}$ and $\psi^{h\mathbb{T}}$ are quasi isomorphisms. The same proof can be used to show that the functors $(-)_{hC_2}$ and $(-)^{hC_2}$ preserve quasi isomorphisms. As $\psi_{h\mathbb{O}} = (\psi_{h\mathbb{T}})_{hC_2}$ and $\psi^{h\mathbb{O}} = (\psi^{h\mathbb{T}})^{hC_2}$ are compositions of these functors, both maps are quasi isomorphisms. □

Comparison with other definitions. In the literature, starting with [Lod87], it is common to define the dihedral homology of a dihedral $\Bbbk$-module $M_\bullet$ as $HD_*(M_\bullet) = Tor_*^{\Delta D}(\Bbbk^\dagger, M_\bullet)$, where $\Bbbk^\dagger$ denotes the trivial (co)dihedral $\Bbbk$-module. Several different chain complexes are available for computing the homology. In particular, every resolution of $\Bbbk^\dagger$ yields such a chain complex. For example, one could resolve all the dihedral groups and patch them together to get a resolution of the trivial module. For the case when 2 is invertible in our base ring $\Bbbk$ this is in fact what Loday did in [Lod87] and a version without this assumption first appeared in [Ldd90]. When working with cyclic homology it is common to take the cyclic analogue of this complex and contract a subcomplex, obtaining the (B, b)-complex.

This procedure can also be applied for dihedral homology to obtain a (B, b) version of dihedral chains, see also [Lod87, Proposition 1.7] and [Ldd93, Lemma 2.2]. In fact, this can be used to see that our definition of dihedral homology is isomorphic to the Tor definition.

In [Ldd93] Lodder discusses several possible definitions for the negative variant of dihedral homology. One of these is called $\mathcal{D}III$, and our definition of negative dihedral homology coincides with the hyperhomology version of this definition. Although it does not seem to be mentioned explicitly, it seems that $HD_*^-(M) = Ext_{\Delta D}^{-*}(\Bbbk^\dagger, M)$.

5. Dihedral Goodwillie isomorphism

This section is a summary of how a dihedral version of the Goodwillie isomorphism is proven in [Dun89]. In [Dun89], all topological space are assumed to be compactly generated and LEC means that the diagonal map is a cofibration. CW complexes are examples of LEC spaces.

Theorem 5.1 ([Dun89] Th 3.6)**.** *Let G be a group-like topological LEC unital monoid with involution, i.e., with a self map $\overline{(-)} : G \to G$ satisfying $\overline{a \cdot b} = \overline{b} \cdot \overline{a}$ and $\overline{e} = e$. Then for $\Bbbk$ a ring we have an isomorphism*

$$HD_*(S_*(G; \Bbbk)) \cong H_*((\mathcal{L}BG)_{h\mathbb{O}}; \Bbbk)$$

Here the differential graded algebra $S_*(G; \Bbbk)$ carries the involution induced by the involution of the monoid G and $\mathcal{L}BG$ has the involution that both reverses the direction of loops and uses $BG \xrightarrow{B\overline{(\cdot)}} BG$.

Proof. The proof can be broken down into a few steps. First we use the Eilenberg–Zilber maps to construct a quasi isomorphism of dihedral chain complexes $B^{cyc}S_*G \xrightarrow{\simeq} S_*(B^{cyc}G)$, which is Proposition 3.5 [Dun89]. Here B^{cyc} denotes the cyclic bar construction, promoted to a dihedral object as in Example 3.5. In the first instance this is done in $(\mathsf{Ch}, \otimes)$ and in the second in $(\mathsf{Top}, \times)$. Then we use [Dun89, p. 3.3]: If $Y_\bullet$ is a (good) dihedral space (e.g., $B^{cyc}G$), then $HD_*(S_*Y_\bullet) \cong H_*(\mathrm{hocolim}_{\Delta D} Y_\bullet)$. This can be seen using a statement about hypertor of functors

$$HD_*(S_*Y_\bullet) \cong \mathbb{T}or_n^{\Delta D}(\Bbbk, S_*Y) \cong H_n(\mathrm{hocolim}_{\Delta D} Y_\bullet; \Bbbk),$$

which is a generalization of Theorem 6.12 of [FL91]. The final ingredient is [Dun89, p. 2.10] $\mathrm{hocolim}_{\Delta D} B^{cyc}G \simeq \mathcal{L}BG_{h\mathbb{O}}$, which follows from the fact that $|B^{cyc}G| \simeq_{\mathbb{O}} \mathcal{L}BG$. □

Proposition 5.2. *There is a natural C_2-equivariant map*

$$\xi \colon B\Omega^{Moore}X \to X$$

that is a weak equivalence if X is connected. Here we use the trivial C_2-action on X and the action on $B\Omega^{Moore}X$ coming from the involution on $\Omega^{Moore}X$ by reversing loops, see also Example 3.3.

Proof. We begin by describing the map as defined in [May75, Lemma 15.4]. It is also shown there that this map is a weak equivalence if X is connected. By viewing the classifying space as the realization of a bar construction, we may define $\xi[\gamma_1, \ldots, \gamma_n; u] = (\gamma_1 \ldots \gamma_n)(\Sigma_{1 \leq i \leq p} u_i a_i)$. Here $\gamma_i \in \Omega^{Moore}X$ of length a_i , $u = (t_0, \ldots, t_p) \in \Delta^p$ and $u_i = t_0 + \ldots + t_{i-1}$. The source carries a C_2-action because it is the realization of a reflexive object, see also Remark 3.8. The generator of C_2 acts as $[\overline{\gamma_n}, \ldots, \overline{\gamma_1}; \overline{u}]$ and a quick calculation shows that $\xi([\overline{\gamma_n}, \ldots, \overline{\gamma_1}; \overline{u}]) = \xi([\gamma_1, \ldots, \gamma_p; u])$. From this calculation it is also clear what to do when X is a C_2-space: One may simply add this action to the involution of the monoid $\Omega^{Moore}X$. □

Corollary 5.3 (Dihedral Goodwillie Isomorphism)**.** *For X a connected LEC space and $\Bbbk$ any ring.*

$$HD_*(S_*(\Omega^{Moore}X); \Bbbk) \cong H_*((\mathcal{L}X)_{h\mathbb{O}}; \Bbbk)$$

Proof. This follows from the theorem above by inserting $G = \Omega^{Moore}X$ and using the equivalence $BG \xrightarrow{\simeq} X$. □

6. The cyclic bar construction and free loop spaces

In this section we establish an equivalence of dihedral objects between the cyclic bar construction of the cochains (Example 3.5) and the cochains of the cosimplicial model for free loop space (Example 3.4). This is done by extending the results of [Ung16] from an equivalence of (co)cyclic chain complex to an equivalence of (co)dihedral chain complexes.

Proposition 6.1. *Let X be a space with finite type homology over a principal ideal domain $\Bbbk$. There is a natural zigzag of equivalences of* dihedral *chain complexes*

$$B^{\text{cyc}}_\bullet S^*(X; \Bbbk) \xleftarrow{\simeq} QB^{\text{cyc}}_\bullet S^*(X; \Bbbk) \xrightarrow{\simeq} S^*(\text{Map}(S^1_\bullet, X); \Bbbk),$$

where $QB^{\text{cyc}}_\bullet S^*(X; \Bbbk)$ is a resolution of the cyclic bar construction.

Remark 6.2. There is a more general statement when working with chains rather than cochains. See Remark 1 in [Ung16].

Proof of Proposition 6.1. To extend the proof of the Main Theorem in [Ung16] two things need to be added. Lemma 1 on [Ung16] should be checked for the morphisms $r_n \in \Delta D^{\text{op}}([n], [n])$, which is elementary. More importantly, one needs a contractible operad $\tilde{\mathcal{S}}$ with a natural action on cochains in such a way that it encodes the cup product in arity two, the involution $\overline{(-)}$ in arity one and insertion of the unit in arity zero. This ensures that the equation $(\star)$ lives entirely inside the operad $\tilde{\mathcal{S}}$. In particular, the action of r_n on the cyclic bar construction is a composition of a permutation of arguments (the operad is symmetric) and termwise application of the involution. The existence of an operad with such an action is established in Proposition 6.4 below. □

Lemma 6.3. *There exists a quasi free unital differential graded algebra (R, δ) over $\Bbbk$ and a natural differential graded R-module structure on $S_*(X)$. The algebra R contains a distinguished element r of degree zero that acts on chains as $r\sigma = \overline{\sigma}$ where $\overline{\sigma}(t_0, \dots, t_n) = (-1)^{n(n+1)/2}\sigma(t_n, \dots, t_0)$.*

Proof. We define $R = \bigcup_l R^l$ where we construct the R^l inductively, starting with R^0 freely generated by an element r of degree zero. From R^l we obtain R^{l+1} by adding a generator $h(a)$ for every word $a \in R^l$, where $h(a)$ is one degree higher than a is. For example, $r^2h(r^{12}h(r)h(rh(r^5)))$ is an element of degree four in R^3. The differential is defined on generators as $\delta h(r^p) = r^p - 1$ and $\delta h(a) = a - h(\delta a)$ if the degree of a is not zero. The map $a \mapsto h(a)$ defines a contracting homotopy.

It now remains to show that we can define the natural module structure on $S_*(X)$. This is done by the method of acyclic models and induction on l, the degree of the operation and the degree of the chain on which the operations act. Note that if a is any natural operation, it is determined by its action on universal simplices $\kappa_n \in S_n(\Delta^n)$ for all n as $a(\sigma) = a(\sigma_*\kappa_n) = \sigma_*(\kappa_n)$ for $\sigma \in S_n(X)$. Also, $a(d\sigma) = \sigma_*a(d\kappa_n) = \sigma_*\Sigma_{i=0}^n\delta^i_*a(\kappa_{n-1})$ where the maps $\delta^i \colon \Delta^{n-1} \to \Delta^n$ are the face inclusions.

Because R^0 is freely generated by r, its action on chains is determined by the formula $r(\sigma) = \overline{\sigma}$. Finding the action of the $h(r^p) \in R^1$ reduces to fixing the action of a single $h(r)$ because $r^2(\sigma) = \sigma$. This operation can either be found using acyclic methods or can be found as the prism operator in the proof of homotopy commutativity of the cup product (see [Hat02, p.211]). We proceed with the inductive step.

Assume we have specified the action of all generators in R^{l-1} and all new generators (elements of the form $h(a)$ for $a \in R^{l-1}$) in R^l of degree m. Let a be a word of degree m in R^{l-1} that is not in R^{l-2}. We need to show the existence of a natural operation associated to $h(a)$ that satisfies $(\delta h(a))(\sigma) = a(\sigma) - h(\delta a)(\sigma)$. Here the first δ should be read as the differential as an operation: $(\delta h(a))(\sigma) = dh(a)(\sigma) - (-1)^{m+1}h(a)(d\sigma)$. Using this, we see that it suffices to specify $h(a)(\kappa_n) \in S_{n+m+1}(\Delta^n)$ for all n, such that

$$(1) \qquad dh(a)(\kappa_n) = a(\kappa_n) - h(\delta a)(\kappa_n) + (-1)^{m+1}\Sigma_{i=0}^n(-1)^i\delta_*^i h(a)(\kappa_{n-1}).$$

We can define such $h(a)(\kappa_n)$ by induction on n: If we assume to have found such $h(a)(\kappa_N)$ for $N < n$, all the terms of the right hand side are elements of $S_{n+m}(\Delta^n)$ that have been found. A small calculation shows that the right hand side is a cycle and as Δ^n is contractible, we see that $h(a)(\kappa_n)$ exists. For the base of this part of the induction one needs to find a $h(a)(\kappa_0) \in S_{m+1}(\Delta^n)$ whose boundary is $a(\kappa_0) - h(\delta a)(\kappa_0)$, which is again possible because Δ^n is contractible. □

Proposition 6.4. *There exists a symmetric, reduced, differential graded operad $\tilde{\mathcal{S}}$ with a natural action on cochains. The operad contains distinguished elements in arity two, one and zero representing the cup product, the involution and insertion of the unit respectively.*

Proof. The operad $\tilde{\mathcal{S}}$ is defined as the pushout of $\mathcal{S} \leftarrow \Bbbk 1 \to R$. Here $\Bbbk 1$ is the initial reduced operad and R is the operad associated to the differential graded algebra of Lemma 6.3. As R is quasi free and contractible, $\Bbbk 1 \hookrightarrow R$ is an acyclic cofibration in the Berger–Moerdijk model structure. Therefore $\mathcal{S} \xrightarrow{\simeq} \tilde{\mathcal{S}}$ is a weak equivalence. An algebra structure on a chain complex for such a coproduct operad is a pair of algebra structures that agree in arity zero. As we know that there are natural action of both $\mathcal{S}$ and R on singular cochains and that they both insert the unit as the arity zero operation, we see that singular cochains carry a natural algebra structure over $\tilde{\mathcal{S}}$.

The distinguished elements in arity two and arity zero are provided by the image of $\mathcal{S} \to \tilde{\mathcal{S}}$. The involution is provided by the action of the generator $r \in R$. □

7. Comparing homotopy orbits

An often used fact is that $X_{hG} \simeq (X_{hN})_{hG/N}$ for X a G-space and $N \triangleleft G$. This is a consequence of the fact that any model for EG is also a model for EN. We use a variation of this fact to see that $X_{h\mathbb{O}} \cong (X_{h\mathbb{T}})_{hC_2}$ in a way that is compatible with the algebraic statement of Proposition 4.5.

Definition 7.1. The two sided bar construction model for $E\mathbb{T}$ comes with a simplicial right $\mathbb{O}$-action that extends the $\mathbb{T}$-action: On $B_n(*, \mathbb{T}, \mathbb{T})$ the action of $\mathbb{O} = \mathbb{T} \rtimes C_2$ is defined as $(z_1, \ldots, z_n)z_0 \cdot (z, \alpha) = (z_1^\alpha, \ldots, z_n^\alpha)z_0^\alpha z^\alpha$.

Proposition 7.2. *Let X be a left $\mathbb{O}$-space. Then there is an equivalence $X_{h\mathbb{O}} \simeq (X_{h\mathbb{T}})_{hC_2}$.*

Proof. Let EC_2 be any contractible space with a free C_2-action. Then C_2 acts diagonally on $EC_2 \times E\mathbb{T}$, whereas $\mathbb{T}$ acts only on $E\mathbb{T}$. Together this gives a free $\mathbb{O}$-action and thus a model for $E\mathbb{O}$. Using the fact that $(-)_{\mathbb{O}} = ((-)_{\mathbb{T}})_{C_2}$, we now see that $X_{h\mathbb{O}} = (EC_2 \times E\mathbb{T}) \times_{\mathbb{O}} X = EC_2 \times_{C_2} (E\mathbb{T} \times_{\mathbb{T}} X) = (X_{h\mathbb{T}})_{hC_2}$. □

Proposition 7.3. *Let X be a left $\mathbb{O}$-space. Then there are equivalences $S_*(X_{h\mathbb{T}}) \simeq (S_*(X))_{h\mathbb{T}}$ and $S_*(X_{h\mathbb{O}}) \simeq (S_*(X))_{h\mathbb{O}}$.*

Proof. On both sides, the homotopy $\mathbb{T}$-orbits can be described by a two sided bar construction. Combining the Eilenberg–Zilber equivalence with the map $\mathbb{T} = H_*(\mathbb{T}) \overset{\simeq}{\hookrightarrow} S_*(\mathbb{T})$, we obtain an equivalence of simplicial chain complexes

$$B_\bullet(\Bbbk, \mathbb{T}, S_*X) \xrightarrow{\simeq} B_\bullet(\Bbbk, S_*\mathbb{T}, S_*\mathbb{T}) \xrightarrow{\simeq} S_*(B_\bullet(*, \mathbb{T}, X)).$$

Passing to the total complexes we obtain a quasi isomorphism

$$(S_*(X))_{h\mathbb{T}} \xrightarrow{\simeq} S_*(X_{h\mathbb{T}}).$$

Although this map is not C_2-equivariant on the nose, it is equivariant up to coherent homotopy, which is enough in order to compare homotopy orbits. More concretely, there exists a C_2-equivariant map

$$B_*(C_2, C_2, B_*(\Bbbk, \mathbb{T}, S_*(X))) \to B_*(\Bbbk, S_*(\mathbb{T}), S_*(X)),$$

and it is clear that this map induces an equivalence on homotopy orbits as claimed. The existence of the homotopy coherent map can be shown using acyclic methods.

The map $\mathbb{T} \hookrightarrow S_*(\mathbb{T})$ does not commute with the C_2-action as $-[\mathbb{T}]$ and $[-\mathbb{T}]$ do not coincide in $S_*(X)$. Here $[-\mathbb{T}]$ is the fundamental cycle with the opposite orientation, given by $[-\mathbb{T}](t_0, t_1) = e^{-2\pi i t_0} = e^{2\pi i t_1}$ as opposed to $[\mathbb{T}](t_0, t_1) = e^{2\pi i t_0}$. The two are however homologous cycles, with the difference given as the boundary of the two-chain $P(t_0, t_1, t_2) = e^{2\pi i t_1}$ in the normalized complex. This gives the zero'th level of a C_2-equivariant map $B_*(C_2, C_2, \mathbb{T}) \xrightarrow{\simeq} S_*(\mathbb{T})$, which exists by an acyclic methods argument. Taking tensor powers we obtain a sequence of maps

$$B_*(C_2, C_2, \mathbb{T})^{\otimes n} \otimes S_*(X) \to (S_*(\mathbb{T}))^{\otimes n} \otimes S_*(X).$$

Using the Eilenberg–Zilber equivalence and the multiplication map $C_2^n \to C_2$ we get

$$B_*(C_2, C_2, \mathbb{T})^{\otimes n} \xrightarrow{\simeq} B_*(C_2^n, C_2^n, \mathbb{T}^{\otimes n}) \xrightarrow{\simeq} B_*(C_2, C_2, \mathbb{T}^{\otimes n}).$$

Combining these maps for all n we get a map

$$B_*(C_2, C_2, B_*(\Bbbk, \mathbb{T}, S_*(X))) \to B_*(\Bbbk, S_*(\mathbb{T}), S_*(X)).$$

By taking C_2 homotopy orbits, we get the desired equivalence

$$(S_*(X))_{h\mathbb{O}} \simeq ((S_*(X))_{h\mathbb{T}})_{hC_2} \simeq S_*(X_{h\mathbb{O}}).$$

□

8. Comparing totalisations

Let $Y^\bullet$ be a codihedral space, for example $\mathrm{Map}(S^1_\bullet, X)$ for a space X. Associated to $Y^\bullet$ are two chain complexes and a natural map between them $\psi\colon S_* \operatorname{tot} Y^\bullet \to \operatorname{Tot}_\Pi S_* Y^\bullet$. This map is defined as $\psi(\sigma) = \Pi_n(\alpha_n)_*(\sigma \times \kappa_n)$, where $\kappa_n \in S_n(\Delta^n)$ is the fundamental simplex and $\alpha_n\colon (\operatorname{tot} Y^\bullet) \times \Delta^n \to Y^n$ is the evaluation map coming from the definition of totalization as the end construction $\operatorname{tot} Y^\bullet = \mathrm{Nat}_\Delta(\delta^\bullet, Y^\bullet)$. Although both sides are $\mathbb{O}$-modules, ψ is only almost an $\mathbb{O}$-map. To fix this, we now introduce a slightly different model for the right hand side. Let $\tilde{S}_n(X)$ denote the *oriented singular n-chains*, defined by quotienting $S_n(X)$ by the relation $g \cdot \sigma \sim sgn(g)\sigma$ where $sgn(g)$ is the sign of a permutation $g \in \Sigma_{n+1}$ that acts by permuting the coordinates simplices. This defines a functor $\tilde{S}\colon \mathsf{Top} \to \mathsf{Ch}$ that is naturally quasi isomorphic to the usual singular chains. For more details on $\tilde{S}$, see [Bar95].

Proposition 8.1. *If ψ is a quasi isomorphism, then $S_* \operatorname{tot} Y^\bullet$ and $\operatorname{Tot}_\Pi S_* Y^\bullet$ are quasi isomorphic as $\mathbb{O}$-modules and*

$$(S_* \operatorname{tot} Y^\bullet)_{h\mathbb{O}} \simeq (\operatorname{Tot}_\Pi S_* Y^\bullet)_{h\mathbb{O}}.$$

Proof. It suffices to show that the composition

$$\tilde{\psi}\colon S_* \operatorname{tot} Y^\bullet \xrightarrow{\psi} \operatorname{Tot}_\Pi S_* Y^n \to \operatorname{Tot}_\Pi \tilde{S}_* Y^n$$

is an $\mathbb{O}$-map. Concretely this means checking that it commutes with the R and B operators on both sides. On $\operatorname{tot} Y^\bullet$, the C_2-action is given by $(Rf)(\underline{t}) = (\rho_n f)(\underline{t}^{\mathrm{op}})$, meaning that $(\alpha_n)_*(R\sigma \times \kappa_n) = (\rho_n \alpha_n)(\sigma \times \kappa^{\mathrm{op}})$. Here $\kappa^{\mathrm{op}}(\underline{t}) = \underline{t}^{\mathrm{op}} = (t_n, \dots, t_0)$. On the other hand, as the C_2-action on $\operatorname{Tot}_\Pi S_* Y^\bullet$ is $(-1)^{n(n+1)/2}\rho_n$ on level n, the R operator on the right hand side gives $R(\alpha_n)_*(\sigma \times \kappa_n) = (-1)^{n(n+1)/2}(\rho_n\alpha_n)_*(\sigma \times \kappa_n)$. By inspecting the definition of the shuffle product, it can be seen that these two expressions are equal when passing to $\tilde{S}_*$.

The analogous check for the B operator involves comparing $\mu_*([\mathbb{T}] \times \iota_{n+1})$ with $B\iota_n$, where $\iota_n \in S_*(\delta^n_C)$ is the fundamental simplex. Again one needs to pass to $\tilde{S}_*$ for the two to be equal. A claim related to ψ commuting with the B operator is on page 417 of [Jon87]. □

Remark 8.2. The condition that ψ is a quasi isomorphism is not always satisfied and is related to the convergence of a generalized Eilenberg–Moore spectral sequence [And72; Bou87]. When $\Bbbk$ is a field and $Y^\bullet = \mathrm{Map}(S^1_\bullet, X)$ for X a simply connected space, the condition is claimed to hold in [And72] and a more detailed discussion is found in [PT03]. Given that Jones' isomorphism has been proven to hold in greater generality in [AF15], it is quite possible that ψ is a quasi isomorphism under weaker hypotheses. This would strengthen our Main Theorem.

9. Proof of the Main Theorem

Main Theorem. *Let $\Bbbk$ be a field and X a simply connected space of finite type over $\Bbbk$. Then there is an isomorphism*

$$H^*(\mathcal{L}X_{h\mathbb{O}}) \cong HD^-_*(S^*(X)).$$

Proof. By Proposition 6.1 there is an equivalence of dihedral chain complexes $B^{\mathrm{cyc}}_\bullet S^*(X) \simeq S^*(\mathrm{Map}(S^1_\bullet, X))$, which by Example 4.3 and a double complex argument gives an equivalence of differential graded $\mathbb{O}$-modules $C_*(S^*(X)) = \mathrm{Tot}_\oplus B^{\mathrm{cyc}}_\bullet S^*(X) \simeq \mathrm{Tot}_\oplus S^*(\mathrm{Map}(S^1_\bullet, X))$. Applying the linear dual of Proposition 8.1 to the codihedral space $Y^\bullet = \mathrm{Map}(S^1_\bullet, X)$ yields an equivalence of differential graded $\mathbb{O}$-modules

$$\mathrm{Tot}_\oplus S^*(\mathrm{Map}(S^1_\bullet, X)) \simeq_{\mathbb{O}} S^*(\mathrm{tot}\,\mathrm{Map}(S^1_\bullet, X)).$$

Note that $\mathrm{tot}\,\mathrm{Map}(S^1_\bullet, X) \cong_{\mathbb{O}} \mathcal{L}X$ by Example 3.9 and that the hypothesis of Proposition 8.1 is satisfied because of Remark 8.2. In all, we now have that $C_*(S^*(X)) \simeq_{\mathbb{O}} S^*(\mathcal{L}X)$ and Proposition 4.7 implies

$$DC^-_*(S^*(X)) = (C_*(S^*(X)))^{h\mathbb{O}} \simeq (S^*(\mathcal{L}X))^{h\mathbb{O}}.$$

After applying the linear dual of Proposition 7.3 we finally see that the last term is equivalent to $(S^*(\mathcal{L}X))^{h\mathbb{O}} \simeq S^*(\mathcal{L}X_{h\mathbb{O}})$ and the theorem follows. □

Remark 9.1. The exact same methods can be used to show a C_2-version of the Jones isomorphism.

$$HR^-_*(S^*(X)) \cong H^*(\mathcal{L}X_{hC_2})$$

The corresponding cohomology theory is called *negative reflexive homology* HR^-_* and is defined as the homology of $(C_*(A))^{hC_2}$.

10. An involutive de Rham isomorphism

The goal of this section is to prove that the de Rham cochain algebra $\Omega^*_{dR}(M)$ and the singular cochain algebra $S^*(M)$ on a compact smooth manifold M are quasi isomorphic as involutive dga's. To do this, we will take a zig-zag witnessing the quasi isomorphism without involutions, give all the terms involutions and check that all the maps in the zig-zag preserve these involutions. In particular this involves upgrading the polynomial de Rham forms $A^*_{PL}(M)$ to an involutive dga. The following pair of theorems is the starting point of the proof.

Theorem 10.1 ([FHT01, Theorem 10.9])**.** *Let $\Bbbk$ be a field of characteristic 0. For K a simplicial set, the natural morphisms of differential graded algebras over $\Bbbk$,*

$$A_{PL}(K) \to (C_{PL} \otimes A_{PL}(K)) \leftarrow C_{PL}(K)$$

are quasi isomorphisms.

Theorem 10.2 ([FHT01, Theorem 11.4])**.** *For a smooth manifold M, the natural morphisms of differential graded algebras over $\Bbbk = \mathbb{R}$,*

$$\Omega^*_{dR}(M) \xrightarrow{\alpha_M} A_{dR}(\mathrm{Sing}^\infty_\bullet(M)) \xleftarrow{\beta_M} A_{PL}(\mathrm{Sing}^\infty_\bullet(M)) \xleftarrow{\gamma_M} A_{PL}(\mathrm{Sing}_\bullet(M))$$

are quasi isomorphisms.

All of the terms except for the smooth forms $\Omega^*_{dR}(M)$ can be defined using the following construction.

Definition 10.3. For every simplicial dga $A_\bullet$, we get an associated *cochain functor* $A(-) = \mathrm{Nat}_{\Delta^{\mathrm{op}}}(-, A)\colon \mathsf{sSet}^{\mathrm{op}} \to \mathsf{dga}$. This construction is covariant in $A_\bullet$, so we have a functor $\mathsf{sSet}^{\mathrm{op}} \times \mathsf{sdga} \to \mathsf{dga}$

Example 10.4. Singular cochains of a space or more generally simplicial cochains of a simplicial set $K_\bullet$ can be viewed as $C_{PL}(K_\bullet)$, where $C_{PL\bullet}$ is the simplicial dga that is $C^*(\Delta[n])$ in simplicial degree n.

Example 10.5. Another important example is the *piecewise linear forms* $A_{PL\bullet}$. In simplicial degree n this is the cdga

$$A_{PL}[n] = \Lambda(t_0, \dots, t_n, dt_0, \dots, dt_n)/(\Sigma t_i - 1).$$

Here the generators t_i are of degree 0 and Λ denotes the free graded commutative algebra. Because $A_{PL\bullet}$ is graded commutative in each simplicial degree, the associated cochain functor lands in cdga's.

Example 10.6. Smooth forms on the geometric simplices also form a simplicial cdga $A_{dR\bullet}$. The map β_M is induced by the inclusion $A_{PL\bullet} \hookrightarrow A_{dR\bullet}$.

Remark 10.7. The map α_M is induced by the maps $\sigma^*\colon \Omega^*_{dR}(M) \to \Omega^*_{dR}(\Delta^n))$ that pull back forms along smooth simplices $\sigma \in \mathrm{Sing}^\infty_n(M)$. The map γ_M is induced by the inclusion of smooth singular simplices into continuous singular simplices $\mathrm{Sing}^\infty_\bullet(M) \hookrightarrow S_\bullet(M)$. The last two maps come from the simplicial maps $A_{PL\bullet} \to C_{PL\bullet} \otimes A_{PL\bullet} \leftarrow C_{PL\bullet}$.

The involutions. From now on, $\Omega^*_{dR}(M)$ will carry as its involution the identity, see Example 2.2. For all the other terms, we extend Definition 10.3 so that it lands in involutive dga's. For this, we need to change the input—instead of simplicial sets, we use reflexive sets $\mathsf{rSet} = \mathsf{Set}^{\Delta R^{\mathrm{op}}}$ and instead of simplicial dga's we use the category **i-rdga** defined below.

Definition 10.8. An *involutive reflexive dga* is a reflexive object in **dga** such that the reflexive structure map r_n is an involution of dga's in every simplicial level n. The category of such objects with morphisms of reflexive dga's is called $\textsf{i-rdga} \subset \mathsf{dga}^{\Delta R^{\mathrm{op}}}$. Note that this is *not* the same as simplicial objects in involutive dga's.

Example 10.9. There is an involutive reflexive structure on $A_{PL\bullet}$ using $r_n\colon t_i \mapsto t_{n-i}$.

Example 10.10. Given $A, B \in \textsf{i-rdga}$, their levelwise tensor product is again an involutive reflexive dga.

Proposition 10.11. *Simplicial cochains $C_{PL\bullet}$ carry the structure of an* ***i-rdga***.

Proof. The standard simplicial sets $\Delta[n]$ carry an involution by reversing order: $\overline{\sigma}(i) = n - \sigma(p-i)$ for $\sigma \in \Delta[n]_p$. If we view these simplices as coming from the Yoneda embedding, we see that $\overline{\sigma \circ \sigma'} = \overline{\sigma} \circ \overline{\sigma'}$ and we see that the construction is both reflexive (in p) and coreflexive (in n). We now define the reflexive structure map on the simplicial dga $C_{PL\bullet}$ to be $\overline{\gamma}(\sigma) = (-1)^{p(p+1)/2}\gamma(\overline{\sigma})$, where γ is a p cochain in simplicial degree n. Unitality, $\overline{\overline{\gamma}} = \gamma$ and graded linearity are clear so we show the remaining properties.

- **Anti-simplicial:** The simplicial structure of C_{PL} is the cosimplicial direction of $\Delta[n]_p$. For example, $\delta^i\colon \Delta[n-1] \to \Delta[n]$ pulls back to $d_i\colon C^*(\Delta[n]) \to C^*(\Delta[n-1])$. It is then an elementary check to see that $\overline{d_i\gamma}(\sigma) = d_{n-i}\overline{\gamma}(\sigma)$ and similarly for the degeneracies.
- **Differential:** First we observe that on the level of simplicial chains we have $\overline{d\sigma} = (-1)^p d\overline{\sigma}$. Then it follows easily that $d\overline{\gamma} = \overline{d\gamma}$.

Multiplication: Because the involution is anti-simplicial, we have the following identities by induction: For a $(p+q)$ simplex σ

$$d_{p+1}\dots d_{p+q}\overline{\sigma} = \overline{d_o^q\sigma}$$

$$\overline{d_{q+1}\dots d_{q+p}\sigma} = d_o^p\overline{\sigma}$$

The cup product of cochains γ_1, γ_2 of degrees p and q is defined as $\gamma_1 \cup \gamma_2(\sigma) = (-1)^{pq}\gamma_1(d_{p+1}\dots d_{p+q}\sigma)\gamma_2(d_o^p\sigma)$. Now we have

$$\begin{aligned}
\overline{\gamma_1 \cup \gamma_2}(\sigma) &= (-1)^{pq}(-1)^{(p+q)(p+q-1)/2}\gamma_1(d_{p+1}\dots d_{p+q}\overline{\sigma})\gamma_2(d_o^p\overline{\sigma}) \\
&= (-1)^{pq}\gamma_1(\overline{d_o^q\sigma})\gamma_2(\overline{d_{q+1}\dots d_{q+p}\sigma}) \\
&= (-1)^{(p+q)(p+q-1)/2+q(q-1)/2+p(p-1)/2}\overline{\gamma_2}\cup\overline{\gamma_1}(\sigma) \\
&= (-1)^{pq}\overline{\gamma_2}\cup\overline{\gamma_1}(\sigma)
\end{aligned}$$

□

Proposition 10.12. *The construction of Definition 10.3 gives a functor* $(\textbf{rSet})^{\mathrm{op}} \times$ ***i-rdga*** $\to$ ***i-dga*** *using the same dga $A(K)$ associated to the underlying simplicial dga of A and underlying simplicial set K. This is defined to carry involution $\overline{\Phi} = (\sigma \mapsto I\Phi(R\sigma))$ for $\Phi \in A(K)$ with I and R the reflexive structure maps r_n of A and K respectively.*

Proof. The fact that the construction is functorial follows immediately from the definition of the morphisms in **rSet** and **i-rdga**. What needs to be checked is that the map described really is an involution on $A(K)$.

Target: The $\overline{\Phi}$ is an element of $A(K) = \mathrm{Nat}^{\mathrm{op}}_{\Delta}(K, A)$, i.e., it is simplicial:

$$\overline{\Phi}(d_i\sigma) = I\Phi(Rd_i\sigma) = I\Phi(d_{n-i}R\sigma) = Id_{n-i}\Phi(R\sigma) = d_iI\Phi(R\sigma) = d_i\overline{\Phi},$$

and similarly for the degeneracy maps.

Differential: The differential of a natural transformation $\Phi \in A(K)$ was defined using the target A. We see that

$$d\overline{\Phi}(\sigma) = d_AI\Phi(R\sigma) = I(d_A\Phi)(R\sigma) = \overline{d\phi}(\sigma).$$

Unitality: Is $\overline{1} = 1 \in A(K)$? The unit of $A(K)$ is defined to send any $\sigma \in K_n$ to the unit in simplicial degree n. So we see that

$$\overline{1}(\sigma) = I1(R\sigma) = 1(R\sigma) = 1(\sigma).$$

Involution: The fact that $\overline{\overline{\Phi}} = \Phi$ follows from the properties $R \circ R = id_K$ and $I \circ I = id_A$.

Multiplication: Let $\Phi, \Psi \in A(K)$ be of degree p and q respectively. We check that

$$\begin{aligned}\overline{\Phi\Psi}(\sigma) &= I(\Phi\Psi)(R\sigma) = I(\Phi(R\sigma)\Psi(R\sigma)) \\ &= (-1)^{pq}(I\Psi(R\sigma))(I\Phi(R\sigma)) = (-1)^{pq}\overline{\Psi}(\sigma)\overline{\Phi}(\sigma) \\ &= (-1)^{pq}\overline{\Psi}\,\overline{\Phi}(\sigma).\end{aligned}$$

□

Proposition 10.13. *The dga of singular cochains $S^*(X) = C_{PL}(\mathrm{Sing}_\bullet X)$ carries an involution given by $\overline{\gamma}(\sigma) = (-1)^{p(p+1)/2}\gamma(\overline{\sigma})$.*

Proof. Combining the **rSet**-structure of $\mathrm{Sing}_\bullet(X)$ with the **i-rdga** structure of C_{PL} supplies the singular cochain dga $S^*(X)$ with an involution. It is useful to see what it does concretely. First we will describe the isomorphism $S^*(X) \cong C_{PL}(\mathrm{Sing}_\bullet(X))$ precisely. Let $\lambda \in C_{PL}(\mathrm{Sing}_\bullet(X))$, say of degree p: A map that assigns to every non-degenerate n simplex in $\mathrm{Sing}_n(X)$ a p cochain in $(C_{PL})_n = C^p(\Delta[n])$. It corresponds to the singular cochain in $S^p(X)$ that sends $\sigma \mapsto \lambda_\sigma(c_p)$ where c_p is the fundamental simplex of $\Delta[p]$, that is $id_p \in \Delta([p],[p])$. The other way around, given a $\gamma \in S^p(X)$, the corresponding element of $C_{PL}(\mathrm{Sing}_\bullet(X))$ sends $\sigma \mapsto C^p(\sigma_*)(\gamma) = (\tau \mapsto \gamma(\sigma_*\tau))$, where $\sigma_*\colon \Delta[n] \to \mathrm{Sing}_\bullet(X)$ using the Yoneda lemma. We now chase the involution.

$$\begin{aligned}\gamma &\mapsto (\sigma \mapsto C^p(\sigma_*)(\gamma)) \\ &\mapsto (\sigma \mapsto \overline{\sigma} \mapsto C^p(\sigma_*)(\gamma) \mapsto (\Delta[n]_p \ni \tau \mapsto (-1)^{p(p+1)/2}\gamma(\overline{\sigma}_*\overline{\tau}))) \\ &\mapsto (\sigma \mapsto (-1)^{p(p+1)/2}\gamma(\overline{\sigma}_*\overline{e_p}) = (-1)^{p(p-1)/2}\gamma(\overline{\sigma}))\end{aligned}$$

In the last line we use the fact that the fundamental simplices are fixed by the involution $\overline{e_p} = e_p$ and that $\overline{\sigma}_* e_p = \overline{\sigma}$. So we see that all the involution does is add a sign and evaluate on the flipped simplex: $\overline{\gamma}(\sigma) = (-1)^{p(p+1)/2}\gamma(\overline{\sigma})$. The exact same holds for smooth simplices. □

Theorem 10.14. *The maps in the zig-zags of Theorems 10.1 and 10.2 are maps of involutive dga's.*

Proof. The only map not given by bifunctoriality of the 'cochain functor' construction is α_M. It is given by sending a form $\omega \mapsto \{\sigma^*\omega\}_{\sigma \in \mathrm{Sing}^\infty_\bullet(M)}$. As $\Omega^*_{dR}(M)$ is graded commutative, the identity map will act as the involution. So to check that α_M respects the involution it is equivalent to check that the image is pointwise fixed by the involution. The involution in the target $A_{dR}(\mathrm{Sing}^\infty_\bullet(M))$ is given by the combination of $I_{A_{dR}}$ and $R_{\mathrm{Sing}^\infty_\bullet}(M)$. The first pulls forms on Δ^n back along ϕ_n (the map that flips coordinates), the second flips coordinates of smooth singular simplices $R\sigma = \sigma \circ \phi = \overline{\sigma}$.

$$\overline{\{\sigma^*\omega\}} = \{I(R\sigma)^*\omega\} = \{\phi^*(\sigma \circ \phi)^*\omega\} = \{\phi^*\phi^*\sigma^*\omega\} = \{\sigma^*\omega\}$$

The map β_M is induced by the inclusion of piecewise linear forms into de Rham forms on Δ^n, $A_{PL} \hookrightarrow A_{dR}$, which is clearly a morphism in **i-rdga**. So β_M respects the involution by functoriality. The same holds for γ_M, which is induced by the morphism $\mathrm{Sing}^\infty_\bullet(M) \hookrightarrow \mathrm{Sing}_\bullet(M)$ in **rSet**.

Finally, it is an elementary check that the tensor product of simplicial dga's can be promoted to a tensor product in **i-dga** (see Example 10.10) and that $A_{PL} \to C_{PL} \otimes A_{PL} \leftarrow C_{PL}$ are morphisms in **i-dga**. Hence the last three maps in the zig-zag respect the involution. □

Corollary 10.15. *De Rham cochains $\Omega^*_{dR}(M)$ with trivial involution and normalized singular cochains $S^*(M)$ with the involution from Proposition 10.13 are quasi isomorphic as involutive dga's.*

Corollary 10.16. *Let M be a simply connected manifold of finite type over $\mathbb{R}$. Then using the trivial involution on $\Omega^*(M;\mathbb{R})$ the following isomorphisms hold.*

$$H^*(\mathcal{L}M;\mathbb{R}) \cong HH_*(\Omega^*(M;\mathbb{R}))$$
$$H^*(\mathcal{L}M_{h\mathbb{T}};\mathbb{R}) \cong HC^-_*(\Omega^*(M;\mathbb{R}))$$
$$H^*(\mathcal{L}M_{h\mathbb{O}};\mathbb{R}) \cong HD^-_*(\Omega^*(M;\mathbb{R}))$$

Note that the first two isomorphisms already follow from Theorem A in [Jon87], combined with the de Rham Theorem. The proofs do imply however, that these two isomorphisms are C_2-equivariant.

Proposition 10.17. *Let $K_\bullet$ be a reflexive set (e.g., $\mathrm{Sing}_\bullet X$) and A an involutive reflexive dga (e.g., A_{PL}). If A is graded commutative in every simplicial degree and the involution $\overline{(-)}$ of Proposition 10.12 is chain homotopic to the identity, then $(A(K), \overline{(-)}) \simeq (A(K), id)$ as involutive dga's.*

Proof. As both $K_\bullet$ and A are reflexive objects, we may form the end construction $\tilde{A}(K) \coloneqq \mathrm{Nat}_{\Delta R}(K_\bullet, A) \subset \mathrm{Nat}_\Delta(K_\bullet, A) = A(K)$. On $\tilde{A}(K)$, the two involutions agree and the inclusion map $\iota\colon \tilde{A}(K) \hookrightarrow A(K)$ is a section of the chain map $\rho\colon \Psi \mapsto \frac{1}{2}(\Psi + \overline{\Psi})$. If h is a chain homotopy between the involution and the identity, then $\frac{1}{2}h$ is a chain homotopy between $\iota \circ \rho$ and the identity. Hence ι is a quasi isomorphism and the result follows from the following zig-zag.

$$(A(K), \overline{(-)}) \hookleftarrow (\tilde{A}(K), id) \hookrightarrow (A(K), id)$$

□

Proposition 10.18. *Let X be a topological space. Then*

$$(A_{PL}(\mathrm{Sing}_\bullet X), \overline{(-)}) \simeq (A_{PL}(\mathrm{Sing}_\bullet X), id)$$

as involutive dga's.

Proof. We need to show that the condition of Proposition 10.17 holds for $K_\bullet = \mathrm{Sing}_\bullet X$ and $A = A_{PL}$. This can be seen by considering the following diagram, which commutes by Theorem 10.1.

$$\begin{array}{ccccc} A_{PL}(\mathrm{Sing}_\bullet X) & \longrightarrow & (C_{PL} \otimes A_{PL}(\mathrm{Sing}_\bullet X)) & \longleftarrow & S^*(X) \\ \big\downarrow \overline{(-)} & & \big\downarrow \overline{(-)} & & \big\downarrow \overline{(-)} \\ A_{PL}(\mathrm{Sing}_\bullet X) & \longrightarrow & (C_{PL} \otimes A_{PL}(\mathrm{Sing}_\bullet X)) & \longleftarrow & S^*(X) \end{array}$$

After taking homology, the vertical morphism on the right hand side is the identity and arrows are isomorphisms. This implies that on homology, the involution $\overline{(-)}$ on $A_{PL}(K)$ is the identity. Over a field, two chain maps are the same on homology if and only if they are chain homotopic and thus it follows that $\overline{(-)}$ is homotopic to the identity. □

The fact above allows one to take any cdga model A for a space X from rational homotopy theory and use the equivalence $(A, id) \simeq (S^*(X;\mathbb{Q}), \overline{(-)})$ to compute $H^*(\mathcal{L}X_{h\mathbb{O}})$ with. In particular we have the following statement.

Corollary 10.19. *If X is a rationally formal simply connected space of finite type, then $S^*(X)$ is formal as an involutive dga and thus*

$$H^*(\mathcal{L}X_{h\mathbb{O}};\mathbb{Q}) \cong HD^-_*(H^*(X;\mathbb{Q})).$$

11. An example calculation

In this section we show that the results from the last section allow one to do concrete calculations. To demonstrate this we calculate Borel equivariant cohomology of $\mathcal{L}S^2$ over the rationals. As S^2 is rationally formal, it is involutively formal over the rationals by Corollary 10.19 and to calculate negative dihedral homology we may use the algebra $H^*(S^2;\mathbb{Q}) = \mathbb{Q}[\alpha]/\alpha^2$ where $|\alpha| = 2$.

The normalized Hochschild complex is generated by classes $\alpha_n = 1 \otimes \alpha^{\otimes n}$ and $\beta_n = \alpha \otimes \alpha^{\otimes n}$ for all $n \geq 0$, which have total degrees $-n$ and $-(n+1)$ respectively. As the internal differential is 0, the total differential is given by $(-1)^{int}\Sigma(-1)^i d_i$. By using that $\alpha^2 = 0$, it is easy to calculate the differential.

The only classes on which the differential is not zero are $d\alpha_n = 2\beta_{n-1}$ for even n.

From this one can see that the Hochschild homology and therefore also $H^*(\mathcal{L}S^2;\mathbb{Q})$ is generated as a graded vector space by the classes

$$\alpha_0, \alpha_1, \alpha_3, \alpha_5, \ldots \quad \textit{and} \quad \beta_0, \beta_2, \beta_4, \beta_6, \ldots$$

That is, there is exactly one class in every negative degree.

By using that the involution is the identity, it can be seen that the R operation is $R = (-1)^{n(n+1)/2} id$ in simplicial degree n. A quick calculation shows that the only classes on which B acts non-trivially are the classes $B\beta_n = (n+1)\alpha_{n+1}$ for n even.

In order to now calculate the homotopy orbits for the C_2-action we can use the following well known fact.

Proposition 11.1. *Let $\Bbbk$ be any ring with $\frac{1}{2} \in \Bbbk$. And let W be a C_2-space. Then $H^*(W_{hC_2};\Bbbk) \cong H^*(W;\Bbbk)^{C_2}$.*

Proof. When 2 is invertible, any element m of a $\Bbbk C_2$-module M can be projected to the invariant element $\frac{1}{2}(m + gm)$, where g is the generator of C_2. Using this, one can check that the functor of C_2-invariants is exact. This implies that the group cohomology over $\Bbbk$ is $H^*(C_2; M) = 0$ for $* > 0$ and $H^0(C_2; M) = M^{C_2}$. The map $W \twoheadrightarrow pt$ induces a fibration

$$EC_2 \times_{C_2} W \twoheadrightarrow EC_2 \times_{C_2} pt \simeq BC_2.$$

The associated Leray–Serre spectral sequence is

$$E_2^{p,q} = H^p(BC_2; H^q(W)) \Longrightarrow H^{p+q}(W_{hC_2})$$

Reinterpreting the twisted coefficients on the E_2 page as group cohomology, we see that the spectral sequence collapses here and read off the conclusion

$$H^q(W_{hC_2}) \cong H^0(C_2; H^q(W)) = H^q(W)^{C_2}.$$

□

Corollary 11.2. *There are isomorphisms*

$$H^*((\mathcal{L}S^2)_{hC_2};\mathbb{Q}) \cong H^*(\mathcal{L}S^2;\mathbb{Q})^{C_2} \cong HH_{-*}(\mathbb{Q}[\alpha]/\alpha^2)^{C_2}.$$

As a graded module, this is generated by the classes $\alpha_0, \alpha_3, \alpha_7, \alpha_{11}, \ldots$ and $\beta_0, \beta_4, \beta_8, \ldots$. In particular, this is four periodic.

In order to now calculate the negative cyclic and negative dihedral homology, we consider the negative cyclic chains as the totalization of a double complex. In general we ought to use the product totalization, but in our case this coincides with the sum totalization because of the coconnectivity of the normalized Hochschild complex. The double complex gives us to converging spectral sequences. In particular, we consider the spectral sequence

$$(E^1, d^1) = (HH_*(\mathbb{Q}[\alpha]/\alpha^2)[u], uB) \Longrightarrow HC_*^-(\mathbb{Q}[\alpha]/\alpha^2).$$

On the E^1 page, there is exactly one generator in each bidegree above or on the diagonal in the third quadrant. By considering the fact that B maps every surviving β_n class to a multiple of α_{n+1}, we see that none of the classes on the interior survive to E^2. On E^2 we are left with the classes $u^p\alpha_0$ and α_q for odd q. Because of their degrees it is possible that d^p maps α_{2p-1} to a multiple of $u^p\alpha_0$. But, because nothing can kill the α_{2p-1} in the double complex, we see that in fact all the differentials must be zero and hence $E^2 = E^\infty$ and we can read off the cyclic homology. And to compute the negative dihedral homology, we can again apply Proposition 11.1. Note that the generator of C_2 acts as $u^p\alpha_0 \mapsto (-1)^p R(\alpha_0) = (-1)^p\alpha_0$.

Theorem 11.3. *As a graded vector space, $H^*((\mathcal{L}S^2)_{h\mathbb{T}}; \mathbb{Q})$ is generated by the classes $\alpha_0, \alpha_1, \alpha_3, \alpha_5, \ldots$ and $u\alpha_0, u^2\alpha_0, u^3\alpha_0, \ldots$. In other words, it is one dimensional in every degree. The cohomology $H^*((\mathcal{L}S^2)_{h\mathbb{O}}; \mathbb{Q})$ is generated by the classes $\alpha_0, \alpha_3, \alpha_7, \alpha_{11}, \alpha_{15}, \ldots$ and $u^2\alpha_0, u^4\alpha_0, u^6\alpha_0, \ldots$ as a graded vector space.*

Remark 11.4. Although $S^*(S^2; \mathbb{F}_2)$ is formal as a dga over $\mathbb{F}_2$, it is not clear to the author whether S^2 is involutively formal over $\Bbbk = \mathbb{F}_2$. Assuming it is, we can again use negative dihedral homology of $\mathbb{F}_2[\alpha]/\alpha^2$ to compute $H^*(\mathcal{L}S^2_{h\mathbb{O}}; \mathbb{F}_2)$. The complex that computes the negative dihedral homology is generated by $v^p u^q \alpha_n$ and $v^p u^q \beta_n$ for all $n, p, q \geq 0$ and the only non-trivial differential is $v^p u^q \beta_n \mapsto v^p u^{q+1}\alpha_{n+1}$. This means that the cohomology is generated by the classes $v^p u^q \beta_n$ for odd n, $v^p u^q \alpha_n$ for even n and $u^q\alpha_n$ for all n. According to a computer calculation for low degrees and [OEIS], this results in Betti numbers that are $\lfloor (* + 2)^2/4 \rfloor$, which is a monotonic sequence.

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