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Equivariant homotopy theory and K-theory of exact categories with duality

PhD thesis by

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Abstract

This thesis has two main parts. The first part, which consists of two papers, is concerned with the role of equivariant loop spaces in the K-theory of exact categories with duality. We prove a group completion-type result for topological monoids with anti-involution. The methods in this proof also apply in the context of K-theory and we obtain a similar result there. We go on to prove equivairant delooping results for Hesselholt and Madsen's Real algebraic K-theory. From these we obtain an equivalence of the fixed points of Real algebraic K-theory with Schlichting's Grothendieck-Witt space. This equivalence implies a group completion result for Grothendieck-Witt-theory, and for Real algebraic K-theory it implies that the analogs of the Cofinalty and Dévissage theorems hold.

The second part of the thesis, which consists of one paper, is about the equivariant homotopy theory of so-called G-diagrams. Here G is a finite group that acts on a small category I. A G-diagram in a category $\mathscr{C}$ is a functor from I to $\mathscr{C}$ together with natural transformations that give a "generalized G-action" on the functor. We give a model structure on the category of I-indexed G-diagrams in $\mathscr{C}$, when the latter is a sufficiently nice model category. Important examples are the categories of topological spaces, simplicial sets and orthogonal spectra with the usual model structures. We formulate a theory of G-linear homotopy functors in terms of cubical G-diagrams. We obtain a new proof of the classical Wirthmüller isomorphism theorem using the fact that the identity functor on orthogonal spectra is G-linear.

Resumé

Denne afhandling har to hoveddele. Den første, som består af to artikler, handler om ækvivariante løkkerums rolle i algebraisk K-teori af eksakte kategorier med daulitet. Vi viser et gruppekompletteringsresultat for topologiske monoider med anti-involuton. Metoderne i dette bevis kan også anvendes i K-teori og vi viser et analogt resultat der. Vi viser ækvivariante afløkningsresultater for Hesselholt og Madsens Reelle algebraiske K-teori. Fra disse får vi en ækvivalens mellem fikspunkterne til Reell algebraisk K-teori og Schlichtings Grothendieck-Witt-rum. Ved hjælp af denne ækvivalens udleder vi både et gruppekompletteringsresultat for Grothendieck-Witt-teori og at Kofinalitet- og Dévissage-sætningerne gælder for Reell algebraisk K-teori.

Den anden del af afhandlingen, som består af én artikel, handler om ækvivariant homotopiteori af såkaldte G-diagrammer. Her er G en endelig gruppe der virker på en lille kategori I. Et G-diagram er en funktor fra I til en kategori $\mathscr{C}$ sammen med naturlige transformationer der giver en "generaliseret G-virkning" på funktoren. Vi giver en modelstruktur på kategorien af G-diagrammer i $\mathscr{C}$ når denne er en tilstrækkelig pæn modelkategori. Viktige eksempler er kategorierne af topologiske rum, simplicielle mængder og ortogonale spektra, med de sædvanlige modelstrukturer. Vi formulerer en teori for G-lineære homotopifunktorer, i form af betingelser for hvordan funktoren virker på kubiske G-diagrammer. Derefter giver et nyt bevis for den klassiske Wirthmüller-isomorfi, ved at bruge at identitetsfunktoren på ortogonale spektra er G-lineær.

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Introduction

This thesis consists of the following three papers:

A. Equivariant loops on classifying spaces

B. Equivariant deloopings in Real algebraic K-theory

C. Homotopy theory of G-diagrams and equivariant excision.

Overview and background

To put the papers in context we begin by giving an overview of the subject and indicate how the papers fit into the bigger picture.

Homotopical group completion

A central topic in homotopy theory is so-called group completion for topological monoids or for H-spaces, which are "monoids up to homotopy". Informally, group completion is a topological version of the process of passing from a monoid to the "closest" group by adding inverses. In topology one often deals with topological monoids, or H-spaces M which do not behave like groups, and one would like a "closest" H-space M' which does behave like a group, at least up to homotopy. Topological group completion is often formulated in terms of the map induced on homology. For the purposes of this introduction we will call an H-map between H-spaces $M \to M'$ a group completion if $\pi_0(M')$ is a group and there is an induced isomorphism on homology rings

$$H_*(M)[\pi_0(M)^{-1}] \xrightarrow{\cong} H_*(M').$$

Exact categories with duality

An exact category, as defined by Quillen, is an additive category $\mathscr{C}$ with a class of sequences $X' \to X \to X''$, called exact, which are required to behave more or less like the exact sequences in an abelian category. A minimal choice of exact sequences is all sequences of the form

$$X \to X \oplus Y \to Y$$

where the maps are the canonical inclusion and projection, respectively, and sequences isomorphic to these. Such sequences are called split-exact, and with this class of exact sequences $\mathscr{C}$ is called a split exact category. We will study exact categories which are also equipped with subcategory of weak equivalences $w\mathscr{C}$ containing the isomorphisms of $\mathscr{C}$. In addition there will be an exact functor $D\colon \mathscr{C}^{op} \to \mathscr{C}$ with a natural weak equivalence $\eta\colon Id_{\mathscr{C}} \to D^2$ satisfying $D(\eta_c) \circ \eta_{Dc} = id_{Tc}$ for all c in $\mathscr{C}$. This structure is called a *duality* on $\mathscr{C}$. A non-degenerate symmetric form in $\mathscr{C}$ is an object c with a weak equivalence $\varphi\colon c \to Dc$ which is self-dual in the sense that $D(\varphi) \circ \eta_C = \varphi$. The category of such forms, with the structure preserving weak equivalences between them is denoted by $Sym(w\mathscr{C})$. If $D^2 = Id_{\mathscr{C}}$ and $\eta = id$ the duality is called strict.

The canonical example to keep in mind is the following: For a commutative ring A let $P(A)$ be the (split exact) category of finitely generated projective A-modules. The weak equivalences in $P(A)$ are the isomorphisms. The duality is the functor $D = Hom(-, A)$ and $\eta_\epsilon\colon P \to DDP$ is the isomorphism $p \mapsto (f \mapsto \epsilon f(p))$. Here ϵ is either 1 or -1 giving symmetric and symplectic forms as objects of $Sym(iP(A))$, respectively. Another example is the category of bounded chain complexes in $P(A)$ with quasi-isomorphisms as weak equivalences and the duality is given by applying D levelwise.

If $(\mathscr{C}, w\mathscr{C}, D, \eta)$ is an exact category with duality and weak equivalences and duality then there is a strictification $(\hat{\mathscr{C}}, w\hat{\mathscr{C}}, \hat{D})$ which has a strict duality. There are weak equivalences $|w\mathscr{C}| \simeq |w\hat{\mathscr{C}}|$ and $|Sym(w\mathscr{C})| \simeq |Sym(w\hat{\mathscr{C}})|$, where in general $|\mathscr{A}|$ denotes the geometric realization of the nerve of $\mathscr{A}$. This is important because when the duality on $\mathscr{C}$ is strict the is a natural C_2-action on $|w\mathscr{C}|$ given by

$$[(c_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} c_n, t_0, \dots t_n)] \mapsto [(Dc_n \xrightarrow{Df_n} \dots \xrightarrow{Df_1} Dc_0, t_n, \dots t_0)],$$

for $(c_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} c_n, t_0, \dots t_n) \in N_n w\mathscr{C} \times \Delta^n$. There is a natural homeomorphism $|Sym(w\mathscr{C})| \cong |w\mathscr{C}|^{C_2}$, so the study of the former space can therefore be phrased in terms of equivariant homotopy theory. Since the duality functor D is additive there is a natural isomorphism $\nu_{X,Y}\colon DX \oplus DY \xrightarrow{\cong} D(X \oplus Y)$ for any pair X, Y of objects of $\mathscr{C}$. We point out that the category $Sym(w\mathscr{C})$ has a sum-operation given by $(X, \varphi) \perp (Y, \psi) = (X \oplus Y, \nu_{X,Y} \circ (\varphi \oplus \psi))$, which is called the *orthogonal sum*.

Algebraic K-theory

Broadly speaking, algebraic K-theory is the study of "generalized" group completion of categories which have a sum-operation and sometimes additional structure such as weak equivalences, exact sequences and so on. By group completion of such a category $\mathscr{C}$ we mean the group completion of the space $|w\mathscr{C}|$, where $w\mathscr{C}$ is often just the groupoid of isomorphisms in $\mathscr{C}$. The group completion consists of an H-space $K(\mathscr{C})$ such that $K_0(\mathscr{C}) = \pi_0 K(\mathscr{C})$ is a group and a map $|w\mathscr{C}| \to K(\mathscr{C})$. This map is a "generalized" group completion in the following sense: Even for exact sequences $X' \to X \to X''$ that do not split, the relation $[X] = [X'] + [X'']$ is required to hold in $K_0(\mathscr{C})$. For a ring A it is customary to write $K(A)$ for $K(P(A))$ and $K_n(A)$ for $\pi_n K(A)$.

Relation to C_2-equivariant loop spaces

Most forms of K-theory or group completion involve loop spaces. A typical approach to group completion of an H-space M is to construct a space B and an H-map $M \to \Omega B$, and prove that it is a group completion. The multiplication on the latter space is given by concatenation of loops. The space B comes in various guises depending on the nature of M. For topologocal monoids one has the classifying space, or bar construction, BM. For nerves of exact categories one has Quillen's Q-construction, and its close relative, the S-construction of Waldhausen. When the exact category in question has a strict duality the nerve has an action of C_2 and one can ask for a C_2-action on $\Omega B = K(\mathscr{C})$ such the map $|w\mathscr{C}| \to \Omega B$ is equivariant. Now we must fix some notation. For a pointed C_2-space X we write $S^{p,q}$ for the one-point compactification $S^{\mathbb{R}^p} = \mathbb{R}^p \cup \{\infty\}$, where C_2 acts by multiplication by -1 in the last q variables. Denote by $\Omega^{p,q}X$ the space of pointed maps $Map_*(S^{p,q}, X)$ with the conjugation action of C_2. Note that the fixed point space $(\Omega^{p,q}X)^{C_2}$ is the space of pointed *equivariant* maps $f\colon S^{p,q} \to X$. In the papers A and B we study and compare various equivariant candidates for the space B which come with equivariant maps

$$|w\mathscr{C}| \to \Omega^{1,1}B \quad \text{or} \quad |w\mathscr{C}| \to \Omega^{2,1}B.$$

Since the fixed point space $|w\mathscr{C}|^{C_2}$ is homeomorphic to the H-space $|Sym(w\mathscr{C})|$ a natural question is whether the maps induced on fixed points are also group completions. If this is the case we call the target space a C_2-equivariant group completion. In general, K-theory spaces are infinite loop spaces with associated spectra. An equivariant K-theory space should be an infinite loop C_2-space and have an associated C_2-spectrum.

Tools for C_2-equivariant loop spaces

To study ordinary loop spaces it is often useful to construct fiber sequences $F \to E \to B$ where the middle space is contractible, because then there is a weak equivalence $F \simeq \Omega B$. This also works for pointed C_2-spaces and gives an equivariant weak equivalence $F \simeq \Omega^{1,0} B$. For the equivariant loop space $\Omega^{1,1} B$ we must take a different approach. We call a square of pointed spaces

$$\begin{tikzcd} X \arrow[r, "p_r"] \arrow[d, "p_l"'] & Y_r \arrow[d, "q_r"] \\ Y_l \arrow[r, "q_l"'] & Z \end{tikzcd}$$

a C_2-square if there are actions of C_2 on X and Z and mutually inverse maps $f\colon Y_l \to Y_r$ and $g\colon Y_r \to Y_l$ which are appropriately compatible with the C_2-actions. In this case the homotopy pullback $Y_r \times^h_Z Y_l = \operatorname{holim}(Y_l \to Z \leftarrow Y_r)$ has a natural action of C_2 and the usual map $X \to Y_r \times^h_Z Y_l$ is equivariant. Moreover, if Y_l, and hence also Y_r, is contractible there is an induced homotopy equivalence

$$Y_r \times^h_Z Y_l \simeq \Omega^{1,1} Z.$$

With some more work one obtains a method for determing when X is weakly C_2-equivalent to $\Omega^{1,1} Z$. These methods are used extensively in the paper B as well as in Emanuele Dotto's thesis [Dot12]. A simplified version is also used in A. The third paper C grew out of a series of discussions with Emanuele Dotto about constructing a general framework for such delooping argumenments.

Linear homotopy functors and equivariant loop spaces

Two of the main questions at the outset of the work on the paper C were as follows: Can one make similar arguments to the one for $\Omega^{1,1}$ sketched above for an arbitrary finite group G and Ω^V for a (sufficiently nice) G-representation V? And, in a different direction, what is the relation to the cube theory in Goodwillie's calculus of functors ([Goo90], [Goo03])? The answer to the first question turns out to be affirmative and this allows us to formulate an answer to the second question. We begin by recalling that a homotopy functor on pointed spaces is a functor $\Phi\colon \mathit{Top}_* \to \mathit{Top}_*$ which preserves weak equivalences. We will only consider *reduced* homotopy functors here, i.e., functors such that $\Phi(*) \simeq *$. Such a functor is called *linear* if for every homotopy cocartesian square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W, \end{tikzcd}$$

the image under Φ

$$\begin{tikzcd} \Phi(X) \arrow[r] \arrow[d] & \Phi(Y) \arrow[d] \\ \Phi(Z) \arrow[r] & \Phi(W), \end{tikzcd}$$

is homotopy cartesian. The canonical example to keep in mind is the functor $X \mapsto \operatorname{hocolim}_n \Omega^n\Sigma^n(X)$. By evaluating Φ on the homotopy pushout square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \arrow[d] \\ CX \arrow[r] & \Sigma X, \end{tikzcd}$$

and using linearity of Φ we get a weak equivalence $\Phi(X) \simeq \Omega\Phi(\Sigma X)$. This procedure can be iterated to get weak equivalences $\Phi(X) \simeq \Omega^n\Phi(\Sigma^n X)$ for all n, so $\Phi(X)$ is an infinite loop space. The paper C gives a theory of equivariant cubes making it possible to build loop spaces and suspension spaces with respect to a large class of representations of G. It also provides a theory of equivariantly linear homotopy functors. If Ψ is such a functor, then in the same way that we got $\Phi(X) \simeq \Omega\Phi(\Sigma X)$ above, we get weak G-equivalences $\Psi(X) \simeq \Omega^V\Psi(\Sigma^V X)$, showing that $\Psi(X)$ is an infinite loop G-space.

A. Equivariant loops on classifying spaces

If M is a topological or simplicial monoid there is a natural map

$$\lambda\colon M \to \Omega BM$$

to the loop space of the classifying space of M. A classical theorem (see e.g. [May75]) states that if M is grouplike, i.e., if the monoid $\pi_0(M)$ is a group, then λ is a weak equivalence. If $\pi_0(M)$ is not a group but instead a central multliplicative subset in the homology ring $H_*(M)$ then by the group completion theorem of McDuff-Segal [MS76] and Quillen [FM94], the map λ induces an isomorphism

$$H_*(M)[\pi_0(M)^{-1}] \xrightarrow{\cong} H_*(\Omega BM).$$

If the monoid M has an anti-invloution $m \mapsto \bar{m}$ such that $\overline{mn} = \bar{n}\bar{m}$ and $\overline{\overline{m}} = m$ then the classifying space inherits an action of C_2. The map λ is equivariant as a map $M \to \Omega^{1,1}BM$ and hence induces a map on fixed point spaces

$$\lambda^{C_2}\colon M^{C_2} \to (\Omega^{1,1}BM)^{C_2}.$$

Nisan Stiennon showed in his thesis [Sti13] that if M is grouplike then λ^{C_2} is a weak equivalence. The first new result of this paper is a description of what happens in the non-grouplike case. Note that the monoid M acts on the fixed points M^{C_2} by $m \cdot n = mn\bar{m}$ inducing an $H_*(M)$-module structure on the homology $H_*(M^{C_2})$.

Theorem (A, 3.12)**:** *Let M be a simplicial monoid with anti-involution such that $\pi_0 M$ is in the center of $H_*(M)$. Then the map λ^{C_2} induces an isomorphism of left $H_*(M)$-modules*

$$H_*(M^{C_2})[\pi_0(M)^{-1}] \xrightarrow{\cong} H_*((\Omega^{1,1}BM)^{C_2}).$$

In general this is not a group completion because M^{C_2} is usually not an H-space.

Let $\mathscr{C}$ be an additive category with weak equivalences strict duality. Then there is an equivariant Γ-space type model $|w\mathscr{C}(S^{1,1})|$ for the "classifying space" of the category $w\mathscr{C}$ and a map

$$|Sym(w\mathscr{C})| \to (\Omega^{1,1}|w\mathscr{C}(S^{1,1})|)^{C_2}$$

analogous to the map λ^{C_2} above. The analog of the action by M on M^{C_2} here is the so-called hyperbolic action which we present in a simplified form. An object c of $\mathscr{C}$ pairs with a symmetric form (d,φ) to give $(d,\varphi) \perp H(c)$. Here $H(c)$ is the so-called hyperbolic form on c. It has underlying object $c \oplus Dc$ and the form is described by the matrix $\left(\begin{smallmatrix} 0 & id_{Dc} \\ \eta_c & 0 \end{smallmatrix}\right)$.

Theorem (A, 5.14)**:** *Let $(\mathscr{C}, w\mathscr{C}, D)$ be an additive category with strict duality and weak equivalences. Then the map $|Sym(w\mathscr{C})| \to (\Omega^{1,1}|w\mathscr{C}(S^{1,1})|)^{C_2}$ induces an isomorphism*

$$H_*(|Sym(w\mathscr{C})|)[\pi_0|w\mathscr{C}|^{-1}] \to H_*((\Omega^{1,1}|w\mathscr{C}(S^{1,1})|)^{C_2})$$

of left $H_*(|w\mathscr{C}|)$-modules.

Here the fixed point space $|Sym(w\mathscr{C})|$ is an H-space and in many cases the map to $(\Omega^{1,1}|w\mathscr{C}(S^{1,1})|)^{C_2}$ is a group completion. This happens for instance if all the Hom-groups of $\mathscr{C}$ are $\mathbb{Z}[\frac{1}{2}]$-modules.

B. Equivariant deloopings in Real algebraic K-theory

Atiyah's Real K-theory functor KR combines into one C_2-equivariant object the cohomology theories KU, KO and Anderson's self-conjugate K-theory KSC. An analog of this has been defined for exact categories with duality by Hesselholt and Madsen in [HM]. They introduce a simplicial construction called the $S^{2,1}$-construction, which is similar to Waldhausen's S-construction. When applied to an exact category with weak equivalences and strict duality, the output is a pointed C_2-space $KR(\mathscr{C}) = \Omega^{2,1}|wS^{2,1}\mathscr{C}|$, which comes with an equivariant map $\lambda_{2,1}\colon |w\mathscr{C}| \to \Omega^{2,1}|wS^{2,1}\mathscr{C}|$. The underlying pointed space of $KR(\mathscr{C})$ has the homotopy type of the Waldhausen K-theory space $K(\mathscr{C}) = \Omega|wS\mathscr{C}|$. If $\mathscr{C}$ is split exact, the map $\lambda_{2,1}$ is an equivariant group completion in the sense explained above. In the 70's Karboubi introduced and studied K-theory of quadratic and symmetric forms. The theory was later recast in a more modern framework by Schlichting in the papers [Sch10a] and [Sch10b]. To an exact category $\mathscr{C}$ with weak ewuivalences and duality it assigns the Grothendieck-Witt space $GW(\mathscr{C})$, and a map $\gamma\colon |Sym(w\mathscr{C})| \to GW(\mathscr{C})$. If the Hom-groups of $\mathscr{C}$ are $\mathbb{Z}[\frac{1}{2}]$-modules, then γ is a group completion (cf. [Sch04]). Grothendieck-Witt theory also has an interpretation as a space of equivariant loops. The duality on $\mathscr{C}$ induces a C_2-action on the usual S-construction. We denote the resulting pointed C_2-space by $|wS^{1,1}\mathscr{C}|$ where the decoration "1, 1" is used because the usual delooping map in K-theory is equivariant as a map

$$|w\mathscr{C}| \to \Omega^{1,1}|wS^{1,1}\mathscr{C}|.$$

There is a natural weak equivalence $(\Omega^{1,1}|wS^{1,1}\mathscr{C}|)^{C_2} \simeq GW(\mathscr{C})$. From the outset the goal of this paper was to generalize foundational theorems of K-theory to KR-theory. The theorems go by the names Cofinality, Dévissage, Resolution and Localization and were originally proved for K-theory by Quillen. The first three have been proved for Grothendieck-Witt theory by Schlichting. Localization was proved by Hornbostel and Schlichting under the assumption that the Hom-groups are $\mathbb{Z}[\frac{1}{2}]$-modules. It is not known whether this assumptoin is necessary. The goals of the paper have mostly been acheived, though not quite in the way that was expected.

Searching for a way to generalize the foundational theorems for the functors $K(-)$ and $GW(-)$ eventually led to the construction of deloopings such as

$$|wS^{1,1}\mathscr{C}| \simeq \Omega^{2,1}|wS^{1,1}S^{2,1}\mathscr{C}| \quad \text{and} \quad |wS^{1,1}S^{2,1}\mathscr{C}| \simeq \Omega^{1,0}|wS^{2,1}S^{2,1}\mathscr{C}|,$$

and so on. These were used to prove the following:

Theorem (4.9 B)**:** *There is a natural weak C_2-equivalence*

$$|wS^{1,1}\mathscr{C}| \xrightarrow{\simeq} \Omega^{1,0}|wS^{2,1}\mathscr{C}|.$$

This result, which is surprising in itself, has a number of interesting consequences. On the one hand, it allows us to transfer our knowledge of $KR(\mathscr{C})^{C_2}$ to $GW(\mathscr{C})$. This shows that in the split-exact case, the map $|Sym(w\mathscr{C})| \to GW(\mathscr{C})$ is a group completion map, even when the Hom-groups are not $\mathbb{Z}[\frac{1}{2}]$-modules. This was previously not known. On the other hand, the theorem allows us to transfer knowledge the other way. In the paper we show in detail how the Cofinality and Dévissage theorems generalize to KR.

C. Homotopy theory of G-diagrams and equivariant excision (Joint with Emanuele Dotto)

Let G be a finite group which acts on a small category I by functors $g_*\colon I \to I$. An I-indexed G-diagram in a category $\mathscr{C}$ is a diagram $X\colon I \to \mathscr{C}$ together with natural transformations $g_X\colon X \to X \circ g_*$, satisfying certain associativity conditions. See also Villarroel-Flores's thesis [VF99], and Jackowski and Słomińska's paper [JS01]. The I-indexed G-diagrams in $\mathscr{C}$ form a category in the evident way. Both the limit and the colimit of a G-diagram X have natural actions of G. If X is a diagram of say, spaces or simplicial sets then this also holds for the homotopy limit and the homotopy colimit. We have already seen one instance of this above for the homotopy pullback in a C_2-square.

In order to study the homotopy theory of G-diagrams in a model category $\mathscr{C}$ we give a model structure on the category $\mathscr{C}_a^I$ of I-indexed G-diagrams provided $\mathscr{C}$ is a so-called G-model catgeory.

Theorem: *Let $\mathscr{C}$ be a G-model category. There is a cofibrantly generated $sSet^G$-enriched model structure on the category of G-diagrams $\mathscr{C}_a^I$ with weak equivalences (resp. fibrations) the maps of G-diagrams $f\colon X \to Y$ such that the value f_i at the object $i \in obI$ is a weak equivalence (resp. fibration) in the model category $\mathscr{C}^{G_i}$ of objects with an action of the stabilizer group G_i.*

Examples include the categories of topological spaces, simplicial sets and orthogonal spectra with the usual model structures. We prove analogs of classical theorems from homotopy theory such as homotopy invariance of homotopy (co)limits and an Elmendorf theorem for G-diagrams.

Our main application of this abstract setup is to construct a theory of cubical G-diagrams and G-linear homotopy functors. In the case of topological spaces our theory turns out to be more or less equivalent to Blumberg's theory of continuous G-functors [Blu06]. We prove a generalized Wirthmüller isomorphism theorem for enriched G-linear homotopy functors. The identity functor on the category of orthogonal spectra is such a functor, and in this case our theorem specializes to the classical Wirthmüller isomorphism theorem.

Perspectives

Localization for KR-theory

The Localization theorem is the crown jewel in Quillen's work on algebraic K-theory. It gives a way to describe the homotopy cofiber on K-theory spectra of the map induced by a localization of sufficiently nice rings. As an example consider the localization map $\mathbb{Z} \to \mathbb{Q}$. By the localization theorem there is an induced cofiber sequence of spectra

$$K(\mathbb{Z}) \to K(\mathbb{Q}) \to \bigvee_p \Sigma K(\mathbb{F}_p).$$

The homotopy type of $K(\mathbb{F}_p)$ were determined by Quillen in [Qui72], so the cofiber sequence tells us a lot about the relation between $K(\mathbb{Z})$ and $K(\mathbb{Q})$. The sequence is an essential ingredient in Quillen's proof that the groups $K_n(A) = \pi_n K(A)$ are finitely generated for all n when A is the ring of integers in a number field [Qui10].

Conjecturally, the localization theorem for KR should give a cofiber sequence of KR-spectra

$$KR(\mathbb{Z}) \to KR(\mathbb{Q}) \to \bigvee_p \Sigma^{1,1} KR(\mathbb{F}_p).$$

Non-equivariantly, this is just Quillen's sequence above, but note the suspension $\Sigma^{1,1}$ which is very different from $\Sigma^{1,0}$. Hornbostel and Schlichting have proved a GW-version of the above sequence when all the involved rings contain $\frac{1}{2}$. The classical localization sequence for Witt groups [MH73, IV.2.1] can be interpreted as a sequence of homotopy groups of KR. Under this interpretation the classical exact sequence is just the exact sequence of homotopy groups of the above cofiber sequence. We mostly know the homotopy type of $KR(\mathbb{F}_p)$ thanks to the work of Quillen in [Qui10] and Friedlander in [Fri76], where they determined the homotopy types of $K(\mathbb{F}_p)$ and $GW(\mathbb{F}_p)$ (for $p \neq 2$), respectively. The Localization theorem for KR would be a very powerful tool for concrete computations in a field that is quite abstract at the moment.

Equivariant Goodwillie calculus

The linearity condition for homotopy functors is expressed in terms of square-shaped diagrams. Such a square is a diagram indexed on the power set $P(S)$ of a set with two elements. Linear functors are also called "excisive" because the collection of functors $\{\pi_n \Phi(-)\}_{n\in\mathbb{N}}$ behaves like a homology theory, in particular satisfying the excision axiom. Higher order excision is expressed using cubical diagrams of higher dimension. These diagrams are indexed on power sets $P(S)$ where S has more than 2 elemtents. Using these one can define, not only linear approximations, but a whole "Taylor tower" of approximations to a functor. In many ways this behaves like the Taylor series of a smooth function. This theory goes by the name "Goodwillie calculus"[Goo03]. It has been used extensively in algebraic K-theory, both to study Waldhausen's A-theory and via the Dundas-McCarthy theorem [DGM13] to study algebraic K-theory of rings.

In the paper C, equivariant linearity is expressed by taking G-diagrams indexed on $P(G_+)$ where G_+ is the G-set G with a disjoint basepoint. Note that the number of G-orbits of G_+ is 2. In the equivariant context the (free) G-orbits in S seem play the role that the elements of S play non-equivariantly. This suggests that a formulation of higher equivariant excision should involve G-cubes indexed on $P(S)$ where S contains several free orbits. It is natural to ask whether "Goodwillie calculus" has an equivariant analog which extends the setup of C. Some progress has been made in this direction by Dotto. There is also a simpler version of this, not using G-diagrams which has been written up by Dotto in [Dot13]. There the relation to the "Real" trace maps from KR to "Real" versions of TC and THH has been studied.

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Equivariant loops on classifying spaces

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Abstract

We compute the homology of the space of equivariant loops on the classifying space of a simplicial monoid M with anti-involution, provided $\pi_0(M)$ is central in the homology ring of M. The proof is similar to McDuff and Segal's proof of the group completion theorem. Then we give an analogous computation of the homology of the C_2-fixed points of a Γ-space-type delooping of an additive category with duality with respect to the sign circle. As an application we show that this fixed point space is sometimes group complete, but in general not.

Contents

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1 Introduction

It is well known that for a group-like simplicial monoid M the natural map

$$\lambda_M \colon |M| \to \Omega|BM|$$

is a weak homotopy equivalence, where B denotes the bar construction and $|-|$ denotes geometric realization. In the non-group-like case the classical group completion theorem of McDuff-Segal [MS76] and Quillen [FM94, Q.4] states that for a simplicial monoid M satisfying certain conditions λ_M induces an isomorphism of $H_*(M)$-algebras

$$H_*(M)[\pi_0(M)^{-1}] \xrightarrow{\cong} H_*(\Omega|BM|).$$

The first goal of this paper is to investigate the corresponding situation when M has an anti-involution, i.e. a map $\overline{(-)} \colon M \to M$ such that $\overline{(mn)} = \bar{n}\bar{m}$. This extra structure allows us to define maps $w_i \colon B_iM \to B_iM$ where $B_iM = M^{\times^i}$ given by

$$w_p(m_1, m_2, \ldots, m_p) = (\overline{m_p}, \ldots, \overline{m_2}, \overline{m_1})$$

which satisfy $w_n \circ w_n = id$ and compatibility relations with the simplicial structure maps (see Section 3). As a consequence the geometric realization $|BM|$ has a natural action by the cyclic group C_2 of order two.

We let $\mathbb{R}^{1,1}$ denote the minus-representation of C_2 on $\mathbb{R}$ and write $S^{1,1}$ for its one-point compactification. For a pointed C_2-space X we write $\Omega^{1,1}X$ for the space $Map_*(S^{1,1}, X)$ with the conjugation action of C_2. In his thesis [Sti13] Nisan Stiennon has shown that λ_M is in fact an equivariant map

$$|M| \to \Omega^{1,1}|BM|,$$

and that if M is group-like, then the induced map on fixed points

$$|M|^{C_2} \to (\Omega^{1,1}|BM|)^{C_2}$$

is a weak equivalence. In view of the group completion theorem it is natural to ask what happens when M is not group-like. The answer is given as follows in Theorem 3.12:

Theorem: *Let M be a simplicial monoid with anti-involution such that π_0M is in the center of $H_*(M)$. Then the map*

$$\lambda_M^{C_2} \colon |M|^{C_2} \to (\Omega^{1,1}|BM|)^{C_2}$$

induces an isomorphism

$$\pi_0(M^{C_2})[\pi_0(M)^{-1}] \xrightarrow{\cong} \pi_0(\Omega^{1,1}|BM|)^{C_2}$$

of left $\pi_0(M)$-sets and an isomorphism of left $H_*(M)$-modules

$$H_*(M^{C_2})[\pi_0(M)^{-1}] \xrightarrow{\cong} H_*((\Omega^{1,1}|BM|)^{C_2}).$$

In [Seg74, 4] Segal proved a variant of the group completion theorem for Γ-spaces. Shimakawa [Shi89] later considered the G-equivariant situation for G a finite group. He described an equivariant delooping machine in terms of Γ_G-spaces and proved a group completion statement for these deloopings provided one is delooping with respect to a representation sphere S^W such that $W^G \neq 0$. In this paper we consider the case $G = C_2$ and $W = \mathbb{R}^{1,1}$. Since $(\mathbb{R}^{1,1})^{C_2} = 0$, Shimakawa's result does not apply. We describe a construction of a Segal-type delooping of an additive category with duality with respect to $S^{1,1}$. The analog of Theorem 3.12 in this setting is as the following (see Theorem 5.14):

Theorem: *Let $(\mathscr{C}, w\mathscr{C}, T)$ be an additive category with strict duality and weak equivalences. Then the map $|NSym(w\mathscr{C})| \to (\Omega^{1,1}|Nw\mathscr{C}(S^{1,1})|)^{C_2}$ induces isomorphisms*

$$\pi_0(NSymw\mathscr{C})[\pi_0 Nw\mathscr{C}^{-1}] \xrightarrow{\cong} \pi_0((\Omega^{1,1}|Nw\mathscr{C}(S^{1,1})|)^{C_2})$$

of monoids and

$$H_*(NSymw\mathscr{C})[\pi_0 Nw\mathscr{C}^{-1}] \to H_*((\Omega^{1,1}|Nw\mathscr{C}(S^{1,1}))^{C_2})$$

of left $H_*(Nw\mathscr{C})$-modules.

As an application we make π_0-computations for symmetric (5.15) and symplectic (5.16) form spaces over $\mathbb{Z}$.

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2 Homology fibrations

In this section we collect some basic facts about homology fibrations of simplicial and bisimplicial sets. We make no claim to originality; the results here can either be found in [PS04], [GJ09, IV.5] or [Jar08] or are easy consequences of the results there.

A map of spaces or simplicial sets inducing an isomorphism on integral homology will be called a *homology equivalence.*

Definition 2.1. A commuting square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

of simplicial sets is called homology cartesian if for any factorization of $f\colon B \to D$ as a trivial cofibration followed by a fibration

$$B \overset{\simeq}{\rightarrowtail} W \twoheadrightarrow D$$

the induced map from A to the pullback $C \times_D W$ is a homology equivalence.

Note that a homotopy cartesian square is automatically homology cartesian. Just as for homotopy cartesian squares it doesn't matter which factorization we use or whether we choose to factor f or g. By analogy with the case of homotopy cartesian squares [GJ09, II.8.22] we have the following Lemma whose proof we omit:

Lemma 2.2: *Let*

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathbf{I}"] & A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathbf{II}"] & A'' \arrow[d] \\ B \arrow[r] & B' \arrow[r] & B'' \end{tikzcd}$$

be a diagram of simplicial sets such that the square **II** *is homotopy cartesian. Then* **I** *is homology cartesian if and only if the outer rectangle* $\mathbf{I} + \mathbf{II}$ *is homology cartesian.*

Let Y be a simplicial set. An m-simplex $\sigma \in Y_m$ of Y corresponds to a unique map $\Delta^m \to Y$ which we will also call σ. The simplices of Y form a category $Simp(Y)$ where an object is a map $\sigma\colon \Delta^m \to Y$, and where a morphism

$$(\sigma\colon \Delta^m \to Y) \to (\tau\colon \Delta^n \to Y)$$

is a map $\alpha\colon [m] \to [n]$ in Δ such that the diagram

$$\begin{tikzcd} \Delta^m \arrow[rr, "\alpha_*"] \arrow[dr, "\sigma"'] & & \Delta^n \arrow[dl, "\tau"] \\ & Y & \end{tikzcd}$$

commutes. Composition is given by composition of maps in Δ.

Let $f\colon X \to Y$ be a map of simplicial sets. Then, for any simplex $\sigma\colon \Delta^m \to Y$ we define $f^{-1}(\sigma)$ to be the pullback in the square

$$\begin{tikzcd} f^{-1}(\sigma) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^m \arrow[r, "\sigma"'] & Y. \end{tikzcd}$$

For a diagram

$$\begin{tikzcd} \Delta^m \arrow[rr, "\alpha_*"] \arrow[dr, "\sigma"'] & & \Delta^n \arrow[dl, "\tau"] \\ & Y & \end{tikzcd}$$

there is an induced map

$$f^{-1}(\alpha_*)\colon f^{-1}(\sigma) \to f^{-1}(\tau).$$

The assignments

$$\sigma \mapsto f^{-1}(\sigma)$$

and

$$(\alpha_*\colon (\sigma\colon \to \Delta^m) \to (\tau\colon \to \Delta^n)) \mapsto \left(f^{-1}(\alpha_*) : f^{-1}(\sigma) \to f^{-1}(\tau)\right)$$

form the object and morphism components, respectively, of a functor

$$f^{-1}\colon Simp(Y) \to sSet.$$

If $g\colon Z \to Y$ is another map to Y, then a map $h\colon X \to Z$ of objects over Y induces a natural transformation

$$h_*\colon f^{-1} \to g^{-1}.$$

Note that the natural map $\operatorname{colim} f^{-1} \to X$ over Y is an isomorphism, in particular $\operatorname{colim} id_Y^{-1} \cong Y$. The homotopy colimit $\operatorname{hocolim} f^{-1}$ is the diagonal of the bisimplicial set $\coprod_* f^{-1}$ (see [GJ09, IV.1.8]) which is given in degree n by $(\coprod_* f^{-1})_n = \coprod_{\sigma \in N_n Simp(Y)} f^{-1}(\sigma(0))$. The canonical maps $f^{-1}(\sigma(0)) \to X$ combine to give a map $\coprod_* f^{-1} \to X$ of bisimplicial sets, where X is constant in the "nerve" simplicial direction. This map induces a weak equivalence on diagonal simplicial sets (see [GJ09, IV.5.1]).

Lemma 2.3: *Let $f\colon X \to Y$ be a map of simplicial sets. The following are equivalent:*

1. *For every simplex $\sigma\colon \Delta^m \to Y$ the pullback diagram*

$$\begin{tikzcd} f^{-1}(\sigma) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^m \arrow[r, "\sigma"'] & Y \end{tikzcd}$$

   is homology cartesian.

2. *For any pair of simplices $\sigma\colon \Delta^m \to Y$ and $\tau\colon \Delta^n \to Y$ and for any diagram*

$$\begin{tikzcd} \Delta^m \arrow[rr, "\alpha_*"] \arrow[dr, "\sigma"'] & & \Delta^n \arrow[dl, "\tau"] \\ & Y & \end{tikzcd}$$

   the induced map on pullbacks along f

$$f^{-1}(\alpha_*)\colon f^{-1}(\sigma) \to f^{-1}(\tau)$$

   is a homology equivalence.

In the proof of this lemma we will use the following result which is proven in [GJ09, IV.5.11], see [PS04] for a different proof of 2.3.

Theorem 2.4: *Let $X\colon I \to sSet$ be a functor such that for any morphism $i \to j$ in I the induced map $X(i) \to X(j)$ is a homology equivalence, then for all objects i of I the pullback diagram*

$$\begin{tikzcd} X(i) \arrow[r] \arrow[d] & \operatorname{hocolim}_I X \arrow[d] \\ * \arrow[r, "i"'] & NI \end{tikzcd}$$

is homology cartesian.

Proof of lemma 2.3. 1 $\implies$ 2: We begin by factoring f as

$$X \overset{g}{\rightarrowtail} W \overset{\bar{f}}{\twoheadrightarrow} Y,$$

where g is a cofibration and a weak equivalence. Condition 1 says precisely that the natural transformation $g_*\colon f^{-1} \to \bar{f}^{-1}$ has components which are homology equivalences. Since $\bar{f}$ is a fibration the functor $\bar{f}^{-1}$ sends all maps in $Simp(Y)$ to weak equivalences. Therefore, a map $\alpha\colon \sigma \to \tau$ in $Simp(Y)$ gives a naturality square

$$\begin{tikzcd} f^{-1}(\sigma) \arrow[r, "f^{-1}(\alpha_*)"] \arrow[d, "g_{*,\sigma}"'] & f^{-1}(\tau) \arrow[d, "g_{*,\tau}"] \\ \bar{f}^{-1}(\sigma) \arrow[r, "\simeq", "\bar{f}^{-1}(\alpha_*)"'] & \bar{f}^{-1}(\tau) \end{tikzcd}$$

where the vertical maps are homology equivalences and the lower horizontal map is a weak equivalence. It follows that $f^{-1}(\alpha_*)$ is a homology equivalence.
2 $\implies$ *1* (cf. [GJ09, IV.5.18]): For every simplex $\sigma\colon \Delta^m \to Y$ there is a diagram of bisimplicial sets

$$\begin{tikzcd}
f^{-1}(\sigma) \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathbf{I}"] & \coprod_* f^{-1} \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\mathbf{II}"] & X \arrow[d, "f"] \\
\Delta^m \arrow[r] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\mathbf{III}"] & \coprod_* id_Y^{-1} \arrow[r, "\simeq"] \arrow[d, "\simeq"] & Y \\
* \arrow[r] & \coprod_{NSimp(Y)} *.
\end{tikzcd}$$

Write $d(\mathbf{I})$ for the square obtained by taking diagonals in the square $\mathbf{I}$ and similarly for the other sub-diagrams. The square $d(\mathbf{I}+\mathbf{III})$ is

$$\begin{tikzcd}
f^{-1}(\sigma) \arrow[r] \arrow[d] & \operatorname{hocolim} f^{-1} \arrow[d] \\
* \arrow[r, "\sigma"'] & NSimp(Y),
\end{tikzcd}$$

which is homology cartesian by Theorem 2.4. Since the square $d(\mathbf{III})$ is homotopy cartesian it follows by Lemma 2.2 that $d(\mathbf{I})$ is homology cartesian. The square $d(\mathbf{II})$ is also homotopy cartesian so it follows, again by Lemma 2.2, that $d(\mathbf{I}+\mathbf{II})$ is homology cartesian. $\square$

Definition 2.5. A map $f\colon X \to Y$ of simplicial sets is called a homology fibration if it satisfies one (and hence both) of the conditions of Lemma 2.3.

Definition 2.6. A map $p\colon E \to B$ of topological spaces is called a homology fibration if for any point $b \in B$ the natural map from the fiber F_b at b to the homotopy fiber hF_b induces an isomorphism on integral homology.

The relation between the two kinds of homology fibrations is given as follows:

Theorem 2.7: *[PS04, 4.4] A map $f\colon X \to Y$ of simplicial sets is a homology fibration if and only if the induced map on realizations $|f|\colon |X| \to |Y|$ is a homology fibration of topological spaces.*

Recall Segal's edgewise subdivision functor $Sd\colon sSet \to sSet$, which has $(SdX)_n = X_{2n+1}$ (see the appendix A and [Seg73] for details). An important property of this construction is that the realization of a simplicial set X is naturally homeomorphic to the realization of its subdivision SdX. Using this, we get the next lemma from Theorem 2.7.

Lemma 2.8: *A map $f\colon X \to Y$ of simplicial sets is a homology fibration if and only if the induced map $Sdf\colon SdX \to SdY$ is a homology fibration.*

The next lemma follows easily from condition 2 of Lemma 2.3.

Lemma 2.9: *Homology fibrations are closed under base change, i.e., the pullback of a homology fibration along any map is a homology fibration.*

Lemma 2.10: *Let $f\colon X \to Y$ be a homology fibration and let $g\colon Z \to Y$ be any map. Then the pullback square*

$$\begin{tikzcd} Z \times_Y X \arrow[r] \arrow[d, "h"'] & X \arrow[d, "f"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

is homology cartesian.

Proof. Factor the map f as

$$X \overset{i}{\rightarrowtail} W \overset{\bar{f}}{\twoheadrightarrow} Y,$$

where i is a trivial cofibration. There is an induced factorization

$$Z \times_Y X \xrightarrow{j} Z \times_Y W \overset{\bar{h}}{\twoheadrightarrow} Z$$

of h and we must show that j is a homology equivalence.

Let

$$H_q(h^{-1}, \mathbb{Z})\colon \mathit{Simp}(Z) \to \mathit{Ab}$$

be the composite functor given by

$$\sigma \mapsto h^{-1}(\sigma) \mapsto H_q(h^{-1}(\sigma), \mathbb{Z}),$$

and similarly for $H_q(\bar{h}^{-1}, \mathbb{Z})$, where Ab is the usual category of abelian groups. The natural transformation $j_*\colon H_q(h^{-1}, \mathbb{Z}) \to H_q(\bar{h}^{-1}, \mathbb{Z})$ is an isomorphism of functors, since f is a homology fibration and the maps from the pullbacks over σ to the pullbacks over $g(\sigma)$ are isomorphisms. Recall that for a functor $F\colon I \to \mathit{Ab}$ the *translation object* EF of F is the simplicial abelian group given in degree n by

$$EF_n = \bigoplus_{i_0 \to \cdots \to i_n} F(i_0)$$

with structure maps as for $\coprod_* F$ (see [GJ09, IV.2.1]). The map

$$E(H_q(h^{-1}, \mathbb{Z})) \to E(H_q(\bar{h}^{-1}, \mathbb{Z}))$$

induced by j_* is an isomorphism. By [GJ09, IV.5.1] there is a first quadrant spectral sequence

$$E_2^{p,q} = \pi_p E(H_q(h^{-1}, \mathbb{Z})) \implies H_{p+q}(Z \times_Y X, \mathbb{Z})$$

and a corresponding one for $\bar{h}^{-1}$ converging to $H_{p+q}(Z \times_Y W, \mathbb{Z})$. The map j induces an isomorphism of E_2-pages and is therefore a homology equivalence by the comparison theorem for spectral sequences. □

For a homology fibration $f\colon X \to Y$ the functor $H_q(f^{-1}, \mathbb{Z})$ sends all maps to isomorphism and hence factors through the groupoid $\mathit{GSimp}(Y)$ obtained from $\mathit{Simp}(Y)$ by inverting all morphisms (see [GJ09, p. 235]). This groupoid is naturally equivalent to the fundamental groupoid of the realization $|Y|$ (see [GJ09, III.1.1]). If for any pair of maps $\xi, \zeta\colon \sigma \to \tau$ in $\mathit{GSimp}(Y)$ the induced maps

$$\xi_*, \zeta_*\colon H_q(f^{-1}(\sigma), \mathbb{Z}) \to H_q(f^{-1}(\tau), \mathbb{Z})$$

agree, we say that the fundamental groupoid acts trivially on the homology of the fibers of f. If Y is connected then for any simplex ρ in Y there is a unique isomorphism of functors

$$(\sigma \mapsto H_q(f^{-1}\sigma, \mathbb{Z})) \cong (\sigma \mapsto H_q(f^{-1}\rho, \mathbb{Z}))$$

whose value at ρ is the identity map.

Lemma 2.11: *Let $f\colon X \to Y$ be a homology fibration such that the fundamental groupoid of Y acts trivially on the homology of the fibers of f. Then, for any homology equivalence $g\colon Z \to Y$ the induced map*

$$g'\colon Z \times_Y X \to X$$

is a homology equivalence.

Proof. Assume, without loss of generality, that Y is connected and choose a fiber F over some vertex of Y. Then, by [GJ09, IV.5.1], there are Serre spectral sequences, for f

$$E_2^{p,q} = H_p(Y, H_q(F)) \implies H_{p+q}(X),$$

and for the pullback of f along g

$$E_2^{p,q} = H_p(Z, H_q(F)) \implies H_{p+q}(Z \times_Y X).$$

The map induced by g' on E_2-pages is an isomorphism by the universal coefficient theorem, and the fact that g is a homology equivalence. It follows that g' is a homology equivalence.

□

We now turn to bisimplicial sets. A map of bisimplicial sets will be called a homology equivalence if the induced map on diagonals is a homology equivalence. Note that any element $\sigma \in Y_{m,n}$, called a bisimplex, is classified by a unique map $\sigma\colon \Delta^{m,n} \to Y$ from a representable bisimplicial set.

Lemma 2.12: *Let $f\colon X \to Y$ be a map of bisimplicial sets. The following are equivalent:*

1. *The diagonal $df\colon dX \to dY$ is a homology fibration.*
2. *For any pair of bisimplices $\sigma\colon \Delta^{m,n} \to Y$ and $\tau\colon \Delta^{p,q} \to Y$ and for any diagram*

$$\begin{tikzcd} \Delta^{m,n} \arrow[rr, "(\alpha,\beta)_*"] \arrow[dr, "\sigma"'] & & \Delta^{p,q} \arrow[dl, "\tau"] \\ & Y & \end{tikzcd}$$

the induced map on pullbacks along f

$$f^{-1}(\alpha, \beta)_*\colon f^{-1}(\sigma) \to f^{-1}(\tau)$$

is a homology equivalence.

Proof. $1 \implies 2$: Given a bisimplex $\sigma\colon \Delta^{m,n} \to Y$ we choose a vertex $v \in Y_{0,0}$ belonging to σ. Since pullbacks commute with diagonals, we get a diagram

$$\begin{tikzcd} df^{-1}(v) \arrow[r] \arrow[d] & df^{-1}(\sigma) \arrow[r] \arrow[d] & dX \arrow[d, "df"] \\ \Delta^0 \arrow[r] & \Delta^m \times \Delta^n \arrow[r] & dY \end{tikzcd}$$

in which the two squares and the outer rectangle are pullback diagrams. The middle vertical map is a homology fibration, by Lemma 2.9, since it is a pullback of df. The lower left map is a weak equivalence, so it follows by Lemma 2.11 that the induced map $df^{-1}(v) \to df^{-1}(\sigma)$ is a homology equivalence. By definition this means that the map $f^{-1}(v) \to f^{-1}(\sigma)$ is a homology equivalence. A map $(\alpha, \beta)\colon \sigma \to \tau$ gives a commuting triangle

$$\begin{tikzcd} & f^{-1}(v) \arrow[dl] \arrow[dr] & \\ f^{-1}(\sigma) \arrow[rr, "f^{-1}(\alpha,\beta)"'] & & f^{-1}(\tau). \end{tikzcd}$$

By the argument above the two downward maps are homology equivalences, so it follows that $f^{-1}(\alpha, \beta)$ is as well.

$2 \implies 1$: The proof follows roughly the same outline as the corresponding proof for simplicial sets. As for simplicial sets there is a category $Simp(Y)$ of bisimplices of Y and condition 2 says that the functor $f^{-1}\colon Simp(Y) \to bisSet$ takes values in homology equivalences. Composing f^{-1} with the diagonal functor $d\colon bisSet \to sSet$ gives a functor $df^{-1}\colon Simp(Y) \to sSet$ taking values in homology equivalences. For a simplex $\sigma\colon \Delta^n \to dY$ there is a diagram of bisimplicial sets like the one in the proof of Lemma 2.3 and by the same argument we conclude that df is a homology fibration. $\square$

Definition 2.13. A map $f\colon X \to Y$ of bisimplicial sets is called a homology fibration if it satisfies one (and hence both) of the conditions of Lemma 2.12.

The exposition of propositions 2.14 and 2.15 below, and their proofs follows [Jar08] closely. Of course any errors or omissions are my own. If X is a bisimplicial set we will write X_n for the vertical simplicial set

$$[p] \mapsto X_{n,p}.$$

Proposition 2.14: *Let $f\colon X \to Y$ be a map of bisimplicial sets such that for each $n \geq 0$ the map $f_n\colon X_n \to Y_n$ is a Kan fibration. Assume that for each $\theta\colon [m] \to [n]$ and each $v \in Y_{n,0}$ the induced map on fibers $f_n^{-1}(v) \to f_m^{-1}(\theta^*(v))$ is a homology equivalence. Then f is a homology fibration.*

Proof. We show that f satisfies condition 2 of Lemma 2.12. Given a bisimplex $\tau\colon \Delta^{p,q} \to Y$ choose a vertex v of Δ^q and let $(id_{[p]}, v)_*\colon \Delta^{p,0} \to \Delta^{p,q}$ be the corresponding map of bisimplicial sets. In level n we can form the iterated pullback

$$\begin{tikzcd} \coprod_{\gamma \in \Delta^p_n} f_n^{-1}(v) \arrow[r, "v_*"] \arrow[d] & \coprod_{\gamma \in \Delta^p_n} f_n^{-1}(((\gamma, id_{[q]})^*\tau)_n) \arrow[r] \arrow[d] & X_n \arrow[d, two heads, "f_n"] \\ \coprod_{\gamma \in \Delta^p_n} \Delta^0 \arrow[r, "\simeq", "\coprod v"'] & \coprod_{\gamma \in \Delta^p_n} \Delta^q \arrow[r, "\coprod ((\gamma, id_{[q]})^*\tau)_n"'] & Y_n \end{tikzcd}$$

where the map v_* is a weak equivalence since f_n is a fibration. This says precisely that the map $f^{-1}((id_{[p]}, v)^*\tau) \to f^{-1}(\tau)$ is a level-wise weak equivalence, and so in particular a homology equivalence. Therefore it suffices to consider diagrams of the form

$$\begin{tikzcd}
\Delta^{m,0} \arrow[rr, "(\alpha,id)_*"] \arrow[dr, "v"'] & & \Delta^{p,0} \arrow[dl, "w"] \\
& Y. &
\end{tikzcd}$$

In level n the induced map $f^{-1}(v) \to f^{-1}(w)$ fits as the top row in the diagram

$$\begin{tikzcd}
\coprod_{\gamma\in\Delta^m_n} f_n^{-1}(((\gamma, id)^*v)_n) \arrow[rr] & & \coprod_{\delta\in\Delta^p_n} f_n^{-1}(((\delta, id)^*w)_n)\ , \\
\coprod_{\gamma\in\Delta^m_n} f_m^{-1}(v) \arrow[u] & \coprod_{\gamma\in\Delta^m_n} f_p^{-1}(w) \arrow[l] \arrow[r] & \coprod_{\delta\in\Delta^p_n} f_p^{-1}(w) \arrow[u]
\end{tikzcd}$$

where the vertical maps and the lower left hand map are homology equivalences by assumption on f. The lower right hand map becomes the weak equivalence

$$\alpha_* \times id\colon \Delta^m \times f_p^{-1}(w) \to \Delta^p \times f_p^{-1}(w)$$

after taking diagonals. $\square$

Theorem 2.15: *Let $f\colon X \to Y$ be a map of bisimplicial sets such that for each $n \geq 0$ the map $f_n\colon X_n \to Y_n$ is a homology fibration and for each $v \in Y_{n,0}$ the induced map on fibers $f_n^{-1}(v) \to f_m^{-1}(\theta^*(v))$ is a homology equivalence. Then f is a homology fibration.*

Proof. We begin by factoring the map $f\colon X \to Y$ as a level-wise trivial cofibration followed by a level-wise fibration $X \xrightarrow{g} W \xrightarrow{h} Y$. Given a bisimplex $\sigma\colon \Delta^{p,q} \to Y$ we get a diagram of bisimplicial sets

$$\begin{tikzcd}
f^{-1}(\sigma) \arrow[r] \arrow[d] & X \arrow[d, "g"] \arrow[dd, bend left=60, "f"] \\
h^{-1}(\sigma) \arrow[r] \arrow[d] & W \arrow[d, "h"] \\
\Delta^{p,q} \arrow[r] & Y,
\end{tikzcd}$$

which in level n looks like

$$\begin{tikzcd}
\coprod_{\theta\in\Delta^p_n} f_n^{-1}(((\theta, id_{[q]})^*\sigma)_n) \arrow[r] \arrow[d] & X_n \arrow[d, "g_n", "\simeq"'] \arrow[dd, bend left=60, "f_n"] \\
\coprod_{\theta\in\Delta^p_n} h_n^{-1}(((\theta, id_{[q]})^*\sigma)_n) \arrow[r] \arrow[d] & W_n \arrow[d, two heads, "h_n"] \\
\coprod_{\theta\in\Delta^p_n} \Delta^q \arrow[r, "\coprod((\theta,id_{[q]})^*\sigma)_n"'] & Yn.
\end{tikzcd}$$

Since f_n is a homology fibration the upper left vertical map induces a homology equivalence on each summand, and is therefore a homology equivalence. This says that the map $f^{-1}(\sigma) \to h^{-1}(\sigma)$ is a level-wise homology equivalence and hence it is a homology equivalence by [GJ09, IV.2.6].

Given a vertex $v \in Y_{n,0}$ and a map $\theta\colon [m] \to [n]$ there is a commuting square of fibers

$$\begin{tikzcd} f_n^{-1}(v) \arrow[r] \arrow[d] & f_m^{-1}(\theta^* v) \arrow[d] \\ h_n^{-1}(v) \arrow[r] & h_m^{-1}(\theta^* v). \end{tikzcd}$$

The vertical maps are homology equivalences since f is a level-wise homology fibration and the upper horizontal map is a homology equivalence by assumption on f. From this we see that lower horizontal map is a homology equivalence which by Proposition 2.14 implies that h is a homology fibration. A map $\sigma \to \tau$ in $Simp(Y)$ induces a square of pullbacks

$$\begin{tikzcd} f^{-1}(\sigma) \arrow[r] \arrow[d] & f^{-1}(\tau) \arrow[d] \\ h^{-1}(\sigma) \arrow[r] & h^{-1}(\tau), \end{tikzcd}$$

where the vertical maps are homology equivalences. Moreover, the lower horizontal map is homology equivalence since h is a homology fibration. Hence the top horizontal map is a homology equivalence and f is a homology fibration. $\square$

For a bisimplicial set X we write Sd_hX for the Segal edgewise subdivision of X in the first (horizontal) variable and Sd_vX for the subdivision in the second (vertical) variable. Clearly $Sd_hSd_vX = Sd_vSd_hX$ and $dSd_hSd_vX = Sd(dX)$.

Lemma 2.16: *Let $f\colon X \to Y$ be a map of bisimplicial sets satisfying the conditions of Theorem 2.15. Then Sd_hf and Sd_vf also satisfy the conditions.*

Proof. We treat Sd_hf first. In level n the map Sd_hf is just the map $f\colon X_{2n+1} \to Y_{2n+1}$ which is a homology fibration by assumption. Assume given a vertex $v \in (Sd_hY)_{n,0} = Y_{2n+1,0}$ and a simplicial structure map $\theta\colon [m] \to [n]$. The induced map $\theta^*\colon (SdY)_n \to (SdY)_m$ is the map simplicial structure map $(\theta \sqcup \theta^{op})^*\colon Y_{2n+1} \to Y_{2m+1}$ so the map on fibers is a homology equivalence by assumption.

Now for the map Sd_vf. For $n \geq 0$ the map $(Sd_vf)_n$ is the subdivision $Sd(f_n)$ of the map $f_n\colon X_n \to Y_n$, so by Lemma 2.8 it is a homology fibration. A vertex $v \in (SdY_n)_0 = Y_{n,1}$ need not come from a vertex in $Y_{n,0}$, but it can be connected to such a vertex by an edge. Since $Sd(f_n)$ is a homology fibration it then follows that the fiber over v is equivalent to the fiber over a vertex in Y_n. This implies that for any simplicial structure map $\theta\colon [m] \to [n]$ the fiber over v maps by a homology equivalence to the fiber over $(Sd\theta^*)(v)$. $\square$

3 Simplicial monoids with anti-involution

Definition 3.1. An anti-involution on a monoid M is a function $m \mapsto \bar{m}$ from M to itself such that $\overline{(\bar{m})} = m$ and $\overline{m \cdot n} = \bar{n} \cdot \bar{m}$ for all $m, n \in M$. A simplicial monoid with anti-involution is a simplicial monoid M with a self-map, which is an anti-involution in each simplicial level.

Given a monoid M we can form the bar construction BM which is the simplicial set $[n] \mapsto M^{\times n}$ with the usual structure maps. If M has the extra structure of an anti-involution we get extra structure on the bar construction as well. The system of maps

$$\{w_i \colon B_i M \to B_i M\}$$

given in level p by

$$w_p(m_1, m_2, \ldots, m_p) = (\overline{m_p}, \ldots, \overline{m_2}, \overline{m_1})$$

together with the simplicial structure maps of BM form a real simplicial set (see appendix A), which we by abuse of language also call BM. Similarly, for a simplicial monoid M with anti-involution we get a functor

$$BM \colon (\Delta R)^{op} \to sSet.$$

We write $\Delta^1_R \boxtimes M$ for the $(\Delta R)^{op} \times \Delta^{op}$-indexed functor $(p,q) \mapsto \Delta^1_p \times M_q$ with the obvious bisimplicial structure maps and real structure maps given by $w_p(\zeta, m) = (\zeta^{op}, \bar{m})$. Since the simplicial set of 1-simplices of BM is just M there is an induced map

$$\Delta^1_R \boxtimes M \to BM.$$

As a consequence, we get a C_2-equivariant map on realizations

$$|\Delta^1| \times |M| \to |BM|$$

which sends the C_2-subspace $|\Delta^1| \times \{e\} \cup \{0,1\} \times |M|$ to the basepoint. Here the C_2 acts on $|\Delta^1|$ by reflection through the midpoint (see Example A.2), so there is an induced C_2-map $S^{1,1} \wedge |M| \to |BM|$ whose adjoint is the canonical map

$$\lambda_M \colon |M| \to \Omega^{1,1}|BM|.$$

Non-equivariantly, the topological monoid $|M|$ acts by left multiplication on itself and acts homotopy associatively on the loop space by $m \cdot \gamma = \lambda(m) * \gamma$, where $*$ means concatenation of loops. Up to homotopy λ commutes with the actions of C_2 and of $|M|$. We are interested in the properties of the map induced by λ on fixed points

$$\lambda_M^{C_2} \colon |M|^{C_2} \to (\Omega^{1,1}|BM|)^{C_2}.$$

The topological monoid $|M|$ acts continuously on the fixed points $|M|^{C_2}$ by $m \cdot n = mn\bar{m}$ and up to homotopy on the fixed points of the loop space by $m \cdot \gamma = \lambda(m) * \gamma * \lambda(\bar{m})$. These actions commute with $\lambda_M^{C_2}$ up to homotopy.

Definition 3.2. Let N be a commutative monoid. An element $s \in N$ is called a cofinal generator if for any $x \in N$ there is an $n \geq 0$ and an element $y \in N$ such that $xy = s^n$. A vertex t in a simplicial monoid M with $\pi_0(M)$ commutative is called a homotopy cofinal generator if its class $[t] \in \pi_0(M)$ is a cofinal generator.

Example 3.3. Let M be a simplicial monoid such that the monoid $\pi_0(M)$ is finitely generated and commutative. Pick vertices $t_1, \ldots, t_n \in M_0$ whose path components $[t_1], \ldots, [t_n]$ generate $\pi_0(M)$. Then the vertex $t = t_1 t_2 \cdots t_n$ is a homotopy cofinal generator of M.

From now on let M denote a simplicial monoid with $\pi_0(M)$ in the center of $H_*(M)$ and let t be a homotopy cofinal generator of M. For a simplicial set X with a left M-action we set

$$X_\infty = \operatorname{hocolim}(X \xrightarrow{t\cdot} X \xrightarrow{t\cdot} X \xrightarrow{t\cdot} \cdots).$$

In particular, we have

$$M_\infty = \operatorname{hocolim}(M \xrightarrow{t\cdot} M \xrightarrow{t\cdot} M \xrightarrow{t\cdot} \cdots).$$

The homology of M is a graded ring. Since $[t]$ is in the center multiplication by $[t]$ on $H_*(M)$ is $H_*(M)$-linear. Hence there is an isomorphism of left $H_*(M)$-modules

$$H_*(M_\infty) \cong \operatorname{colim}(H_*(M) \xrightarrow{[t]\cdot} H_*(M) \xrightarrow{[t]\cdot} H_*(M) \xrightarrow{[t]\cdot} \cdots).$$

Lemma 3.4: *The map $M \to M_\infty$ including M at the start of the diagram induces an isomorphism of $H_*(M)$-algebras*

$$H_*(M)[\pi_0(M)^{-1}] \to H_*(M_\infty).$$

Proof. Since $\pi_0(M)$ is central in $H_*(M)$ there is an isomorphism

$$\operatorname{colim}(H_*(M) \xrightarrow{[t]\cdot} H_*(M) \xrightarrow{[t]\cdot} H_*(M) \xrightarrow{[t]\cdot} \cdots) \cong H_*(M)[t^{-1}]$$

and $H_*(M) \to H_*(M)[t^{-1}]$ is the localization map. Since $[t]$ is a cofinal generator of $\pi_0(M)$ the further localization map

$$H_*(M)[t^{-1}] \to H_*(M)[\pi_0(M)^{-1}]$$

is an isomorphism. □

It follows that the vertices M_0 of M act on M_∞ by homology equivalences. The following can also be found in e.g., [GJ09, IV.5.15].

Lemma 3.5: *Let X be a simplicial set with a right action of M such M_0 acts by homology equivalences. Then the canonical map $p\colon B(X, M, *) \to BM$ satisfies the conditions of Theorem 2.15. In particular, it is a homology fibration.*

Proof. In each level $n \geq 0$ the map

$$p_n\colon X \times M^{\times n} \to M^{\times n}$$

is a homology fibration because the induced map on realizations is, by Theorem 2.7.

Let $v \in M_0^{\times n}$ be a vertex and let $\theta\colon [m] \to [n]$ be a map in Δ. Note that the fiber over any vertex is isomorphic to X. We must show that the map on fibers

$$p_n^{-1}(v) \to p_m^{-1}(\theta^*(v))$$

is a homology equivalence. Since θ^* can be factored into face and degeneracy maps we reduce to these cases. If $\theta^* = d_j$ with $j \neq 0$ then the map $p_n^{-1}(v) \to p_{n-1}^{-1}(\theta^*(v))$, similarly for $\theta^* = s_i$. Otherwise, if $\theta^* = d_0$, then the induced map corresponds to acting on X by an element of M_0 and is therefore a homology equivalence. □

Lemma 3.6: *Let $sSet - M$ be the category of simplicial sets with right M-action and equivariant maps and let $G\colon I \to sSet - M$ be a functor. If X is a simplicial set with left M-action there is a natural isomorphism of simplicial sets*

$$dB(\operatorname{hocolim} G, M, X) \cong \operatorname{hocolim} dB(G, M, X).$$

Proof. Both simplicial sets are obtained by taking iterated diagonals of the trisimplicial set $B(\coprod_* G, M, X)$ given by

$$[p],[q],[r] \mapsto \left(\coprod_{\sigma \in N_r(I)} G(\sigma(0))_q \right) \times M_q^{\times^p} \times X_q.$$

□

Corollary 3.7: *For any simplicial set X with a left M-action there is an isomorphism*

$$dB(M_\infty, M, X) \cong (dB(M, M, X))_\infty.$$

Theorem 3.8 (Group completion)**:** *(cf. [MS76], [FM94] and [PS04]) Let M be a simplicial monoid such that $\pi_0 M$ is in the center of $H_*(M)$. Then there is an isomorphism of left $H_*(M)$-algebras*

$$H_*(M)[\pi_0(M)^{-1}] \xrightarrow{\cong} H_*(\Omega|BM|).$$

Proof. Assume first that M has a homotopy cofinal generator t. Then, by lemma 3.5 the map $B(M_\infty, M, *) \to BM$ is a homology fibration with fiber M_∞. Taking $X = *$ in Corollary 3.7 shows that the simplicial set $dB(M_\infty, M, *)$ is a homotopy colimit of contractible spaces and hence is contractible. From this we get a homology equivalence $|M_\infty| \to \Omega|BM|$ and we conclude by Lemma 3.4.

For general M we let $F(M)$ denote the poset of submonoids of M with finitely generated monoid of path components. Then there is an isomorphism of simplicial monoids $\operatorname{colim}_{M_i \in F(M)} M_i \cong M$ and the colimit is filtering. The functors $|-|$, B, Ω, $H_*(-)$ and inverting π_0 commute with the filtering colimit in question so the result now follows since each $M_i \in F(M)$ has a homotopy cofinal generator by 3.3. □

Inspired by Theorem 3.8 we will now proceed to analyze the map $\lambda_M^{C_2}$. It becomes easier to work with the anti-involution when we take the Segal subdivision in the horizontal (i.e., bar construction-) direction of BM. The output is the bisimplicial set $Sd_h BM$ which has a simplicial action of C_2 and whose fixed points we will now describe. An element in level (p,q) of $Sd_h BM$ is a tuple

$$(m_1, \ldots, m_{2p+1}) \in M_q^{\times^{2p+1}},$$

and the action of the non-trivial element in C_2 is

$$(m_1, \ldots, m_p, m_{p+1}, m_{p+2}, \ldots, m_{2p+1}) \mapsto (\overline{m_{2p+1}}, \ldots, \overline{m_{p+2}}, \overline{m_{p+1}}, \overline{m_p}, \ldots, \overline{m_1}).$$

The fixed points of this action are of the form

$$(m_1, \ldots, m_p, m_{p+1}, \overline{m_p}, \ldots, \overline{m_1}), \text{ where } m_{p+1} = \overline{m_{p+1}}.$$

Here, the last p factors are redundant and projection on the first $p+1$ factors gives a bijection

$$b_{p,q}\colon (M_q^{\times^{2p+1}})^{C_2} \xrightarrow{\cong} M_q^{\times^p} \times M_q^{C_2}.$$

The monoid M_q acts on $M_q^{C_2}$ on the left by $(m,n) \mapsto m \cdot n \cdot \overline{m}$. Both this action and the description of the fixed points are compatible with the simplicial structure maps of M. Combining this with the fact that

$$d_p(m_1, \dots, m_p, m_{p+1}, \overline{m_p}, \dots, \overline{m_1}) = (m_1, \dots, m_p \cdot m_{p+1} \cdot \overline{m_p}, \dots, \overline{m_1}),$$

we get the following:

Lemma 3.9: *Let M be a simplicial monoid with anti-involution. Then the maps $b_{p,q}$ determine natural isomorphism of bisimplicial sets*

$$b\colon (Sd_h BM)^{C_2} \xrightarrow{\cong} B(*, M, M^{C_2}).$$

The map $p\colon B(M_\infty, M, *) \to BM$ induces a map

$$Sd_h p\colon Sd_h B(M_\infty, M, *) \to Sd_h BM$$

on subdivisions. Since p satisfies the conditions of Theorem 2.15, the map $Sd_h p$ does as well, by Lemma 2.16. Therefore $Sd_h p$ is a homology fibration.

Lemma 3.10: *The pullback of $Sd_h p$ along the inclusion*

$$B(*, M, M^{C_2}) \hookrightarrow Sd_h BM$$

is isomorphic to $B(M_\infty, M, M^{C_2})$.

The proof is straightforward. It now follows from Lemma 2.10 that the square

$$\begin{tikzcd} B(M_\infty, M, M^{C_2}) \arrow[r] \arrow[d] & Sd_h B(M_\infty, M, *) \arrow[d] \\ B(*, M, M^{C_2}) \arrow[r] & Sd_h BM \end{tikzcd} \tag{*}$$

becomes homology cartesian after taking diagonals. We consider M^{C_2} as a bisimplicial set which is constant in the first variable. Define the map

$$i\colon M^{C_2} \to B(M, M, M^{C_2})$$

level-wise by

$$m \mapsto (e, e, \dots, e, m).$$

This map has a retraction r given by

$$r(m_0, m_1, \dots, m_p, m) = m_0 \cdot m_1 \cdots m_p \cdot m \cdot \overline{m_p} \cdots \overline{m_1} \cdot \overline{m_0}$$

and there is a standard simplicial homotopy $r \circ i \simeq id$. The map $M \to M_\infty$ of Lemma 3.4 induces a map

$$j\colon B(M, M, M^{C_2}) \to B(M_\infty, M, M^{C_2}).$$

Lemma 3.11: *The map $j \circ i\colon M^{C_2} \to B(M_\infty, M, M^{C_2})$ induces an isomorphism of left $\pi_0(M)$-sets*

$$\pi_0(M^{C_2})[\pi_0(M)^{-1}] \xrightarrow{\cong} \pi_0(dB(M_\infty, M, M^{C_2}))$$

and an isomorphism of left $H_*(M)$-modules

$$H_*(M^{C_2})[\pi_0(M)^{-1}] \xrightarrow{\cong} H_*(dB(M_\infty, M, M^{C_2})).$$

Proof. We present the argument for homology, the one for π_0 is similar. By Corollary 3.7 there is an isomorphism $(dB(M, M, M^{C_2}))_\infty \cong dB(M_\infty, M, M^{C_2})$. In the diagram

$$\begin{tikzcd}
dB(M,M,M^{C_2}) \arrow[r, "t\cdot"] \arrow[d, "dr"] & dB(M,M,M^{C_2}) \arrow[r, "t\cdot"] \arrow[d, "dr"] & dB(M,M,M^{C_2}) \arrow[r, "t\cdot"] \arrow[d, "dr"] & \cdots \\
M^{C_2} \arrow[r, "t\cdot"] & M^{C_2} \arrow[r, "t\cdot"] & M^{C_2} \arrow[r, "t\cdot"] & \cdots
\end{tikzcd}$$

the vertical maps are weak equivalences and hence induce a weak equivalence of homotopy colimits $dB(M, M, M^{C_2})_\infty \xrightarrow{r_\infty} M^{C_2}_\infty$. In homology we get a sequence of isomorphisms of left $H_*(M)$-modules

$$H_*(B(M_\infty, M, M^{C_2})) \xrightarrow{\cong} H_*(M^{C_2}_\infty) \xrightarrow{\cong} H_*(M^{C_2})[\pi_0(M)^{-1}].$$

□

Let (X, x) be a based C_2-space with $\sigma\colon X \to X$ representing the action of the non-trivial element of C_2. The homotopy fiber hF_{ι_X} of the canonical inclusion $\iota_X\colon X^{C_2} \hookrightarrow X$ of the fixed points can be identified with the space of paths $\chi\colon [0, \frac{1}{2}] \to X$ such that $\chi(0) = x$ and $\chi(\frac{1}{2}) \in X^{C_2}$. There is a map

$$b_X\colon hF_{\iota_X} \to (\Omega^{1,1}X)^{C_2}$$

given by $b_X(\chi) = \chi * (\sigma \circ \overline{\chi})$ where $*$ is the concatenation operation and $\overline{\chi}$ is the path $t \mapsto \chi(1-t)$. This map is a homeomorphism with inverse given by restricting loops to $[0, \frac{1}{2}]$.

Now we apply geometric realization to the square (*) to obtain a homology cartesian square of spaces

$$\begin{tikzcd}
{|B(M_\infty, M, M^{C_2})|} \arrow[r] \arrow[d] & {|B(M_\infty, M, *)|} \arrow[d] \\
{|B(*, M, M^{C_2})|} \arrow[r] & {|BM|.}
\end{tikzcd}$$

The space $|B(M_\infty, M, *)|$ is contractible and so $|B(M_\infty, M, M^{C_2})|$ is homology equivalent to the homotopy fiber of the composite

$$|B(*, M, M^{C_2})| \cong |BM|^{C_2} \hookrightarrow |BM|.$$

By the discussion above, this space is homeomorphic to $(\Omega^{1,1}|BM|)^{C_2}$ and so we get a homology equivalence

$$g\colon |B(M_\infty, M, M^{C_2})| \to (\Omega^{1,1}|BM|)^{C_2}.$$

Theorem 3.12: *Let M be a simplicial monoid with anti-involution such that $\pi_0 M$ is in the center of $H_*(M)$. Then the map*

$$\lambda_M^{C_2} \colon M^{C_2} \to (\Omega^{1,1}|BM|)^{C_2}$$

induces an isomorphism

$$\pi_0(M^{C_2})[\pi_0(M)^{-1}] \xrightarrow{\cong} \pi_0(\Omega^{1,1}|BM|)^{C_2}$$

of left $\pi_0(M)$-sets and an isomorphism of left $H_*(M)$-modules

$$H_*(M^{C_2})[\pi_0(M)^{-1}] \xrightarrow{\cong} H_*((\Omega^{1,1}|BM|)^{C_2}).$$

Proof. Assume first that M has a homotopy cofinal generator $t \in M_0$. By Lemma 3.11 the map

$$|j \circ i| \colon |M^{C_2}| \to |B(M_\infty, M, M^{C_2})|$$

induces the desired localization map on homology. Since the homology equivalence $g \colon |B(M_\infty, M, M^{C_2})| \to (\Omega^{1,1}BM)^{C_2}$ is induced by the contracting homotopy on $|B(M_\infty, M, *)|$ which is homotopic to a homotopy that induces the map $\lambda_M^{C_2}$ we conclude that $\lambda_M^{C_2}$ also induces the desired map on homology.

If M does not have a cofinal generator we reduce to the above case by a colimit argument as in the proof of theorem 3.8. The above proof easily generalizes to prove the π_0-statement of the theorem. $\square$

4 Categories with duality

In this section we summarize some facts we will need later. Again we make no claim of originality and the reader can consult [Dot12], [Sch10a] or [HM] for details. To avoid set-theoretic problems we fix two Grothendieck universes $\mathscr{U} \in \mathscr{V}$ and we will assume without further mention that all categories in this section and the next are $\mathscr{V}$-small.

Definition 4.1. A ($\mathscr{V}$-small) category with duality is a triple $(\mathscr{C}, T, \eta)$ where $\mathscr{C}$ is a ($\mathscr{V}$-small) category, $T \colon \mathscr{C}^{op} \to \mathscr{C}$ is a functor and $\eta \colon id \to T \circ T^{op}$ is a natural transformation such that for each c in $\mathscr{C}$ the composite map

$$Tc \xrightarrow{\eta_{Tc}} TT^{op}Tc \xrightarrow{T(\eta_c)} Tc$$

is the identity on Tc. If $\eta = id$, so that $T \circ T^{op} = Id_{\mathscr{C}}$, then the duality is said to be strict.

Example 4.2. A monoid M can be thought of as a category $\mathscr{C}_M$ with one object $*$ and $Hom_{\mathscr{C}_M}(*, *) = M$ as monoids. Then a duality T, η on $\mathscr{C}_M$ is the same as a monoid map $t \colon M^{op} \to M$, i.e., such that $t(mn) = t(n)t(m)$ for all $m, n \in M$, and an element $\eta \in M$ such that $\eta t^2(m) = m\eta$ for all $m \in M$ and $t(\eta)\eta = e$. The duality is strict if and only if t is an anti-involution on M.

The main example of interest to us is the following (see e.g. [Wal70]).

Example 4.3. A Wall anti-structure is a triple (R, α, ε) where R is a ring in the smaller universe $\mathscr{U}$, α is an additive map $R \to R$ such that $\alpha(rs) = \alpha(s)\alpha(r)$ and ε is a unit in R such that $\alpha^2(r) = \varepsilon r \varepsilon^{-1}$ and $\alpha(\varepsilon) = \varepsilon^{-1}$. For an anti-structure (R, α, ε) there is a naturally associated category with duality $P(R, \alpha, \varepsilon)$ with underlying category $P(R)$ the category of finitely generated projective (f.g.p) $\mathscr{U}$-small right R-modules. The duality functor on $P(R, \alpha, \varepsilon)$ is $Hom_R(-, R)$ where for an f.g.p. module P we give $Hom_R(P, R)$ the right (!) module structure given by $(fr)(p) = \alpha(r)f(p)$. The map

$$\eta_P \colon P \xrightarrow{\cong} Hom_R(Hom_R(P, R), R)$$

is the isomorphism given on elements $p \in P$ by $\eta_P(p)(f) = \alpha(f(p))\varepsilon$. It is straightforward to check that the equation $(\eta_P)^* \circ \eta_{Hom_R(P,R)} = id_{Hom_R(P,R)}$ holds for all f.g.p. modules P. The smallness conditions on R and the modules ensures that $P(R)$ is small with respect to the larger universe $\mathscr{V}$.

Definition 4.4. A duality preserving functor

$$(F, \xi)\colon (\mathscr{C}, T, \eta) \longrightarrow (\mathscr{C}', T', \eta')$$

consists of a functor $F\colon \mathscr{C} \to \mathscr{C}'$ and a natural transformation

$$\xi\colon F \circ T \to T' \circ F^{op}$$

such that for all c in $\mathscr{C}$ the diagram

$$\begin{tikzcd} F(c) \arrow[r, "\eta'_{F(c)}"] \arrow[d, "F(\eta_c)"'] & T'(T')^{op}F(c) \arrow[d, "T'(\xi_c)"] \\ FTT^{op}(c) \arrow[r, "\xi_{T(c)}"'] & T'F^{op}T^{op}(c) \end{tikzcd}$$

commutes.

Composition is given by $(G, \zeta) \circ (F, \xi) = (G \circ F, \zeta_F \circ G(\xi))$. An *equivalence of categories with duality* is a duality preserving functor

$$(F, \xi)\colon (\mathscr{C}, T, \eta) \longrightarrow (\mathscr{C}', T', \eta')$$

such that there is a duality preserving functor $(F', \xi')\colon (\mathscr{C}', T', \eta') \longrightarrow (\mathscr{C}, T, \eta)$ and natural isomorphisms $u\colon F' \circ F \xrightarrow{\cong} Id_{\mathscr{C}}$ and $u'\colon F \circ F' \xrightarrow{\cong} Id_{\mathscr{C}'}$ satisfying $\xi'_{F(c)} \circ F'(\xi_c) = T(u_c) \circ u_{T(c)}$ for c in $\mathscr{C}$ and similarly for u'.

Definition 4.5. Let $(\mathscr{C}, T, \eta)$ be a category with duality. The category $Sym(\mathscr{C}, T, \eta)$ of symmetric forms in $(\mathscr{C}, T, \eta)$ is given as follows:

- The objects of $Sym(\mathscr{C}, T, \eta)$ are maps $f\colon a \to Ta$ such that $f = Tf \circ \eta_a$.
- A morphism from $f\colon a \to Ta$ to $f'\colon a' \to Ta'$ is a map $r\colon a \to a'$ in $\mathscr{C}$ such that the diagram

  $$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "r"'] & Ta \\ a' \arrow[r, "f'"'] & Ta' \arrow[u, "Tr"'] \end{tikzcd}$$

  commutes.
- Composition is given by ordinary composition of maps in $\mathscr{C}$.

The reason for the name "symmetric form" in the preceding definition is the following. Let (R, α, ε) be a Wall-anti-structure. The category $SymP(R, \alpha, \varepsilon)$ has as objects maps $\varphi\colon P \to Hom_R(P, R)$ such that the adjoint map $\tilde{\varphi}\colon P \otimes_{\mathbb{Z}} P \to R$ is a biadditive form on P satisfying

$$\begin{aligned} \tilde{\varphi}(pr, qs) &= \alpha(r)\tilde{\varphi}(p, q)s \\ \tilde{\varphi}(q, p) &= \alpha(\tilde{\varphi}(p, q))\varepsilon, \end{aligned}$$

for $r, s \in R$ and $p, q \in P$. A map

$$h\colon (P \xrightarrow{\varphi} Hom_R(P, R)) \to (P' \xrightarrow{\varphi'} Hom_R(P', R))$$

is an R-module homomorphism $h\colon P \to P'$ such that $\tilde{\varphi}'(h(p), h(q)) = \tilde{\varphi}(p, q)$ for all $p, q \in P$. An object $\varphi\colon P \to Hom_R(P, R)$ such that φ is an isomorphism is called non-degenerate.

Definition 4.6. (cf. [HM]) For a category with duality $(\mathscr{C}, T, \eta)$ the strictification $\mathscr{D}(\mathscr{C}, T, \eta)$ has objects triples (c, c', f) where $f\colon c' \to Tc$ is a map and morphisms from (c, c', f) to (d, d', g) are pairs $(r\colon c \to d, s\colon d' \to c')$ such that the diagram

$$\begin{tikzcd} c' \arrow[r, "f"] & Tc \\ d' \arrow[u, "s"] \arrow[r, "g"'] & Td \arrow[u, "Tr"'] \end{tikzcd}$$

commutes. Composition is given by composition in each component. The duality on $\mathscr{D}(\mathscr{C}, T, \eta)$ is given by sending an object $f\colon c' \to Tc$ to the composite $c \xrightarrow{\eta_c} TT^{op}c \xrightarrow{Tf} Tc'$ and $(r\colon c \to d, s\colon d' \to c')$ to $(s\colon d' \to c', r\colon c \to d)$.

It is easy to see that the duality on $\mathscr{D}(\mathscr{C}, T, \eta)$ is strict. There are duality preserving functors

$$(I, \iota)\colon (\mathscr{C}, T, \eta) \to \mathscr{D}(\mathscr{C}, T, \eta)$$

given by $I(c) = (c, Tc, id_{T(c)})$, $I(f) = (f, Tf)$ and $\iota_c = (id_{T(c)}, \eta_c)$ and

$$(K, \kappa)\colon \mathscr{D}(\mathscr{C}, T, \eta) \to (\mathscr{C}, T, \eta),$$

given by $K(c, c', f) = c$, $K(r, s) = r$ and $\kappa_{(c,c',f)} = f$. These induce homotopy inverse weak equivalences $N\mathscr{C} \simeq N\mathscr{D}\mathscr{C}$ and $NSym(\mathscr{C}) \simeq NSym(\mathscr{D}\mathscr{C})$. Both the construction $\mathscr{D}$ and the functors K and I are functorial in $(\mathscr{C}, T, \eta)$ for duality preserving functors.

A strict duality T on a category $\mathscr{C}$ gives a map

$$NT\colon (N\mathscr{C})^{op} = N(\mathscr{C}^{op}) \xrightarrow{\cong} N\mathscr{C}$$

such that $NT \circ (NT)^{op} = id_{N\mathscr{C}}$. We know from Lemma A.1 that this is equivalent to extending the simplicial structure of $N\mathscr{C}$ to a real simplicial structure. It follows that the realization has an induced C_2-action given by

$$[(c_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} c_n, t_0, \dots t_n)] \mapsto [(c_n \xrightarrow{Tf_n} \dots \xrightarrow{Tf_1} Tc_0, t_n, \dots t_0)],$$

for $(c_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} c_n, t_0, \dots t_n) \in N_n\mathscr{C} \times \Delta^n$. Thus, from the topological perspective the effect of the $\mathscr{D}$-construction is to replace the geometric realization $|N\mathscr{C}|$, which has an action of C_2 in the homotopy category, by the bigger space $|N\mathscr{D}\mathscr{C}|$ which has a continuous action of C_2. The actions are compatible in the sense that the maps $|NI|$ and $|NK|$ are mutually inverse isomorphisms of C_2-objects in the homotopy category.

Definition 4.7. Let $\mathscr{C}$ be a category. Its subdivision $Sd\mathscr{C}$ is a category given as follows: An object of $Sd\mathscr{C}$ is a morphism $f\colon a \to b$ in $\mathscr{C}$ and a map from $f\colon a \to b$ to $g\colon c \to d$ is a pair (h, i) of maps such that the following diagram commutes

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "h"'] & b \\ c \arrow[r, "g"'] & d. \arrow[u, "i"'] \end{tikzcd}$$

Composition is given by $(h', i') \circ (h, i) = (h' \circ h, i \circ i')$.

Note that $SdN\mathscr{C} = NSd\mathscr{C}$. If $(\mathscr{C}, T, \eta)$ is a category with duality then there is an induced functor

$$SdT\colon Sd\mathscr{C} \to Sd\mathscr{C}$$

given by $SdT(a \xrightarrow{f} b) = Tb \xrightarrow{Tf} Ta$ and $SdT(h, i) = (Ti, Th)$. If T is a strict duality then $Sym(\mathscr{C})$ is the category fixed under the C_2-action defined by SdT, so using the strictification $\mathscr{D}$ every Sym-category is up to homotopy a category of fixed points.

5 K-theory of additive categories with duality

Definition 5.1. Let $\mathscr{C}$ be a category and let X_1 and X_2 be objects of $\mathscr{C}$. A biproduct diagram for the pair (X_1, X_2) is a diagram

$$X_1 \underset{i_1}{\overset{p_1}{\leftrightarrows}} Y \underset{i_2}{\overset{p_2}{\rightleftarrows}} X_2 \tag{1}$$

in $\mathscr{C}$ such that $p_j \circ i_j = id_{X_j}$, the p_j-s express Y as the product of X_1 and X_2 and the i_j-s express Y as a coproduct of X_1 and X_2.

If $\mathscr{C}$ is a category which has a zero object and each pair of objects has a biproduct diagram in $\mathscr{C}$ the hom-sets of $\mathscr{C}$ naturally inherit the structure of commutative monoids such that composition is bilinear [ML98, VIII,2]. We call such a category $\mathscr{C}$ *additive* if the hom-sets are abelian groups, not just monoids. A functor between additive categories is called additive if it preserves biproducts and zero-objects. Additive functors induce group homomorphisms on hom-groups.

Let X be a finite pointed set. The category $Q(X)$ is defined as follows: The objects in $Q(X)$ are the pointed subsets $U \subseteq X$. A morphism $U \to V$ of pointed subsets is a pointed subset of the intersection $U \cap V$. The composition of two subsets $A \subseteq U \cap V$ and $B \subseteq V \cap W$ is $A \cap B \subseteq U \cap W$. Note that $A \subseteq U \cap V$ can be thought of both as a map from U to V and as a map from V to U this gives an isomorphism $Q(X) \cong Q(X)^{op}$.

Definition 5.2. Let $\mathscr{C}$ be an additive category and X a finite pointed set. A sum-diagram in $\mathscr{C}$ indexed by X is a functor

$$A\colon Q(X) \to \mathscr{C}$$

such that for any pointed subset $U \subseteq X$ the maps $A(U) \to A(\{u, *\})$ induced by the pointed subsets $\{u, *\} \subseteq U$ induce an isomorphism

$$A(U) \xrightarrow{\cong} \prod_{u \in U \setminus \{*\}} A(\{u, *\}).$$

We write $\mathscr{C}(X)$ for the full subcategory of sum-diagrams in the functor category $Fun(Q(X), \mathscr{C})$.

A *pointed* category is a category $\mathscr{C}$ with a chosen object $0_{\mathscr{C}}$. When $\mathscr{C}$ is additive $0_{\mathscr{C}}$ will always be a zero-object, but in general it need not be. We say that a functor between pointed categories is pointed if it preserves the chosen objects. Many of the constructions we will do in the following rely on having chosen base points. To avoid confusion and to make our constructions functorial we will usually work with pointed categories.

For a pointed additive category and a finite pointed set X we require that the elements of $\mathscr{C}(X)$ be pointed, i.e., that the send the subset $\{*\}$ to $0_{\mathscr{C}}$. We write $\mathscr{C}^X$ for the (pointed) category $Fun_*(X, \mathscr{C})$ of pointed functors from X to $\mathscr{C}$, where we think of X as a discrete category. There is a natural evaluation functor $e_X \colon \mathscr{C}(X) \to \mathscr{C}^X$ given on objects by $e_X(A)(x) = A(\{x, *\})$ and similarly for morphisms. The following lemma is easily verified.

Lemma 5.3: *Let $\mathscr{C}$ be a pointed additive category. For any finite pointed set X the functor*

$$e_X \colon \mathscr{C}(X) \to \mathscr{C}^X$$

is an equivalence of categories.

A pointed map $f \colon X \to Y$ induces a pushforward functor $f_* \colon \mathscr{C}(X) \to \mathscr{C}(Y)$ given by

$$(f_*(A))(U) = A(f^{-1}(U \setminus \{*\}) \cup \{*\}).$$

Given two composable maps f and g of finite pointed sets it is not hard to see that $(f \circ g)_* = f_* \circ g_*$, so that we get a functor

$$\mathscr{C}(-) \colon FinSet_* \to Cat_*,$$

where $FinSet_*$ is the category of finite sets and pointed maps and Cat_* is the category of $\mathscr{V}$-small pointed categories and pointed functors between them. This notion coincides up to suitable equivalence with Segal's Γ-category construction [Seg74]. If S is a pointed simplicial set which is finite in each simplicial level we can regard it as a functor $S \colon \Delta^{op} \to FinSet_*$ and form the composite functor $\mathscr{C}(S)$ which is a simplicial pointed category, i.e. a simplicial object in Cat_*.

Definition 5.4. An additive category with weak equivalences is a pair $(\mathscr{C}, w\mathscr{C})$ where $\mathscr{C}$ is an additive category and $w\mathscr{C} \subseteq \mathscr{C}$ is a subcategory such that all isomorphisms are in $w\mathscr{C}$ and such that if f and g are in $w\mathscr{C}$ then their coproduct $f \oplus g$ is in $w\mathscr{C}$.

A map $F \colon (\mathscr{C}, w\mathscr{C}) \to (\mathscr{C}', w'\mathscr{C}')$ of additive categories with weak equivalences is an additive functor which preserves weak equivalences. It is an equivalence of additive categories with weak equivalences if it has an additive inverse which preserves weak equivalences. If $\mathscr{C}$ is pointed we take $w\mathscr{C}$ to be pointed with the same chosen object as $\mathscr{C}$.

Let $(\mathscr{C}, w\mathscr{C})$ be a pointed additive category with weak equivalences and X a finite pointed set. Then $\mathscr{C}(X)$ is additive and we define the subcategory $w\mathscr{C}(X) \subseteq \mathscr{C}(X)$ to have the same objects as $\mathscr{C}(X)$ and morphisms that are pointwise in $w\mathscr{C}$. It is a subcategory of weak equivalences in $\mathscr{C}(X)$. If $f \colon X \to Y$ is a pointed map the functor f_* maps $w\mathscr{C}(X)$ into $w\mathscr{C}(Y)$, so there is an induced functor

$$w\mathscr{C}(-) \colon FinSet_* \to Cat_*.$$

As in Lemma 5.3 the functor $we_X \colon w\mathscr{C}(X) \to w\mathscr{C}^X$ induced by e_X is an equivalence of categories. We write S^1 for the simplicial circle $\Delta^1/\partial\Delta^1$, with basepoint $[\partial\Delta^1]$. Segal showed in [Seg74] that the space

$\Omega|Nw\mathscr{C}(S^1)|$ is a model for the algebraic K-theory of $(\mathscr{C}, w\mathscr{C})$, analogous to the space $\Omega|BM|$ for a simplicial monoid M.

The functor $w\mathscr{C} \to w\mathscr{C}(S^1_1)$ sending an object c to the diagram with value c on the non-trivial subset of S^1_1 and $0_{\mathscr{C}}$ on $\{*\}$ is an equivalence of categories. There is an induced map

$$\Delta^1 \boxtimes Nw\mathscr{C} \to Nw\mathscr{C}(S^1)$$

of bisimplicial sets which in turn induces a map

$$\lambda_{\mathscr{C}} \colon |Nw\mathscr{C}| \to \Omega|Nw\mathscr{C}(S^1)|$$

of spaces. In [Seg74, 4] Segal proves a group completion theorem for the map $\lambda_{\mathscr{C}}$ analogous to 3.8. We will mimic the treatment of the monoid case above to reprove Segal's result and extend it to an equivariant statement analogous to Theorem 3.12 in the case that $\mathscr{C}$ has an additive duality.

Lemma 5.5: *(see e.g. [HM]) Let $(\mathscr{C}, w\mathscr{C})$ be an additive category with weak equivalences. Then there is a pointed additive category with weak equivalences $(\mathscr{C}', w'\mathscr{C}')$ and an additive equivalence $F \colon (\mathscr{C}, w\mathscr{C}) \to (\mathscr{C}', w'\mathscr{C}')$ such that $(\mathscr{C}', w'\mathscr{C}')$ has a coproduct functor*

$$\oplus \colon \mathscr{C}' \times \mathscr{C}' \to \mathscr{C}'$$

making $\mathscr{C}'$ a strictly unital, strictly associative symmetric monoidal category.

The construction $w\mathscr{C}(S^1)$ makes sense also for non-pointed $\mathscr{C}$ but one must choose a basepoint for $\Omega|Nw\mathscr{C}(S^1)|$ and $\lambda_{\mathscr{C}}$ to be defined. This can be done is such a way that the induced map $F_{S^1} : w\mathscr{C}(S^1) \to w'\mathscr{C}'(S^1)$ gives a homotopy equivalence on realizations and there is a commutative diagram

$$\begin{tikzcd}
|Nw\mathscr{C}| \arrow[r, "\lambda_{\mathscr{C}}"] \arrow[d, "F"'] & \Omega|Nw\mathscr{C}(S^1)| \arrow[d, "\Omega|NF_{S^1}|"] \\
|Nw'\mathscr{C}'| \arrow[r, "\lambda'_{\mathscr{C}}"'] & \Omega|Nw'\mathscr{C}'(S^1)|
\end{tikzcd}$$

of spaces, in which the vertical maps are homotopy equivalences and H-maps. From now on we assume, without loss of generality, that $(\mathscr{C}, w\mathscr{C})$ is pointed and has a coproduct functor $\oplus$ as in Lemma 5.5.

The path components of the nerve $Nw\mathscr{C}$ will be called weak equivalence classes. The set $\pi_0 Nw\mathscr{C}$ of such classes is a commutative monoid under the operation $[a] + [b] = [a \oplus b]$. We assume that the $\pi_0 Nw\mathscr{C}$ has a cofinal generator represented by an object t of $\mathscr{C}$. Then there is a functor $t \oplus - \colon \mathscr{C} \to \mathscr{C}$ which restricts to an endofunctor on $w\mathscr{C}$. By analogy with the monoid case above we form the diagram

$$w\mathscr{C} \xrightarrow{t\oplus -} w\mathscr{C} \xrightarrow{t\oplus -} w\mathscr{C} \xrightarrow{t\oplus -} \cdots$$

of categories. We define $\underline{\mathbb{N}}$ to be the poset category of natural numbers with the usual ordering

$$0 \to 1 \to 2 \to \cdots$$

so that the above diagram of categories becomes a functor $D \colon \underline{\mathbb{N}} \to Cat$ in the obvious way. Now set $w\mathscr{C}_\infty = \underline{\mathbb{N}} \wr D$, where $\wr$ denotes the Grothendieck construction (see e.g. [Tho79]). The objects of the category $w\mathscr{C}_\infty$ are pairs $(m,c) \in \mathbb{N} \times ob\mathscr{C}$ and a map $(n,c) \to (n+k,d)$ is a map $(t \oplus -)^k(c) \to d$ in $w\mathscr{C}$ (see also [Gra76, p.8]). Thomason [Tho79, 1.2] constructs a natural weak equivalence

$$\operatorname{hocolim}(ND) \to Nw\mathscr{C}_\infty.$$

Since the nerve $Nw\mathscr{C}$ is a simplicial monoid, its homology $H_*(Nw\mathscr{C})$ is a ring under the induced Pontrjagin product. The following is a special case of Lemma 3.4.

Lemma 5.6: *The canonical functor $w\mathscr{C} \to w\mathscr{C}_\infty$ sending an object c to $(0,c)$ induces an isomorphism of $H_*(Nw\mathscr{C})$-algebras*

$$H_*(Nw\mathscr{C})[\pi_0(Nw\mathscr{C})^{-1}] \xrightarrow{\cong} H_*(Nw\mathscr{C}_\infty).$$

We now recall the simplicial path construction (see [Wal85, 1.5] for details). Define the shift functor $P\colon \Delta \to \Delta$ by $P([n]) = [0] \sqcup [n] = [n+1]$ and $P(\alpha) = id_{[0]} \sqcup \alpha$. For a simplicial object $X\colon \Delta^{op} \to \mathscr{A}$ the (simplicial) path object PX on X is defined as $PX = X \circ P^{op}$. The natural transformation $\delta^0\colon Id_\Delta \to P$ given on objects by $\delta^0\colon [n] \to [n+1]$ gives a natural map $d_0\colon PX \to X$. For a simplicial set X there is a natural map $PX \to X_0$ onto the vertices of X which is a simplicial homotopy equivalence [Wal85, 1.5.1]. In the case of the simplicial circle the map $d_0\colon PS^1 \to S^1$ induces a map $w\mathscr{C}(PS^1) \to w\mathscr{C}(S^1)$ of simplicial categories which we will also call d_0. There is a simplicial homotopy equivalence $PS^1 \xrightarrow{\simeq} *$ which induces a weak equivalence $Nw\mathscr{C}(PS^1) \xrightarrow{\simeq} Nw\mathscr{C}(*) \simeq *$ of bisimplicial sets. Let $\zeta_n \in \Delta^1_n$ be the element such that $\zeta_n(0) = 0$ and $\zeta_n(i) = 1$ for $i \geq 1$. We denote its image in the quotient set $\Delta^1_n/\partial\Delta^1_n$ by z_n and write $\tilde{c}_{n+1}$ for the diagram in $\mathscr{C}((PS^1)_n) = \mathscr{C}(S^1_{n+1})$ whose value is $c \in ob\mathscr{C}$ on all pointed subsets containing z_{n+1} and $0_\mathscr{C}$ on the other subsets. The maps between c's in $\tilde{c}_{n+1}$ are all identities and the remaining maps are zero. The functor $d_0\colon \mathscr{C}(S^1_{n+1}) \to \mathscr{C}(S^1_n)$ restricts diagrams to the part away from z_{n+1}, so $d_0(\tilde{c}_{n+1}) = 0_{w\mathscr{C}(S^1_n)}$, the 0-diagram. Now let $c = t$, a homotopy cofinal generator. Adding the object $\tilde{t}_{n+1}$ from the left gives a functor

$$\tilde{t}_{n+1} \oplus -\colon w\mathscr{C}(S^1_{n+1}) \to w\mathscr{C}(S^1_{n+1}).$$

We define $w\mathscr{C}(S^1_{n+1})_\infty$ to be the Grothendieck construction on the diagram

$$w\mathscr{C}(S^1_{n+1}) \xrightarrow{\tilde{t}_{n+1}\oplus -} w\mathscr{C}(S^1_{n+1}) \xrightarrow{\tilde{t}_{n+1}\oplus -} w\mathscr{C}(S^1_{n+1}) \xrightarrow{\tilde{t}_{n+1}\oplus -} \cdots.$$

Since 0 is a strict unit in $\mathscr{C}$ the system functors $\{\tilde{t}_{n+1} \oplus -\}_{n\geq 0}$ commutes with the structure maps of $w\mathscr{C}(PS^1)$, and the map $d_0\colon w\mathscr{C}(PS^1) \to w\mathscr{C}(S^1)$. Therefore the $w\mathscr{C}(S^1_{n+1})_\infty$'s assemble to a simplicial category $w\mathscr{C}(PS^1)_\infty$ with a map $d_{0,\infty}\colon w\mathscr{C}(PS^1)_\infty \to w\mathscr{C}(S^1)$. The inclusion of $w\mathscr{C}(S^1_{n+1})$ in the first spot of the diagram gives a map $w\mathscr{C}(PS^1) \to w\mathscr{C}(PS^1)_\infty$ such that the diagram

$$\begin{tikzcd} w\mathscr{C}(PS^1) \arrow[rr] \arrow[dr, "d_0"'] & & w\mathscr{C}(PS^1)_\infty \arrow[dl, "d_{0,\infty}"] \\ & w\mathscr{C}(S^1) & \end{tikzcd}$$

commutes.

Proposition 5.7: *The induced map on nerves*

$$Nd_{0,\infty}\colon Nw\mathscr{C}(PS^1)_\infty \to Nw\mathscr{C}(S^1)$$

is a homology fibration of bisimplicial sets.

Proof. We will show that the map satisfies the conditions of Theorem 2.15. First, we verify that it is a level-wise homology fibration. The functor $\tilde{t}_{n+1} \oplus -$ commutes with $(d_{0,\infty})_n$ so the evaluation functors give a commuting square

$$\begin{tikzcd} w\mathscr{C}(S^1_{n+1})_\infty \arrow[r, "e_{n,\infty}"] \arrow[d] & w\mathscr{C}_\infty \times w\mathscr{C}^{\times^n} \arrow[d, "p"] \\ w\mathscr{C}(S^1_n) \arrow[r, "e_n"'] & w\mathscr{C}^{\times^n}, \end{tikzcd} \tag{2}$$

where the horizontal arrows are equivalences of categories. Assume given a simplex $\sigma\colon \Delta^m \to Nw\mathscr{C}(S^1_n)$ and consider the resulting diagram

$$\begin{tikzcd} \Delta^m \arrow[r, "\sigma"] \arrow[d, "id"'] & Nw\mathscr{C}(S^1_n) \arrow[d, "Ne_n"] & Nw\mathscr{C}(S^1_{n+1})_\infty \arrow[l, "N(d_{0,\infty})_n"'] \arrow[d, "Ne_{n,\infty}"] \\ \Delta^m \arrow[r, "N(e_n\circ\sigma)"'] & Nw\mathscr{C}^{\times^n} & Nw\mathscr{C}_\infty \times Nw\mathscr{C}^{\times^n}. \arrow[l, "p"] \end{tikzcd}$$

Since the vertical maps are weak equivalences the induced map on homotopy pullbacks is a weak equivalence. The map p is obviously a homology fibration, so it suffices to show that the map on actual pullbacks is a weak equivalence. Nerves commute with limits, so this pullback can be taken in Cat_* where it is straightforward to check that the map on pullbacks is an equivalence of categories.

To see that the second condition of 2.15 holds we observe that the fiber over an object c in $w\mathscr{C}(S^1_n)$ is equivalent to $w\mathscr{C}_\infty$. Now we conclude by Lemma 5.6 in the same way as in the proof of Lemma 3.5. □

The proof of the following theorem (cf. [Seg74, 5], [FM94, Q.9]) is similar to that of 3.8.

Theorem 5.8 (K-theoretic group completion)**:** *The map $\lambda_{\mathscr{C}}$ induces an isomorphism of $H_*(Nw\mathscr{C})$-algebras*

$$H_*(Nw\mathscr{C})[\pi_0(Nw\mathscr{C})^{-1}] \xrightarrow{\cong} H_*(\Omega|Nw\mathscr{C}(S^1)|).$$

We now turn to additive categories with duality.

Definition 5.9. An additive category with duality and weak equivalences is a tuple $(\mathscr{C}, T, \eta, w\mathscr{C})$ such that:

- T is additive and η is takes values in weak equivalences,
- T and η give a duality on $\mathscr{C}$,
- T sends (opposites of) weak equivalences to weak equivalences,
- $(\mathscr{C}, w\mathscr{C})$ is an additive category with weak equivalences.

Example 5.10. Let (R, α, ε) be a Wall-anti-structure. Then the category $P(R, \alpha, \varepsilon)$ becomes an additive category with duality and weak equivalences if we take the weak equivalences to be the isomorphisms. In applications it is sometimes useful to work with bounded chain complexes in $P(R, \alpha, \varepsilon)$ with quasi-isomorphisms as weak equivalences, see e.g. Schlichting's paper [Sch10b].

To get a strict duality we can apply the functor $\mathscr{D}$; the category $\mathscr{D}\mathscr{C}$ is additive since $\mathscr{C}$ is so. Taking the weak equivalences in $\mathscr{D}\mathscr{C}$ to be pairs of maps in $w\mathscr{C}$ gives $\mathscr{D}(\mathscr{C}, T, \eta)$ the structure of an additive category with duality and weak equivalences which is a functorial and better behaved replacement of $(\mathscr{C}, T, \eta, w\mathscr{C})$. There is a commutative square of H-spaces and H-maps

$$\begin{tikzcd} {|Nw\mathscr{C}|} \arrow[r, "\lambda_{\mathscr{C}}"] \arrow[d] & \Omega|Nw\mathscr{C}(S^1)| \arrow[d] \\ {|Nw\mathscr{D}\mathscr{C}|} \arrow[r, "\lambda_{\mathscr{D}\mathscr{C}}"] & \Omega|Nw\mathscr{D}\mathscr{C}(S^1)| \end{tikzcd}$$

where the vertical maps are weak equivalences. Note that Lemma 5.5 also applies to additive categories with duality and weak equivalences, so that we may assume that our categories have a strict duality T, a duality preserving direct sum functor $(- \oplus -)$ which is strictly associative and strictly unital and that the unit 0 is fixed under the duality.

Remark 5.11. Let $\nu\colon T(-) \oplus T(-) \to T(- \oplus -)$ be the canonical natural isomorphism. The category $Sym(w\mathscr{C})$ has a functorial sum operation $\perp$ called the orthogonal sum given by

$$(f\colon c \to T(c)) \perp (g\colon d \to Td) = c \oplus d \xrightarrow{f \oplus g} T(c) \oplus T(d) \xrightarrow{\nu_{c,d}} T(c \oplus d).$$

Under the induced operation the set $\pi_0(Sym(w\mathscr{C}))$ becomes a commutative monoid with unit represented by the 0-form $0 \to 0$. For any object $c \xrightarrow{f} d$ of $Sd(w\mathscr{C})$ we can form the *hyperbolic form* $H(f)$ on $c \xrightarrow{f} d$ which is the object

$$c \oplus T(d) \xrightarrow{\begin{pmatrix} 0 & T(f) \\ \eta_d \circ f & 0 \end{pmatrix}} T(c) \oplus TT(d) \xrightarrow{\nu_{c,T(d)}} T(c \oplus T(d))$$

of $Sym(w\mathscr{C})$. This is also compatible with maps in $Sdw\mathscr{C}$. Together the functors $\perp$ and H give an action of $Sdw\mathscr{C}$ on $Sym(w\mathscr{C})$, in the sense of [Gra76, p.2], which analogous to the action of M on M^{C_2} of section 3.

Let X be a pointed C_2-set with $\sigma\colon X \to X$ representing the action of the non-trivial group element. The category $Q(X)$ inherits a strict duality t by taking $t(U) = \sigma(U)$ and similarly for morphisms. If $\mathscr{C}$ is an additive category with weak equivalences and strict duality there is an induced duality T_X on $w\mathscr{C}(X)$ given by taking a diagram

$$A\colon Q(X) \to \mathscr{C}$$

to the composite diagram

$$Q(X) \xrightarrow{t^{op}} Q(X)^{op} \xrightarrow{A^{op}} \mathscr{C}^{op} \xrightarrow{T} \mathscr{C}.$$

Clearly the duality T_X is strict and functorial in both X and $(\mathscr{C}, T)$. Let $n_+ = \{0, 1, \dots, n\}$ based at 0 with the action of C_2 taking an element $k \geq 1$ to $n - k + 1$ and fixing 0. If $X = 2_+$ then the action interchanges the two non-trivial elements and the duality on $\mathscr{C}(2_+)$ sends the diagram

$$\begin{tikzcd} X \arrow[r, shift left, leftarrow, "p_1"] \arrow[r, shift right, "i_1"'] & Y \arrow[r, shift left, "p_2"] \arrow[r, shift right, leftarrow, "i_2"'] & X' \end{tikzcd}$$

to the diagram

$$TX' \underset{Tp_2}{\overset{Ti_2}{\leftrightarrows}} TY \underset{Tp_1}{\overset{Ti_1}{\leftrightarrows}} TX.$$

We will always give $w\mathscr{C}^{\times n}$ the strict duality given on objects by

$$(X_1, \dots, X_n) \mapsto (TX_n, \dots, TX_1)$$

and similarly for maps. The evaluation map

$$e_n \colon w\mathscr{C}(n_+) \to w\mathscr{C}^{\times n}$$

is compatible with these dualities and is an equivalences of categories with duality.

We will now use the real simplicial set $S^{1,1} = \Delta^1_R/\partial\Delta^1_R$ to describe an action of C_2 on the algebraic K-theory space of an additive category with strict duality and weak equivalences. For each $m \geq 0$ and each non-basepoint simplex $x \in S^{1,1}_m$ there is a unique m-simplex $\xi \in \Delta^1_m$ mapping to x under the quotient map. The simplices Δ^1_m are linearly ordered by $\xi \leq \zeta \iff \xi(i) \leq \zeta(i)$ for all i, and this gives a linear ordering of $S^{1,1}_m \setminus \{*\}$ which is reversed by the real simplicial structure map w_m. For each $n \geq 0$ the category $w\mathscr{C}(S^{1,1}_n)$ inherits a duality T_n from the action of w_n and the duality T. There are induced maps

$$w_{m,n} \colon N_n w\mathscr{C}(S^{1,1}_m) \to N_n w\mathscr{C}(S^{1,1}_m)$$

given by

$$w_{m,n}(A_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} A_n) = (T_m A_n \xrightarrow{T_m f_n} \dots \xrightarrow{T_m f_1} T_m A_0),$$

which satisfy the relations $w_{m,n} \circ w_{m,n} = id$ and $w_{m,n} \circ (\alpha, \beta)^* = (\alpha^{op}, \beta^{op})^* \circ w_{p,q}$ for maps $(\alpha, \beta) \colon ([m], [n]) \to ([p], [q])$ in $\Delta \times \Delta$. After subdivision these assemble to a map of bisimplicial sets

$$W \colon SdNw\mathscr{C}(SdS^{1,1}) \to SdNw\mathscr{C}(SdS^{1,1})$$

which in level (m, n) is the map $Nw_{2m+1,2n+1}$.

The bisimplicial set $SdNw\mathscr{C}(SdS^{1,1})$ is naturally isomorphic to $NSdw\mathscr{C}(SdS^{1,1})$ and under this identification the map W comes from a map of simplicial categories

$$\tilde{W} \colon Sdw\mathscr{C}(SdS^{1,1}) \to Sdw\mathscr{C}(SdS^{1,1})$$

which squares to the identity and hence defines an action of C_2 on $Sdw\mathscr{C}(SdS^{1,1})$. Let

$$e^{1,1}_n \colon w\mathscr{C}(S^{1,1}_n) \to w\mathscr{C}^{\times n}$$

be the evaluation map which preserves the ordering of the underlying indexing set. It is an equivalence of categories with duality and it induces a functor

$$Sde^{1,1}_{2n+1} \colon Sdw\mathscr{C}(S^{1,1}_{2n+1}) \to Sdw\mathscr{C}^{\times 2n+1}$$

which is C_2-equivariant. The category $Sdw\mathscr{C}^{\times 2n+1}$ has the action given by

$$(f_1, \dots, f_{2n+1}) \mapsto (Tf_{2n+1}, \dots, Tf_1),$$

so a fixed object is of the form $(f_1, \ldots, f_n, f_{n+1}, Tf_n, \ldots, Tf_1)$ with $Tf_{n+1} = f_{n+1}$. We see that the last n factors are redundant, so evaluation followed by projection on the first $n+1$ coordinates defines a functor

$$Sym(w\mathscr{C}(S^{1,1}_{2n+1})) \to Sdw\mathscr{C}^{\times^n} \times Sym(w\mathscr{C})$$

which is an equivalence of categories.

The map $d_0\colon PS^1 \to S^1$ induces a map $Sdd_0\colon SdPS^1 \to SdS^1$ and hence a map of simplicial categories

$$Sdw\mathscr{C}(SdPS^1) \to Sdw\mathscr{C}(SdS^1).$$

Define $Pb(\mathscr{C}, T, w\mathscr{C})$ to be the pullback in the diagram

$$\begin{tikzcd} Pb(\mathscr{C}, T, w\mathscr{C}) \arrow[r] \arrow[d] & Sdw\mathscr{C}(SdPS^1) \arrow[d] \\ Sym(w\mathscr{C}(SdS^{1,1})) \arrow[r] & Sdw\mathscr{C}(SdS^1) \end{tikzcd}$$

of simplicial categories (without C_2-actions) where the bottom map is the inclusion functor and the right hand vertical map is induced by Sdd_0. Note that the evaluation map gives an equivalence of categories (without duality)

$$Pb(\mathscr{C}, T, w\mathscr{C})_n \simeq Sdw\mathscr{C} \times Sdw\mathscr{C}^{\times^n} \times Sym(w\mathscr{C}).$$

Thinking of $Sym(w\mathscr{C})$ as a constant simplicial category we define a map of simplicial categories

$$i\colon Sym(w\mathscr{C}) \to Pb(\mathscr{C}, T, w\mathscr{C})$$

which in level n sends an object $f\colon a \to Ta$ to the sum-diagram with value $f\colon a \to Ta$ on subsets containing the unique non-trivial fixed point of $S^{1,1}_{2n+1}$ and $id\colon 0 \to 0$ on subsets not containing it. The morphisms in $i_n(f)$ are identities or 0 as for $\tilde{t}_n$.

Lemma 5.12: *The map i induces a homology equivalence on nerves.*

Proof. Under the equivalences $Pb(\mathscr{C}, T, w\mathscr{C}) \simeq Sdw\mathscr{C} \times Sdw\mathscr{C}^{\times^n} \times Sym(w\mathscr{C})$ the functor i_n corresponds to the inclusion of $Sym(w\mathscr{C})$ by

$$(f\colon a \to Ta) \mapsto (id\colon 0 \to 0, id\colon 0 \to 0, \ldots, id\colon 0 \to 0, f\colon a \to Ta).$$

Let k be a field and let $h_*(-)$ be homology with coefficients in k. We take R to be the graded ring $h_*(NSdw\mathscr{C})$ and P to be the graded left R-module $h_*(NSym(w\mathscr{C}))$, where the action comes from the one described in Remark 5.11. Since k is a field the Künneth formula gives an isomorphism between the simplicial graded k-vector space $[n] \mapsto h_*(Pb(\mathscr{C}, T, w\mathscr{C})_n)$ and the bar construction $B(R, R, P)$. The map in homology induced by i is map $P \to B(R, R, P)$ which in degree n is the map $P \to R \otimes R^{\otimes^n} \otimes P$ given on generators by

$$p \mapsto 1 \otimes 1 \otimes \cdots \otimes 1 \otimes p.$$

This is a quasi-isomorphism of simplicial graded k-vector spaces. Hence, using the spectral sequence

$$E_2^{p,q} = H_p(h_q(X)) \implies h_{p+q}(dX)$$

for bisimplicial sets X (see e.g., [GJ09, IV.2]) we get an isomorphism on homology with k coefficients and, since this holds for any field k, an isomorphism on homology with integral coefficients. □

Now assume that $\mathscr{C}$ has an object t whose class in $\pi_0 Nw\mathscr{C}$ is a cofinal generator. By a colimit argument as in the proof of Theorem 3.8 this can be done without loss of generality. The subdivision of the functor $t \oplus -\colon w\mathscr{C} \to w\mathscr{C}$ is the functor that adds $t \xrightarrow{id} t$ to objects $a \to b$ of $Sdw\mathscr{C}$. Similarly, the subdivision $Sd\tilde{t}_n \oplus -$ of the functor $\tilde{t} \oplus -$, defined earlier, adds the map of sum-diagrams $\tilde{t}_n \xrightarrow{id} \tilde{t}_n$ to objects $A \to B$ in $Sdw\mathscr{C}(S^{1,1}_n)$. For each $n \geq 0$ there is a diagram

$$Sdw\mathscr{C}(S^1_n) \xrightarrow{Sd\tilde{t}_n \oplus -} Sdw\mathscr{C}(S^1_n) \xrightarrow{Sd\tilde{t}_n \oplus -} \dots$$

and we define $Sdw\mathscr{C}_\infty$ and $Sdw\mathscr{C}(S^1_n)_\infty$ to be the Grothendieck constructions on the diagrams as before. Also the map

$$(Sdd_0)_*\colon Sdw\mathscr{C}(S^1_{2n+2}) \to Sdw\mathscr{C}(S^1_{2n+1})$$

commutes with the maps $Sd\tilde{t}_n \oplus -$ and just as before there is an induced map

$$Sdw\mathscr{C}(SdPS^1)_\infty \to Sdw\mathscr{C}(SdS^1)$$

which induces a homology fibration on nerves. The maps $Sd\tilde{t}_n \oplus -$ also induce a map on the pullback $Pb(\mathscr{C}, T, w\mathscr{C})$ which commutes with the projection to $Sym(w\mathscr{C}(SdS^{1,1}))$. There results a pullback square of simplicial categories

$$\begin{tikzcd}
Pb(\mathscr{C}, T, w\mathscr{C})_\infty \arrow[r] \arrow[d] & Sdw\mathscr{C}(SdPS^1)_\infty \arrow[d] \\
Sym(w\mathscr{C}(SdS^{1,1})) \arrow[r] & Sdw\mathscr{C}(SdS^1)
\end{tikzcd}$$

where the vertical maps induce homology fibrations on nerves. The inclusion of $Pb(\mathscr{C}, T, w\mathscr{C})$ into $Pb(\mathscr{C}, T, w\mathscr{C})_\infty$ at the start of the diagram defining the latter will be called j.

Lemma 5.13: *The map*

$$j \circ i\colon Sym(w\mathscr{C}) \to Pb(\mathscr{C}, T, w\mathscr{C})_\infty$$

induces an isomorphism

$$H_*(NSym(w\mathscr{C})[\pi_0 Nw\mathscr{C}^{-1}] \xrightarrow{\cong} H_*(NPb(\mathscr{C}, T, w\mathscr{C})_\infty)$$

of left $H_*(NSdw\mathscr{C})$-modules.

Proof. By Lemma 5.12 the map i induces an isomorphism $H_*(NSym(w\mathscr{C})) \cong H_*(NPb(\mathscr{C}, T, w\mathscr{C}))$ of left $H_*(NSdw\mathscr{C})$-modules. The map

$$Sd\tilde{t}\colon Pb(\mathscr{C}, T, w\mathscr{C}) \to Pb(\mathscr{C}, T, w\mathscr{C})$$

induces left multiplication by $[t]$ on $H_*(NPb(\mathscr{C}, T, w\mathscr{C}))$, and by Thomason's theorem [Tho79, 1.2] we get a sequence of isomorphisms

$$\begin{aligned}
H_*(NPb(\mathscr{C}, T, w\mathscr{C})_\infty) &\cong \operatorname{colim}\left(H_*\left(NSym(w\mathscr{C})\right) \xrightarrow{[t]\cdot} H_*\left(NSym(w\mathscr{C})\right) \xrightarrow{[t]\cdot} \dots \right) \\
&\cong H_*(NSym(w\mathscr{C}))[t^{-1}] \\
&\cong H_*(NSym(w\mathscr{C}))[\pi_0 Nw\mathscr{C}^{-1}]
\end{aligned}$$

of left $H_*(NSdw\mathscr{C})$-modules as desired.

$\square$

The proof of the following statement is similar to that of Theorem 3.12. We use that there is a natural ring isomorphism $H_*(NSdw\mathscr{C}) \cong H_*(Nw\mathscr{C})$.

Theorem 5.14: *Let $(\mathscr{C}, w\mathscr{C}, T)$ be an additive category with strict duality and weak equivalences. Then the map $|NSym(w\mathscr{C})| \to (\Omega^{1,1}|Nw\mathscr{C}(S^{1,1})|)^{C_2}$ induces isomorphisms*

$$\pi_0(NSym(w\mathscr{C}))[\pi_0 Nw\mathscr{C}^{-1}] \xrightarrow{\cong} \pi_0((\Omega^{1,1}|Nw\mathscr{C}(S^{1,1})|)^{C_2})$$

of monoids and

$$H_*(NSym(w\mathscr{C}))[\pi_0 Nw\mathscr{C}^{-1}] \to H_*((\Omega^{1,1}|Nw\mathscr{C}(S^{1,1})|)^{C_2})$$

of left $H_*(Nw\mathscr{C})$-modules.

We write $i\mathscr{A}$ for the subcategory og isomorphisms in the category $\mathscr{A}$. For a Wall-anti-structure (R, α, ε), see Examples 4.3 and5.10, we set

$$K_0^{1,1}(R, \alpha, \varepsilon) = \pi_0(\Omega^{1,1}|NSym(i\mathscr{D}P(R, \alpha, \varepsilon))(S^{1,1})|)^{C_2}.$$

We will now investigate the two fundamental cases when $R = \mathbb{Z}$ and $\alpha = id_{\mathbb{Z}}$. They are $\varepsilon = 1$ and $\varepsilon = -1$. In the first case observe that $Sym(iP(\mathbb{Z}, id_{\mathbb{Z}}, 1)$ is the category of non-degenerate symmetric bilinear form spaces over $\mathbb{Z}$.

Proposition 5.15: *The monoid $K_0^{1,1}(\mathbb{Z}, id_{\mathbb{Z}}, 1)$ is not a group.*

Proof. By Theorem 5.14 there is an isomorphism

$$K_0^{1,1}(\mathbb{Z}, id_{\mathbb{Z}}, 1) \cong \pi_0|NSym(i\mathscr{D}P(\mathbb{Z}, id_{\mathbb{Z}}, 1))|[\pi_0(Ni\mathscr{D}(P(\mathbb{Z})))^{-1}]$$

and the right hand side is isomorphic to the monoid $M := \pi_0 NSym(iP(\mathbb{Z}, id_{\mathbb{Z}}, 1))[\pi_0(Ni(P(\mathbb{Z})))^{-1}]$. We will show that the latter is not a group by finding an element that does not have an inverse.

The n-th hyperbolic space H^n is the symmetric bilinear form space with underlying abelian group $\mathbb{Z}^{2n}$ and the symmetric form given by the matrix

$$\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},$$

where I_n denotes the $n \times n$ identity matrix. The monoid $\pi_0(Ni(P(\mathbb{Z})))$, which is isomorphic to $\mathbb{N}$, acts on $\pi_0 NSym(iP(\mathbb{Z}, id_{\mathbb{Z}}, 1))$ by adding hyperbolic spaces H^n via the orthogonal sum. Let $\langle 1 \rangle$ denote the group $\mathbb{Z}$ with the symmetric bilinear form given by ordinary multiplication. Assume that $[\langle 1 \rangle]$ has an inverse in M. Elements of M can be represented as differences $[a] - [H^m]$ where a is in $Sym(iP(\mathbb{Z}, id_{\mathbb{Z}}, 1)$. An inverse for $[\langle 1 \rangle]$ is a difference $[a] - [H^m]$ such that $[\langle 1 \rangle] + [a] - [H^m] = 0$ in M, or equivalently such that for some n the equation

$$[\langle 1 \rangle] + [a] + [H^n] = [H^m] + [H^n]$$

holds in $\pi_0 NSym(iP(\mathbb{Z}, id_{\mathbb{Z}}, 1))$. Since $H^m \perp H^n \cong H^{m+n}$, this means that we have an isomorphism

$$\langle 1 \rangle \perp a \perp H^n \cong H^{m+n}.$$

On the left hand side the element $(1, 0, 0)$ pairs with itself to $1 \in \mathbb{Z}$ under the bilinear form. However, by direct calculation one can see that on the right hand side any element in H^{m+n} gives an even number when paired with itself. Hence no such isomorphism can exist and $K_0^{1,1}(\mathbb{Z}, id_{\mathbb{Z}}, 1)$ is not a group.

□

The second case is $Sym(iP(\mathbb{Z}, id_{\mathbb{Z}}, -1))$, the category of non-degenerate symplectic bilinear form spaces over $\mathbb{Z}$. We write ${}_{-1}H^n(\mathbb{Z})$ for the symplectic form module with matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

By [MH73, 4, 3.5] any symplectic form module over $\mathbb{Z}$ is isomorphic to ${}_{-1}H^n(\mathbb{Z})$ for a uniquely determined $n \geq 0$. We call this number the rank of the symplectic module. The corresponding rank map

$$\pi_0|NSym(iP(\mathbb{Z}, id_{\mathbb{Z}}, -1)| \to \mathbb{N}$$

is an isomorphism of monoids.

Proposition 5.16: *The rank map induces an isomorphism*

$$K_0^{1,1}(\mathbb{Z}, id_{\mathbb{Z}}, -1) \cong \mathbb{Z}.$$

A Appendix: The category ΔR

The category ΔR has the same objects as the finite ordinal category Δ but more morphisms. In addition to the maps of Δ there is for each $n \geq 0$ a morphism $\omega_n \colon [n] \to [n]$, which should be thought of as reversing the ordering on $[n]$. The maps satisfy the relations

$$\omega_n \circ \omega_n = id_{[n]} \tag{3}$$
$$\omega_n \circ \sigma^j = \sigma^{n-j} \circ \omega_{n+1} \tag{4}$$
$$\omega_n \circ \delta^i = \delta^{n-i} \circ \omega_{n-1} \tag{5}$$

for $0 \geq i, j \geq n$. Following [HM] a functor from $(\Delta R)^{op}$ to sets is called a real simplicial set and similarly for functors into other categories. The maps induced in a real simplicial object by ω_n is denoted by w_n.

If we restrict a real simplicial set X to Δ^{op} the realization $|(X|_{\Delta^{op}})|$carries an action of C_2 which for $(x, t_0, \dots, t_n) \in X_n \times \Delta^n$ acts by

$$[(x, t_0, \dots, t_n)] \mapsto [(w_n(x), t_n, \dots, t_0)],$$

(see [HM] for details). Recall the functor

$$(-)^{op} \colon \Delta \to \Delta$$

which is the identity on objects and which sends $\delta^i \colon [n] \to [n+1]$ to $(\delta^i)^{op} = \delta^{n+1-i}$ and $\sigma^j \colon [n] \to [n-1]$ to $(\sigma^j)^{op} = \sigma^{n-1-j}$. Clearly $(-)^{op} \circ (-)^{op} = Id_{\Delta}$. Let $\mathscr{A}$ be any category. Given a simplicial object $X \colon \Delta^{op} \to \mathscr{A}$ its opposite is defined by

$$X^{op} = X \circ (-)^{op}.$$

This defines a functor on the functor category $\mathscr{A}^{\Delta^{op}}$ which squares to the identity.

Lemma A.1: *Giving an extension of a functor $X \colon \Delta^{op} \to \mathscr{A}$ to the category ΔR^{op} is equivalent to giving a map $\omega \colon X^{op} \to X$ such that $\omega \circ \omega^{op} = id_X$.*

Now recall the functor $Sd\colon \Delta \to \Delta$ given by $Sd[n] = [2n+1]$ and $Sd(\theta) = \theta \sqcup \theta^{op}$. The Segal edgewise subdivision of X is defined by $SdX = X \circ Sd$. It gives an endofunctor on $\mathscr{A}^{\Delta^{op}}$ and it is not hard to see that $Sd \circ (-)^{op} = Sd$, so that $SdX^{op} = SdX$ for any simplicial object X. Given any real simplicial object $X\colon (\Delta R)^{op} \to \mathscr{A}$ we can regard it as a simplicial object $X|_{\Delta^{op}}$ with a map $\omega\colon X^{op} \to X$ as in Lemma A.1. On the subdivision we get a map

$$Sd(\omega)\colon SdX^{op} = SdX \to SdX$$

such that $Sd(\omega)^2 = id_{SdX}$. In other words, $Sd(\omega)$ defines an action of C_2 on SdX. For a real simplicial set X the natural homeomorphism $|Sd(X|_{\Delta^{op}})| \xrightarrow{\cong} |X|_{\Delta^{op}}|$ of [Seg73, A.1] is C_2-equivariant.

Example A.2. The representable functor $\Delta^1_R = hom_{\Delta R}(-, [1])$ realizes to the topological 1-simplex with the action of C_2 given by reflection through the middle point. Its boundary $\partial\Delta^1_R$ realizes to the two end points, which are interchanged by the C_2-action. By abuse of notation we write $S^{1,1}$ for the real simplicial set $\Delta^1_R/\partial\Delta^1_R$. The realization $|S^{1,1}|$ is C_2-homeomorphic to the usual $S^{1,1}$.

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Equivariant deloopings in Real Algebraic K-theory

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Abstract

This paper is about Real algebraic K-theory which is an algebraic analogue of Atiyah's Real K-theory, defined by Hesselholt and Madsen. Real algebraic K-theory is a functor, denoted by KR, which associates to an exact category $\mathscr{C}$ with duality and weak equivalence a C_2-space whose underlying homotopy type is that of the usual Waldhausen K-theory of $\mathscr{C}$. We first describe Hesselholt and Madsen's definition of KR in terms of the $S^{2,1}$-construction, which is similar in spirit to Waldhausen's S-construction. Then we prove two equivariant delooping result for both Waldhausen's original construction and for the $S^{2,1}$-construction. Using this we show that the two constructions yield the same result up to homotopy, which shows that the fixed points of KR-theory agree with Grothendieck-Witt-theory, or hermitian K-theory, as studied by Karoubi, Schlichting, Hornbostel and others. As applications of this we translate some classical results of Quillen and Schlichting into equivariant language.

Contents

1 Introduction

In his paper "K-theory and Reality" [Ati66] Atiyah defined an equivariant cohomology theory called "Real K-theory" or KR-theory which combines into one C_2-equivariant object the (non-equivariant) cohomology theories KO, KU and Anderson's so called self-conjugate K-theory KSC. Recently Hesselholt and Madsen have defined a similar C_2-equivariant version of algebraic K-theory called Real algebraic K-theory or algebraic KR-theory. Its input is an exact category $\mathscr{C}$ with weak equivalences and duality and its output is a C_2-space $KR(\mathscr{C})$. The underlying pointed space of $KR(\mathscr{C})$ has the homotopy type of the algebraic K-theory space of $\mathscr{C}$ in the sense of Quillen [Qui73] and Waldhausen [Wal85].

In the present paper we investigate the relation of algebraic KR-theory to Waldhausen K-theory and to Grothendieck-Witt theory GW, also called hermitian K-theory, which goes back to Karoubi's work in the 70's and in recent years to the work of Schlichting, Hornbostel and others. After giving the necessary setup for exact categories and categories with duality we recall Waldhausen's S-construction for an exact category with weak equivalences and strict duality. This is essentially the same construction as the one used by Schlichting in [Sch10b] to define $GW(\mathscr{C})$ except that we do not need to apply any subdivisions and the geometric realization $|wS^{1,1}\mathscr{C}|$ is a pointed C_2-space. We will use the notation $S^{p,q}$ for the p-sphere $S^{\mathbb{R}^p}$ where C_2 acts by switching signs on the last q-coordinates. The reason for the superscript "$1,1$" in $|wS^{1,1}\mathscr{C}|$ is the equivariant delooping map

$$|w\mathscr{C}| \to \Omega^{1,1}|wS^{1,1}\mathscr{C}|,$$

where in general, for a pointed C_2-space X we define $\Omega^{p,q}X = Map_*(S^{p,q}, X)$ with the conjugation action of C_2. In order to define algebraic KR-theory we introduce a second S-construction called $S^{2,1}\mathscr{C}$, which was first defined by Hesselholt and Madsen in [HM]. It is similar in spirit to the $S^{1,1}$-construction discussed above and comes with a delooping map

$$|w\mathscr{C}| \to \Omega^{2,1}|wS^{2,1}\mathscr{C}|$$

with respect to the regular representation sphere $S^{2,1}$ of C_2.

In the next section we prove that the two S-construction give equivariant deloopings with respect to each other. The proof is an equivariant version of Waldhausen's delooping argument using a path constructions and additivity in the equivariant setting. The approach is inspired by the one used in the PhD-thesis of Dotto [Dot12] to construct equivariant deloopings for "Real Topological Hochschild Homology", or THR. Putting several of these deloopings together we conclude in 4.8 that the KR-spectrum, defined by iterating the $S^{2,1}$-construction on $\mathscr{C}$, is a positively fibrant ρ_{C_2}-spectrum. This mirrors the non-equivariant behavior of Waldhausen's K-theory spectrum. Another important outcome of the delooping results is Theorem 4.9 which gives a natural weak C_2-equivalence

$$|wS^{1,1}\mathscr{C}| \simeq \Omega^{1,0}|wS^{2,1}\mathscr{C}|.$$

From this we see that KR-theory, which is defined in terms of the right hand space agrees up to homotopy with the K-theory and Grothendieck-Witt theory which are defined in terms of the left hand space. Hesselholt and Madsen give a group completion result, see Theorem 3.5 or [HM], for KR-theory of split exact categories and the above result extends this to Grothendieck-Witt theory, i.e., it shows that $GW(\mathscr{C})$ is a group completion of the category of symmetric forms in $\mathscr{C}$. Previously this was only known under the hypothesis that 2 acts invertibly on the Hom-groups of $\mathscr{C}$, see [Sch04, Corollary 4.6].

The last section is devoted to reformulating two classical results in algebraic K-theory for algebraic KR-theory. The first one is the Cofinality Theorem, see [Gra76, p.8] for K-theory and [Sch10a, 5.2] for Grothendieck-Witt theory. The second result is the Dévissage Theorem, again see [Qui73, 5.4] for the K-theory version and [Sch10a, 6.1] for the Grothendieck-Witt result. The results in this section are not essentially new but we feel there is something to be gained by formulating them in the equivariant context. The reader familiar with the classical theorems in algebraic K-theory may wonder how the Resolution Theorem and the Localization Theorem generalize to KR-theory. For Resolution the reader should consult Schlichting's formulation [Sch10b, Lemma 9] which carries over easily to KR-theory by Theorem 4.10. Localization is proved for Grothendieck-Witt theory by Hornbostel and Schlichting in [HS04] under the assumption that 2 is a unit in all the involved rings. The author plans to address localization in KR-theory in a future paper.

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2 Definitions and setup

2.1 Real simplicial sets

Recall that the *opposite* of a map $\alpha\colon [m] \to [n]$ in the finite ordinal category Δ is the map $\bar{\alpha}$ which is given by $\bar{\alpha}(i) = n - \alpha(m-i)$. A real simplicial set is a simplicial set X with additional structure maps $w_n\colon X_n \to X_n$ satisfying the relations

$$w_n^2 = id \tag{1}$$
$$\alpha^* \circ w_n = w_m \circ \bar{\alpha}^*, \tag{2}$$

for all maps $m, n \geq 0$ and $\alpha\colon [m] \to [n]$. This can be encoded by defining the Real simplex category ΔR which is the category Δ extended by morphisms $\omega_n\colon [n] \to [n]$ satisfying relations dual to (1) and (2) above. Note that in ΔR the equation

$$\omega_n \circ \alpha \circ \omega_m = \bar{\alpha}$$

holds for all maps $\alpha\colon [m] \to [n]$ from Δ. A Real simplicial set can equivalently be defined as a functor $X\colon \Delta R^{op} \to Set$. The geometric realization of a Real simplicial set carries a natural C_2-action given by

$$[x, t_0, t_1, \ldots, t_n] \mapsto [w_n(x), t_n, \ldots, t_1, t_0], \text{ for } (x, t_0, t_1, \ldots, t_n) \in X_n \times \Delta^n.$$

We write $|X|_R$ for this C_2-space and $|X|$ for the realization of the underlying simplicial set, considered as an ordinary space. A map $f\colon X \to Y$ of Real simplicial sets is called a weak equivalence if it induces an equivariant homotopy equivalence of C_2-spaces after realization. The geometric realization of a Real simplicial set is a C_2-CW complex. It follows by the equivariant Whitehead theorem that $f\colon X \to Y$ is a weak equivalence if and only if the maps $|f|\colon |X| \to |Y|$ on underlying spaces and $|f|_R^{C_2}\colon |X|_R^{C_2} \to |Y|_R^{C_2}$ on fixed points are weak eqiuvalences.

Recall Segal's edgewise subdivision Sd which takes a simplicial set X to the simplicial set SdX given by $(SdX)_n = X_{2n+1}$ with maps $\alpha\colon [m] \to [n]$ inducing $(\alpha \amalg \bar{\alpha})^*\colon X_{2n+1} \to X_{2m+1}$(see [Seg73, Appendix 1]). Where $\alpha \amalg \bar{\alpha}$ is the map $[2m+1] \to [2n+1]$ that acts by α on the first $m+1$ elements and $\bar{\alpha}$ on the last $m+1$ elements. If X is a real simplicial set then it is easy to see that SdX has a simplicial C_2-action induced from the maps w_n on X. For a simplicial set (or space) X Segal gives a natural homeomorphism $|SdX| \xrightarrow{\cong} |X|$ (see [Seg73, A.1]). When X is Real simplicial this map is easily seen to be equivariant, so we get:

Lemma 2.1: *Let X be a real simplicial set. The natural map*

$$|SdX| \to |X|_R$$

is a C_2-equivariant homeomorphism.

A Real bisimplicial set is a bisimplicial set X with structure maps $w_{p,q}\colon X_{p,q} \to X_{p,q}$ satisfying $w_{p,q}^2 = id$ and $w_{p,q} \circ (\alpha, \beta)^* = (\bar{\alpha}, \bar{\beta}) \circ w_{m,n}$ for all maps $(\alpha, \beta)\colon ([m], [n]) \to ([p], [q])$ in $\Delta \times \Delta$. Note that a Real bisimplicial set is *not* the same as a functor from $\Delta R^{op} \times \Delta R^{op}$ to sets. However, the diagonal of a Real bisimplicial set is a Real simplicial set. A real simplicial space will be simply a functor from ΔR^{op} to the category Top of compactly generated weak Hausdorff spaces and continuous maps.

2.2 Exact categories with weak equivalences

Recall from [Qui73, 2], [Sch10a, p.4] that an exact category is a pair $(\mathscr{C}, \mathscr{E})$ where $\mathscr{C}$ is an additive category and $\mathscr{E}$ is a class of sequences of maps

$$A \xrightarrow{i} B \xrightarrow{p} C \tag{3}$$

in $\mathscr{C}$ which are called *exact*. Morphisms, such as i above, that appear as the first map in some exact sequence will be called admissible monomorphism and will be depicted by $\rightarrowtail$. Dually, a map which appears as the last map in an exact sequence will be called an admissible epimorphism and will be depicted by $\twoheadrightarrow$. The class $\mathscr{E}$ is required to satisfy the following properties (cf. [Sch10a, p.4]):

1. In an exact sequence (3), the map i is a kernel of p and p is a cokernel of i.
2. If a sequence is isomorphic to an exact sequence, then it is an exact sequence.
3. Admissible monomorphisms are closed under composition and admissible epimorphisms are closed under composition.
4. Any diagram $Z \leftarrow X \rightarrowtail Y$ can be completed to a pushout square

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d] & Y \arrow[d] \\ Z \arrow[r, tail, "j"'] & W \end{tikzcd}$$

   and the map j is an admissible monomorphism.
5. Any diagram $Z \twoheadrightarrow X \leftarrow Y$ can be completed to a pullback square

$$\begin{tikzcd} W \arrow[r, two heads, "q"] \arrow[d] & Y \arrow[d] \\ Z \arrow[r, two heads] & X \end{tikzcd}$$

   and the map q is an admissible epimorphism.
6. For any pair X, Y of objects of $\mathscr{C}$ the sequence

$$X \xrightarrow{i_1} X \oplus Y \xrightarrow{p_2} Y \tag{4}$$

   is exact, where i_1 and p_2 are the canonical inclusion and projection maps, respectively.

An exact sequence is called split exact if it is isomorphic to a sequence of the form (4). If all sequences in an exact category $\mathscr{C}$ are split exact then $\mathscr{C}$ is called a split exact category. There is a unique split exact structure on any additive category $\mathscr{C}$ and we will therefore suppress the class $\mathscr{E}$ from the notation when discussing split exact categories.

A subcategory of weak equivalences in an exact category is a subcategory $w\mathscr{C} \subseteq \mathscr{C}$ which contains all isomorphisms in $\mathscr{C}$, has the 2 out of 3 property for composition of maps and which is closed under pushouts along admissible monomorphisms and pullbacks along admissible epimorphisms.

To avoid set theoretic problems we fix two Grothendieck universes $\mathscr{U}$ and $\mathscr{V}$ such that $\mathscr{U} \in \mathscr{V}$. All exact categories will be small relative to $\mathscr{V}$. To achieve this we will always assume that groups, rings, modules and so on which inhabit our exact categories are in the smaller universe $\mathscr{U}$. The nerves of our exact categories and their geometric realizations will then be simplicial sets and topological spaces in $\mathscr{V}$.

2.3 Categories with duality

Definition 2.2. A category with duality is a triple $(\mathscr{C}, D, \eta)$ where $D\colon \mathscr{C}^{op} \to \mathscr{C}$ is a functor and η is a natural transformation $Id_{\mathscr{C}} \to D \circ D^{op}$ such that for each object c of $\mathscr{C}$ the equation $\eta_{Dc} = D(\eta_c) \circ \eta_{Dc}$ holds.

The duality is called *strict* if $D \circ D^{op} = Id_{\mathscr{C}}$ and $\eta = id$, in this case we will omit η from the notation. The category $Sym(\mathscr{C}, D, \eta)$ of symmetric forms in $\mathscr{C}$ has as objects pairs (c, φ) where c is an object of $\mathscr{C}$ and $\varphi\colon c \to Dc$ is a map such that $\varphi = D\varphi \circ \eta_c$. A map from (c, φ) to (d, ψ) is morphism $f\colon c \to d$ such that $\varphi = Df \circ \psi \circ f$, and composition is given by ordinary composition in $\mathscr{C}$. A duality preserving functor between categories with duality is a pair $(F, \xi)\colon (\mathscr{C}, D, \eta) \to (\mathscr{C}', D', \eta')$ where F is a functor $F\colon \mathscr{C} \to \mathscr{C}'$ and $\xi\colon F \circ D \to D' \circ F$ is a natural transformation such that for each c in $\mathscr{C}$ the equation $\xi_{Dc} \circ F(\eta_c) = D'(\xi_c) \circ \eta'_{F(c)}$ holds. Such a functor induces a functor $Sym(F, \xi)\colon Sym(\mathscr{C}, D, \eta) \to Sym(\mathscr{C}', D', \eta')$ which is given on objects by $Sym(F, \xi)(c, \varphi) = (Fc, \xi_c \circ F\varphi)$.

The subdivision $Sd\mathscr{C}$ of $\mathscr{C}$ is the category whose objects are maps $f\colon c \to c'$ in $\mathscr{C}$ and where a morphism from $f\colon c \to c'$ to $g\colon d \to d'$ is a pair $(h\colon c \to d, i\colon d' \to c')$ such that $f = i \circ g \circ h$. The nerve $NSd\mathscr{C}$ is Segal's edgewise subdivision $SdN\mathscr{C}$ of the nerve of $\mathscr{C}$. Since there is a natural isomorphism of categories $Sd(\mathscr{C}^{op}) \cong Sd\mathscr{C}$ a duality D on $\mathscr{C}$ gives a covariant functor $SdD\colon Sd\mathscr{C} \to Sd\mathscr{C}$. The category $Sym(\mathscr{C}, D, \eta)$ is a kind of subcategory of fixed points in $Sd\mathscr{C}$. In the case of a strict duality this is literally true; we have $SdD^2 = Id_{Sd\mathscr{C}}$ and $Sym(\mathscr{C}, D, id)$ is the fixed category under the action defined by SdD. Moreover, $SdN\mathscr{C}$ is a C_2-simplicial set and $Sym(\mathscr{C}, D, id) = SdN\mathscr{C}^{C_2}$.

A real simplicial category is a simplicial category $\mathscr{C}$ with duality maps in each simplicial degree $D_n\colon \mathscr{C}_n^{op} \to \mathscr{C}_n$ satisfying the relations $D_n^2 = Id$ and $D_m \circ (\alpha^*)^{op} = (\bar{\alpha})^* \circ D_n$ for maps $\alpha\colon [m] \to [n]$. The nerve of a Real simplicial category is a Real bisimplicial set with $w_{p,q} = N_p D_q$.

2.4 Exact categories with weak equivalences and duality

Definition 2.3. An exact category with weak equivalences and duality is a tuple $(\mathscr{C}, \mathscr{E}, w\mathscr{C}, D, \eta)$ such that

1. the pair $(\mathscr{C}, \mathscr{E})$ is an exact category,
2. the triple $(\mathscr{C}, D, \eta)$ a category with duality,
3. the transformation η takes values in $w\mathscr{C}$,
4. the functor D preserves exact sequences and weak equivalences.

We write $Sym(\mathscr{C}, w)$ for the category $Sym(w\mathscr{C}, D|_{w\mathscr{C}}, \eta|_{w\mathscr{C}})$. An object in $Sym(\mathscr{C}, w)$ will be called a non-degenerate symmetric form in $\mathscr{C}$. Such a form is a pair (X, φ), where $\varphi\colon X \xrightarrow{\sim} DX$ is a weak equivalence. Note that in the equation $\varphi = D\varphi \circ \eta_X$ all maps are weak equivalences.

Example 2.4. Let A be a ring with anti-involution, that is, an additive self-map $a \mapsto \bar{a}$ such that $\overline{a \cdot b} = \bar{b} \cdot \bar{a}$ and $\bar{\bar{a}} = a$ for all $a, b \in A$. We write $P(A)$ for the category of finitely generated projective right A-modules and for each P in $P(A)$ we give $P^* = Hom_A(P, A)$ the right(!) module structure $(\lambda \cdot a)(p) = \bar{a}\lambda(p)$, for $\lambda \in P^*$. We give $P(A)$ the split exact structure. Note, however, that since every epimorphism onto a projective module splits, a sequence $P' \to P \to P''$ in $P(A)$ is split exact if and only if it is exact as a sequence of A-modules.

Let ϵ be either 1 or -1 and define ${}_\epsilon\eta$ to be the natural transformation with components ${}_\epsilon\eta_P\colon P \to P^{**}$ given by

$$p \mapsto (\lambda \mapsto \epsilon\overline{\lambda(p)}).$$

We take the weak equivalences in $P(A)$ to be the isomorphisms. The category of these will be denoted by $iP(A)$. With these choices the tuple $(P(A), iP(A), (-)^*, {}_\epsilon\eta)$ is a (split) exact category with weak equivalences and duality.

Next we consider the category $Sym(P(A), i)$ of non-degenerate symmetric forms in $P(A)$. Under the $(Hom, \otimes)$-adjunction for A-modules an object $(P, \varphi\colon P \xrightarrow{\cong} P^*)$ corresponds to a biadditive map $\tilde{\varphi}\colon P \times P \to A$ satisfying

$$\tilde{\varphi}(pa, qb) = \bar{a}\tilde{\varphi}(p, q)b \text{ and } \tilde{\varphi}(p, q) = \epsilon\overline{\tilde{\varphi}(q, p)}$$

for $p, q \in P$ and $a, b \in A$. Such maps are called non-degenerate ϵ-hermitian forms. A map in $Sym(P(A), i)$ is precisely an A-linear map on the underlying modules which preserves the ϵ-hermitian form.

Example 2.5. Let fAb be the category of finite abelian groups. We set $G^* = Hom_{Ab}(G, \mathbb{Q}/\mathbb{Z})$ the Pontryagin dual of G. The natural map $\eta_G\colon G \to G^{**}$ is given by $g \mapsto (\lambda \mapsto \lambda(g))$. We take weak equivalences to be isomorphisms and sequences to be exact if they are exact in the usual sense. These choices give the structure of an exact category with weak equivalences and duality on fAb.

The category $Sym(fAb, i)$ is the category of finite abelian groups G with a map $\varphi\colon G \xrightarrow{\cong} G^*$ such that the adjoint map $\tilde{\varphi}\colon G \otimes G \to \mathbb{Q}/\mathbb{Z}$ is a non-degenerate bilinear form. Such forms are sometimes called 'linking forms', since they appear as linking forms on homology groups of manifolds.

Here is an example where the weak equivalences are not isomorphisms.

Example 2.6. (cf. [Sch10b]) Let A be as in Example 2.4 and consider the category $Ch^b(P(A))$ of bounded chain complexes in $P(A)$. This becomes an exact category when we take the exact sequences to be the ones that are level-wise split exact. We define the weak equivalences in $Ch^b(P(A))$ to be the quasi-isomorphisms, i.e., the maps inducing isomorphisms on homology. The duality structure can be extended from $P(A)$ by taking a chain complex $(C., d)$ to the chain complex given by $D(C.)_n = (C_{-n})^*$ with n-th differential $D(d_{-n})$. This amounts to "flipping" the chain complex upside down and dualizing in each degree. The transformation η can be taken to be ${}_\epsilon\eta$ in each degree.

The category $Sym(Ch^b(P(A), q)$ is a chain level version of $Sym(P(A), i)$. Non-degeneracy here means that an object is quasi-isomorphic to its dual. This does however imply a stricter notion of duality on homology. For details and generalizations the reader should consult subsection 6.1 of [Sch10b].

The structure preserving maps for exact categories with weak equivalences and duality are pairs $(F, \xi)\colon (\mathscr{C}, w\mathscr{C}, \mathscr{E}, D, \eta) \to (\mathscr{C}', w'\mathscr{C}', \mathscr{E}', D', \eta')$ where $F\colon \mathscr{C} \to \mathscr{C}'$ is an exact functor that preserves weak equivalences and the pair (F, ξ) is a duality preserving functor from $(\mathscr{C}, D, \eta)$ to $(\mathscr{C}', D', \eta')$ such that ξ takes values in weak equivalences.

We will assume that we have chosen for each pair X, Y of objects of $\mathscr{C}$ a biproduct diagram

$$X \underset{i_1}{\overset{p_1}{\leftrightarrows}} X \oplus Y \underset{i_2}{\overset{p_2}{\rightleftarrows}} Y .$$

These choices determine a coproduct functor $\oplus\colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ and a natural isomorphism $\nu\colon D(-) \oplus D(-) \to D(- \oplus -)$. This structure allows us to define the *orthogonal sum*

$$\perp\colon Sym(\mathscr{C}, w) \times Sym(\mathscr{C}, w) \to Sym(\mathscr{C}, w)$$

given by $(c, \varphi) \perp (d, \psi) = (c \oplus d, \nu_{c,d} \circ (\varphi \oplus \psi))$ which, along with the appropriate choices of unit etc., makes $Sym(\mathscr{C}, w)$ into a symmetric monoidal category.

2.5 Strictification

In the next sections it will be necessary to have strict dualities on the categories we work with, but as can be seen from the examples above the dualities arising in nature are not strict in general. To remedy this failure we will use the following strictification procedure: (cf. [Sch10b, Lemma 4]) Given an exact category with duality and weak equivalences $(\mathscr{C}, w\mathscr{C}, \mathscr{E}, D, \eta)$ its strictification will be an exact category with strict duality and weak equivalences $(\hat{\mathscr{C}}, w\hat{\mathscr{C}}, \hat{\mathscr{E}}, \hat{D})$ which we will now define. The objects of $\hat{\mathscr{C}}$

are triples (c, c', f) where $f\colon c' \xrightarrow{\sim} Dc$ is a weak equivalence. A map from (c, c', f) to (d, d', g) is a pair $(r\colon c \to d, s\colon d' \to c')$ such that $f \circ s = Dr \circ g$, and composition is defined component-wise. The duality $\hat{D}$ is defined on objects by $\hat{D}(c, c', f) = (c', c, Df \circ \eta_c)$ and on morphisms by $\hat{D}(r, s) = (s, r)$. It is not hard to see that $\hat{D}^2 = Id_{\hat{\mathscr{C}}}$. The weak equivalences in $w\hat{\mathscr{C}}$ are the pairs (r, s) where both r and s are weak equivalences in $\mathscr{C}$ and a sequence is exact if it is exact component-wise. Note that this construction is functorial in $(\mathscr{C}, w\mathscr{C}, \mathscr{E}, D, \eta)$ and that there is a natural duality preserving functor

$$(J, \iota)\colon (\mathscr{C}, w\mathscr{C}, \mathscr{E}, D, \eta) \to (\hat{\mathscr{C}}, w\hat{\mathscr{C}}, \hat{\mathscr{E}}, \hat{D})$$

where J is given by $J(c) = (Dc, c, id_{Dc})$ on objects and $J(f) = (Df, f)$ on morphisms. The transformation ι has components $\iota_c = (id_{Dc}, \eta_c)$. There is also a functor

$$(K, \kappa)\colon (\hat{\mathscr{C}}, w\hat{\mathscr{C}}, \hat{\mathscr{E}}, \hat{D}) \to (\mathscr{C}, w\mathscr{C}, \mathscr{E}, D, \eta)$$

given by $K(c, c', f) = c$ on morphisms $K(r, s) = r$ on morphisms and $\kappa_{(c,c',f)} = f$. The functors (J, ι) and (K, κ) are mutual inverses up to natural weak equivalence. This implies in particular that the induced functors $w\mathscr{C} \to w\hat{\mathscr{C}}$ and $Sym(\mathscr{C}, w) \to Sym(\hat{\mathscr{C}}, w)$ become homotopy equivalences after geometric realization.

A pointed category is a category $\mathscr{C}$ with a chosen object c. A functor between pointed categories will be called pointed if it preserves the chosen object. In the presence of a duality we will also need our categories to be pointed in the sense that the chosen object c is fixed under the duality, i.e., $Dc = c$. For an exact category with weak equivalences the chosen object will be a zero-object. In this case we will consider the nerves $N\mathscr{C}$ and $Nw\mathscr{C}$ as pointed at the vertex corresponding to the chosen object. A choice 0 of zero-object in $\mathscr{C}$ determines a 0-object $\hat{0} = (D0, 0, id_{D0})$ in the strictification $\hat{\mathscr{C}}$ which satisfies $\hat{D}\hat{0} = \hat{0}$ and which makes the functor J is pointed on underlying categories (without duality).

3 K-theory and S-constructions

3.1 Lower K-theory and the hyperbolic sequence

Recall that the Grothendieck group of an exact category with weak equivalences, denoted by $K_0(\mathscr{C}) = K_0(\mathscr{C}, w, \mathscr{E})$ is the abelian group generated isomorphism classes $[X]$ of objects of $\mathscr{C}$ modulo the relations

1. if $f\colon X \xrightarrow{\sim} Y$ is a weak equivalence then $[X] = [Y]$,
2. if $X \rightarrowtail Y \twoheadrightarrow Z$ is an exact sequence then $[Y] = [X] + [Z]$.

Similarly, the Grothendieck-Witt group $GW_0(\mathscr{C}) = GW_0(\mathscr{C}, \mathscr{E}, w, D, \eta)$ of an exact category with weak equivalences and duality is the abelian group generated isomorphism classes $[X, \varphi]$ of non-singular symmetric forms modulo the following relations (cf. [Sch10b, §2.3, Def. 1])

1. $[X, \varphi] + [Y, \psi] = [(X, \varphi) \perp (Y, \psi)]$,
2. if $f\colon X \to Y$ is a weak equivalence then $[Y, \psi] = [X, f^*\psi f]$,
3. if the diagram

$$\begin{tikzcd}
X_0 \arrow[r, tail, "i"] \arrow[d, "\varphi_0", "\sim"'] & X_1 \arrow[r, two heads, "p"] \arrow[d, "\varphi_1", "\sim"'] & X_2 \arrow[d, "\varphi_2", "\sim"'] \\
DX_2 \arrow[r, tail, "Dp"] & DX_1 \arrow[r, two heads, "Di"] & DX_0
\end{tikzcd}$$

has exact rows and horizontal weak equivalences and is self-dual in the sense that (X_1, φ_1) is a non-singular symmetric form and the equations $\varphi_0 = D\varphi_2 \circ \eta_{c_0}$ and $\varphi_2 = D\varphi_0 \circ \eta_{c_2}$ hold, then

$$[X_1, \varphi_1] = \left[X_0 \oplus X_2, \nu_{X_0, X_2} \circ \begin{pmatrix} 0 & \varphi_2 \\ \varphi_1 & 0 \end{pmatrix} \right].$$

For any object X of $\mathscr{C}$ one has the *hyperbolic form* on X given by

$$H(X) = \left(X \oplus DX, \nu_{X,DX} \circ \begin{pmatrix} 0 & id_{DX} \\ \eta_X & 0 \end{pmatrix} \right).$$

The hyperbolic map $H\colon K_0(\mathscr{C}) \to GW_0(\mathscr{C})$ is given on generators by $H([X]) = [H(X)]$. To see that this defines a map $K_0(\mathscr{C}) \to GW_0(\mathscr{C})$ we must check that H is compatible with the relations 1 and 2 for K_0. The first case holds, since a weak equivalence $f\colon X \xrightarrow{\sim} Y$ gives rise to the diagram

$$\begin{tikzcd}
X \oplus DX \arrow[d, "{\left(\begin{smallmatrix} 0 & id_{DX} \\ \eta_X & 0 \end{smallmatrix}\right)}"'] & X \oplus DY \arrow[l, "id_X \oplus Df"'] \arrow[r, "f \oplus id_{DY}"] \arrow[d, "{\left(\begin{smallmatrix} 0 & Df \\ \eta_X \circ f & 0 \end{smallmatrix}\right)}"'] & Y \oplus DY \arrow[d, "{\left(\begin{smallmatrix} 0 & id_{DY} \\ \eta_Y & 0 \end{smallmatrix}\right)}"'] \\
DX \oplus D^2X \arrow[r, "id \oplus D^2 f"'] & DX \oplus D^2Y & DY \oplus D^2Y \arrow[l, "Df \oplus id"]
\end{tikzcd}$$

which, upon composing the vertical maps with the appropriate ν's gives a zig-zag of weak equivalence connecting $H(X)$ and $H(Y)$. For relation 2 observe that from an exact sequence

$$X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$$

one can construct a diagram

$$\begin{tikzcd}
X \oplus DZ \arrow[r, "i \oplus Dp"] \arrow[d, "{\left(\begin{smallmatrix} 0 & id_{DZ} \\ \eta_X & 0 \end{smallmatrix}\right)}"'] & X \oplus DY \arrow[r, "p \oplus Di"] \arrow[d, "{\left(\begin{smallmatrix} 0 & id_{DY} \\ \eta_Y & 0 \end{smallmatrix}\right)}"'] & Z \oplus DX \arrow[d, "{\left(\begin{smallmatrix} 0 & id_{DX} \\ \eta_Z & 0 \end{smallmatrix}\right)}"'] \\
DZ \oplus D^2X \arrow[r, "Dp \oplus D^2 i"'] & DY \oplus D^2Y \arrow[r, "Di \oplus D^2 i"'] & DX \oplus D^2Z
\end{tikzcd}$$

which after composing with ν's becomes as in relation 3 for $GW_0(\mathscr{C})$. It follows that

$$H([Y]) = H([X \oplus Z]) = H([X]) + H([Y])$$

where the last equality holds since $H(X \oplus Y) \cong H(X) \perp H(Y)$ in $Sym(\mathscr{C}, w)$.

The cokernel of the hyperbolic map is called the *Witt group* of $\mathscr{C}$ and is denoted by $W(\mathscr{C})$ (cf. [Sch10b, §2.3]). The *hyperbolic sequence* is the resulting exact sequence of abelian groups

$$K_0(\mathscr{C}) \xrightarrow{H} GW_0(\mathscr{C}) \longrightarrow W(\mathscr{C}) \longrightarrow 0. \tag{5}$$

It is natural in $\mathscr{C}$ and is of fundamental importance in the study of Witt, and Grothendieck-Witt groups.

Remark 3.1. In the case when every exact sequence in $\mathscr{C}$ splits both $K_0(\mathscr{C})$ and $GW_0(\mathscr{C})$ simply group completions of the monoids of weak equivalence classes of objects with respect to the operations $\oplus$ and $\perp$, respectively. This is shown e.g. in [Sch10a, Corollary 2.10] in the case when the weak equivalences are isomorphisms.

3.2 The $S^{1,1}$-construction

From now on $(\mathscr{C}, \mathscr{E}, D, w\mathscr{C}, 0)$ will denote a pointed exact category with weak equivalences and strict duality. The $S^{1,1}$-construction is our name for Waldhausen's S-construction (see [Wal85, 1.3]) applied to $(\mathscr{C}, D, w\mathscr{C}, 0)$ with dualities in each level making it a real simplicial category.

Let $\mathbb{R}^{p,q}$ denote the C_2-representation on $\mathbb{R}^p$ where C_2 acts trivially on the first $p-q$ coordinates and acts by multiplication with -1 on the last p coordinates. We write $S^{p,q}$ for the one-point compactification of $\mathbb{R}^{p,q}$. Thus $S^{1,1}$ is the one-dimensional sign circle and $S^{2,1}$ is the sphere of the regular representation $\rho_{C_2} = \mathbb{R}^{2,1}$. The superscript "1, 1" on $S^{1,1}\mathscr{C}$ is used because the $S^{1,1}$-construction gives an equivariant delooping with respect to the sign representation circle $S^{1,1}$.

We will consider the objects $[n]$ of the finite ordinal category Δ as categories in the usual way, depicted diagrammatically as

$$0 \to 1 \to \cdots \to n.$$

The morphisms in Δ are functors between these categories. We write $Cat([m],[n])$ for the category of functors from $[m]$ to $[n]$. The object set of $Cat([m],[n])$ is just the set Δ^n_m of m-simplices in the n-simplex and there is a map $\alpha \to \beta$ if $\alpha(i) \leq \beta(i)$ for all $0 \leq i \leq m$. There is a unique isomorphism $[n]^{op} \cong [n]$, which sends i to $n-i$. This induces a duality $\alpha \mapsto \bar{\alpha}$ on the category $Cat([m],[n])$ which is given by $\bar{\alpha}(i) = n - \alpha(m-i)$.

Define $S^{1,1}_n\mathscr{C}$ to be the full subcategory of the functor category $Cat(Cat([1],[n]),\mathscr{C})$ of objects X such that

- for all $0 \leq i \leq n$ the value $X_{i,i}$ is 0, the chosen zero-object of $\mathscr{C}$
- for all $i \leq j \leq k$ the sequence $X_{i,j} \rightarrowtail X_{i,k} \twoheadrightarrow X_{j,k}$ is exact.

For a map $\alpha\colon [p] \to [n]$ the induced functor $Cat(Cat([1],[n]),\mathscr{C}) \to Cat(Cat([1],[p]),\mathscr{C})$ restricts to a functor

$$\alpha^*\colon S^{1,1}_n\mathscr{C} \to S^{1,1}_p\mathscr{C}$$

and thus $S^{1,1}\mathscr{C} = ([n] \mapsto S^{1,1}_n\mathscr{C})$ is a simplicial category. There is a canonical duality D_n on $S^{1,1}_n\mathscr{C}$ given by $(D_nX)_{i,j} = D(X_{n-j,n-i})$. The action of D_3 on an object X of $S^{1,1}_3\mathscr{C}$ can be pictured as follows: The diagram

$$\begin{tikzcd} X_{0,1} \arrow[r, tail] & X_{0,2} \arrow[r, tail] \arrow[d, two heads] & X_{0,3} \arrow[d, two heads] \\ & X_{1,2} \arrow[r, tail] & X_{1,3} \arrow[d, two heads] \\ & & X_{2,3} \end{tikzcd} \quad \text{is mapped to} \quad \begin{tikzcd} DX_{2,3} \arrow[r, tail] & DX_{1,3} \arrow[r, tail] \arrow[d, two heads] & DX_{0,3} \arrow[d, two heads] \\ & DX_{1,2} \arrow[r, tail] & DX_{0,2} \arrow[d, two heads] \\ & & DX_{0,1} \end{tikzcd}$$

where we have omitted the diagonal elements $X_{i,i}$ which are 0 by definition. The interaction of the dualities D_n with the simplicial structure maps is given by

$$D_p \circ (\alpha^*)^{op} = (\bar{\alpha})^* \circ D_n$$

for maps $\alpha\colon [p] \to [n]$, thus $S^{1,1}\mathscr{C}$ is a real simplicial category.

The categories $S^{1,1}_n\mathscr{C}$ are themselves pointed exact categories with weak equivalences and strict duality when we make the following choices: The duality is the functor D_n defined above. Exact sequences and weak equivalences are defined pointwise and we take the zero-object to be the constant diagram on the chosen zero-object of $\mathscr{C}$.

To increase readability we will write $|w\mathscr{C}|$ for the geometric realization $|Nw\mathscr{C}|$ and for simplicial categories such as $wS^{1,1}\mathscr{C}$ we will write $|wS^{1,1}\mathscr{C}|$ for the geometric realization of the diagonal $|dNwS^{1,1}\mathscr{C}|$. There is an isomorphism of categories with duality $w\mathscr{C} \xrightarrow{\cong} wS^{1,1}_1\mathscr{C}$ sending an object X to the diagram with value X on $0 \leq 1$. This induces a C_2-equivariant map

$$\Delta^1_R \times |w\mathscr{C}|_R \to |wS^{1,1}\mathscr{C}|_R$$

where Δ^1_R here means the topological 1-simplex with the flip action of C_2. Since $S^{1,1}_0\mathscr{C} = *$ this map factors over the smash product $S^{1,1} \wedge |w\mathscr{C}|_R$ and by adjunction there is an induced C_2-equivariant map

$$|w\mathscr{C}|_R \to \Omega^{1,1}|wS^{1,1}\mathscr{C}|_R,$$

which, as we shall see later, behaves like an equivariant group completion map.

In the paper [Sch10b, 2.7] Marco Schlichting defines the Grothendieck-Witt space of an exact category with duality and a chosen zero-object as the homotopy fiber of the composite map $|Sym(SdS\mathscr{C}, w)| \xrightarrow{u} |Sd(wSdS\mathscr{C})| \xrightarrow{l} |wS\mathscr{C}|$. Here Sd denotes Segal's edgewise subdivision, which is applied both in the "S-direction" and in the "nerve direction" to get a simplicial category $Sd(wSdS\mathscr{C})$. Its nerve $NSd(wSdS\mathscr{C})$ is the bisimplicial set obtained from $N(wS\mathscr{C})$ by subdiving both simplicial directions. The map u is the geometric realization of the map which level-wise is the inclusion $Sym(S_{2n+1}\mathscr{C}, w) \hookrightarrow SdwS_{2n+1}\mathscr{C}$ of symmetric forms into the subdivision. The map l is the "last vertex map" induced by the inclusion of $[n]$ into the first summand of $[n] \amalg [n]^{op}$. It is a weak equivalence and so the natural map from the homotopy fiber hF_u of u to $GW(\mathscr{C}) = GW(\mathscr{C}, \mathscr{E}, w\mathscr{C}, D, \eta, 0)$ is a weak equivalence as well. The combined effect of u and l should be thought of as taking a symmetric form (c, φ) to the underlying object c.

Lemma 3.2: *Let $(\mathscr{C}, \mathscr{E}, w\mathscr{C}, D, \eta)$ be an exact category with weak equivalences and duality and a zero object 0 which is not necessarily fixed under D. There is a natural zig-zag of weak equivalences connecting $GW(\mathscr{C}, \mathscr{E}, w\mathscr{C}, D, \eta, 0)$ and $(\Omega^{1,1}|wS^{1,1}\hat{\mathscr{C}}|)^{C_2}$.*

Proof. Let $J\colon \mathscr{C} \to \hat{\mathscr{C}}$ be the strictification map as defined in section 2.5 and consider the following diagram of pointed spaces

$$\begin{tikzcd}
GW(\mathscr{C}) \arrow[r] & {|Sym(SdS\mathscr{C}, w)|} \arrow[r, "l\circ u"] & {|wS\mathscr{C}|} \\
hF_u \arrow[u, "\simeq"] \arrow[r] \arrow[d, "\simeq"'] & {|Sym(SdS\mathscr{C}, w)|} \arrow[u, "="] \arrow[r, "u"] \arrow[d, "\simeq"] & {|SdwSdS\mathscr{C}|} \arrow[u, "l"'] \arrow[d, "\simeq"] \\
hF_{\hat{u}} \arrow[r] \arrow[d, "\cong"'] & {|Sym(SdS\hat{\mathscr{C}}, w)|} \arrow[r, "\hat{u}"] \arrow[d, "\cong"] & {|SdwSdS\hat{\mathscr{C}}|} \arrow[d, "\cong"'] \\
(\Omega^{1,1}|wS^{1,1}\hat{\mathscr{C}}|)^{C_2} \arrow[r] & {|wS^{1,1}\hat{\mathscr{C}}|^{C_2}} \arrow[r] & {|wS^{1,1}\hat{\mathscr{C}}|.}
\end{tikzcd}$$

The first three horizontal sequences are fiber sequences by definition and the lower sequences is the fiber sequence obtained by mapping the cofiber sequence of C_2-spheres

$$C_{2+} \wedge S^0 \to S^{0,0} \to S^{1,1}$$

into the C_2-space $|wS^{1,1}\hat{\mathscr{C}}|$ and taking fixed points. The middle vertical maps are induced by J and the lower right hand vertical map is the natural homeomorphism $|SdX| \xrightarrow{\cong} |X|$ for X equal to the diagonal simplicial set $Sd(dNwS\hat{\mathscr{C}}) = dNSdwSdS\hat{\mathscr{C}}$. This map identifies $NSym(SdS\hat{\mathscr{C}}, w)$ with the fixed points $(dNSdwSdS\hat{\mathscr{C}})^{C_2}$, giving the lower middle vertical homeomorphism. The lower left vertical map is the induced homeomorphism of homotopy fibers.

Now the zig zag of weak equivalences of homotopy fibers is the desired one. □

Remark 3.3. In [Sch10b] Schlichting shows how the hyperbolic sequence

$$K_0(\mathscr{C}) \to GW_0(\mathscr{C}) \twoheadrightarrow W_0(\mathscr{C}) \to 0$$

is isomorphic to segment

$$\pi_1|wS\mathscr{C}| \to \pi_0 GW(\mathscr{C}) \twoheadrightarrow \pi_0|Sym(SdS\mathscr{C}, w)| \to \pi_0|wS\mathscr{C}|$$

of the exact homotopy sequence of the top fiber sequence in the proof above. The proof given here shows how the hyperbolic sequence can be reinterpreted as a sequence of equivariant homotopy groups of the C_2-space $|wS^{1,1}\hat{\mathscr{C}}|$.

3.3 The $S^{2,1}$-construction

Now we introduce the $S^{2,1}$-construction. A four-term sequence

$$A \rightarrowtail B \to C \twoheadrightarrow D$$

will be called exact if the map $B \to C$ factors as $B \twoheadrightarrow E \rightarrowtail C$ such that the sequences $A \rightarrowtail B \twoheadrightarrow E$ and $E \rightarrowtail C \twoheadrightarrow D$ are exact. Define the category $S_n^{2,1}\mathscr{C}$ to be the full subcategory of $Cat(Cat([2],[n]),\mathscr{C})$ of diagrams X such that

- if $i = j$ or $j = k$ then $X_{i,j,k}$ equals 0, the chosen zero-object of $\mathscr{C}$,
- for all $i \leq j \leq k \leq l$ the four term sequence $X_{i,j,k} \rightarrowtail X_{i,j,l} \to X_{i,k,l} \twoheadrightarrow X_{j,k,l}$ is exact.

The categories $S_n^{2,1}\mathscr{C}$ for small n can be described as follows: Both $S_0^{2,1}\mathscr{C}$ and $S_1^{2,1}\mathscr{C}$ have one object and one morphism. The category $S_2^{2,1}\mathscr{C}$ is isomorphic to $\mathscr{C}$ and $S_3^{2,1}\mathscr{C}$ can be described as the category of four term exact sequences as defined above. An object of $S_4^{2,1}\mathscr{C}$ can be pictured as

$$\begin{tikzcd}
 & & & & & & X_{2,3,4} \\
 & & & & X_{1,2,4} \arrow[rr] & & X_{1,3,4} \arrow[u, two heads] \\
 & & & X_{1,2,3} \arrow[ur, tail] & & & \\
 & & X_{0,1,4} \arrow[rr] & & X_{0,2,4} \arrow[uu, two heads] \arrow[rr] & & X_{0,3,4} \arrow[uu, two heads] \\
 & X_{0,1,3} \arrow[ur, tail] \arrow[rr] & & X_{0,2,3} \arrow[uu, two heads] \arrow[ur, tail] & & & \\
X_{0,1,2} \arrow[ur, tail] & & & & & &
\end{tikzcd}$$

where we have omitted the terms in the diagram which are 0 by definition. See also Dotto's thesis [Dot12] for more pictures of the structure of $S_n^{2,1}\mathscr{C}$.

The categories $S_n^{2,1}\mathscr{C}$ become pointed exact categories with strict duality in the same way as $S_n^{1,1}\mathscr{C}$ are and $[n] \mapsto wS_n^{2,1}\mathscr{C}$ is a real simplicial category in the same way. As in the $S^{1,1}$-case the isomorphism $w\mathscr{C} \xrightarrow{\cong} wS_2^{2,1}\mathscr{C}$ induces an equivariant map

$$|w\mathscr{C}| \to \Omega^{2,1}|wS^{2,1}\mathscr{C}|,$$

since both $S_0^{2,1}\mathscr{C}$ and $S_1^{2,1}\mathscr{C}$ have one object and one morphism.

Definition 3.4. The Real algebraic K-theory space of $\mathscr{C}$ is the pointed C_2-space given by

$$KR(\mathscr{C}) = \Omega^{2,1}|wS^{2,1}\mathscr{C}|.$$

A symmetric ρ_{C_2}-spectrum is a sequence of pointed spaces $\{X_n\}_{n\geq 0}$ with pointed actions of $C_2 \times \Sigma_n$ on X_n along with based structure maps $X_n \wedge S^{2,1} \to X_{n+1}$ such that the iterated structure maps

$$X_m \wedge \underbrace{S^{2,1} \wedge \cdots \wedge S^{2,1}}_{n} \to X_{m+n}$$

are $C_2 \times \Sigma_m \times \Sigma_n$-equivariant (cf. [Man04]).

If $(\mathscr{C}, w, D, 0)$ is a pointed exact category with weak equivalences and strict duality then its Real algebraic K-theory spectrum, denoted $\underline{KR}(\mathscr{C}) = \underline{KR}(\mathscr{C}, \mathscr{E}, w\mathscr{C}, D, 0)$ is the symmetric ρ_{C_2}-spectrum given by

$$\underline{KR}(\mathscr{C})_n = |w(S^{2,1})^{(n)}\mathscr{C}| = |w\underbrace{S^{2,1} \dots S^{2,1}}_{n}\mathscr{C}|$$

with structure maps induced by the inclusion of the 2-simplices and with Σ_n acting by permuting the $S^{2,1}$-factors.

The following result from [HM] generalizes Remark 3.1:

Theorem 3.5: *[HM] If $(\mathscr{C}, D, 0)$ is a pointed split exact category with strict duality then the map*

$$|w\mathscr{C}| \to \Omega^{2,1}|wS^{2,1}\mathscr{C}|$$

induces isomorphisms

$$H_*(|w\mathscr{C}|)[\pi_0|w\mathscr{C}|^{-1}] \xrightarrow{\cong} H_*(\Omega^{2,1}|wS^{2,1}\mathscr{C}|)$$

and

$$H_*(|Sym(\mathscr{C}, w)|)[\pi_0|Sym(\mathscr{C}, w)|^{-1}] \xrightarrow{\cong} H_*((\Omega^{2,1}|wS^{2,1}\mathscr{C}|)^{C_2}).$$

3.4 Additivity

The constructions $S^{1,1}\mathscr{C}$ and $S^{2,1}\mathscr{C}$ are clearly functorial with respect to functors preserving 0-objects, weak equivalences and exact sequences. We will now use this to state two additivity results which are proved in [HM].

Definition 3.6. Let $(\mathscr{C}, D, \eta)$ be a category with duality. Then we write $H(\mathscr{C}, D, \eta)$ for the category $\mathscr{C} \times \mathscr{C}$ equipped with the duality $D_H(c, c') = (Dc', Dc)$ and with (η, η) as the double-dual isomorphism.

We will often suppress D and η from the notation and simply write $H(\mathscr{C})$ for $H(\mathscr{C}, D, \eta)$. If $\mathscr{C}$ is an exact category with weak equivalences and duality we extend this structure to $H(\mathscr{C})$ by defining weak equivalences and exact sequences component-wise. If 0 is our chosen zero-object of $\mathscr{C}$ we take $(0, 0)$ to be the chosen zero-object of $H(\mathscr{C})$.

Projecting onto the diagonal defines duality preserving functors

$$\Delta_{2k} \colon S^{1,1}_{2k}\mathscr{C} \to H(\mathscr{C})^{\times k}$$

and

$$\Delta_{2k+1} \colon S^{1,1}_{2k+1}\mathscr{C} \to H(\mathscr{C})^{\times k} \times \mathscr{C}.$$

Since the $S^{1,1}$- and $S^{2,1}$-constructions both commute non-equivariantly with products one can easily prove the following:

Lemma 3.7: *The projection maps induce weak equivalences of pointed C_2-spaces*

$$|wS^{1,1}H(\mathscr{C})|_R \to |wS^{1,1}(\mathscr{C})| \times |wS^{1,1}(\mathscr{C})|$$

and

$$|wS^{2,1}H(\mathscr{C})|_R \to |wS^{2,1}(\mathscr{C})| \times |wS^{2,1}(\mathscr{C})|$$

where C_2 acts on the products by interchanging the two factors.

The following additivity theorem is proved in [HM]. The $|wS^{1,1}(-)|$-case also follows from Schlichting's additivity theorem [Sch10b, Theorem 4].

Theorem 3.8 (Additivity for $S^{1,1}_k$)**:** *The functors $|wS^{1,1}(-)|$ and $|wS^{2,1}(-)|$ take the functors Δ_n to weak equivalences.*

The functor $d_0 \circ d_{n+2} \colon S^{2,1}_{n+2}\mathscr{C} \to S^{2,1}_n\mathscr{C}$ sends a diagram X to the diagram obtained by "deleting" all the entries indexed by tuples $i \leq j \leq k$ such that at least one of i, j, k is 0 or $n+2$. Following Dotto's thesis [Dot12, p. 129] we define the functor $I_n \colon S^{1,1}_{2n+1}\mathscr{C} \to S^{2,1}_{n+2}\mathscr{C}$ on objects by

$$I_n(X)_{i,j,k} = \begin{cases} X_{j-1,k-1} & \text{if } i = 0 \text{ and } k \neq n+2 \\ X_{j-1,j+n} & \text{if } i = 0 \text{ and } k = n+2 \\ X_{i+n,j+n} & \text{if } i \neq 0 \text{ and } k \neq n+2 \\ 0 & \text{otherwise} \end{cases}$$

and similarly on morphisms. The functor $I_2 \colon S^{1,1}_5\mathscr{C} \to S^{2,1}_4\mathscr{C}$ may be pictured as sending the diagram

$$\begin{tikzcd}
X_{0,1} \arrow[r, tail] & X_{0,2} \arrow[r, tail] \arrow[d, two heads] & X_{0,3} \arrow[r, tail] \arrow[d, two heads] & X_{0,4} \arrow[r, tail] \arrow[d, two heads] & X_{0,5} \arrow[d, two heads] \\
& X_{1,2} \arrow[r, tail] & X_{1,3} \arrow[r, tail] \arrow[d, two heads] & X_{1,4} \arrow[r, tail] \arrow[d, two heads] & X_{1,5} \arrow[d, two heads] \\
& & X_{2,3} \arrow[r, tail] & X_{2,4} \arrow[r, tail] \arrow[d, two heads] & X_{2,5} \arrow[d, two heads] \\
& & & X_{3,4} \arrow[r, tail] & X_{3,5} \arrow[d, two heads] \\
& & & & X_{4,5}
\end{tikzcd}$$

to the diagram

$$\begin{tikzcd}
& & & & & X_{4,5} \\
& & & & X_{3,4} \arrow[r] & X_{4,5} \arrow[u, two heads] \\
& & & 0 \arrow[ur, tail] & & \\
& & X_{0,3} \arrow[rr] & & X_{1,4} \arrow[uu, two heads] \arrow[r] & X_{2,5} \arrow[uu, two heads] \\
& X_{0,2} \arrow[ur, tail] \arrow[rr] & & X_{1,2} \arrow[uu, two heads] \arrow[ur, tail] & & \\
X_{0,1} \arrow[ur, tail] & & & & &
\end{tikzcd}$$

See [Dot12, p.131] for a picture of this map when $n = 7$. The maps in the diagram $I_n(X)$ are simply composites of maps in X. Note that by construction the composite $d_0 \circ d_{n+2} \circ I_n$ equals the constant map on the zero object. There is some loss of information when applying I_n but the original diagram can be reconstructed "up to extensions". More precisely one has the following additivity theorem from [HM]:

Theorem 3.9 (Additivity for $S^{2,1}_{n+2}$)**:** *The sequence*

$$S^{1,1}_{2n+1}\mathscr{C} \xrightarrow{I_n} S^{2,1}_{n+2}\mathscr{C} \longrightarrow S^{2,1}_{n}\mathscr{C}$$

becomes a split fiber sequence after applying $|wS^{1,1}(-)|$ or $|wS^{2,1}(-)|$. In particular, there are induced weak equivalences

$$|wS^{1,1}(S^{2,1}_{n+2}\mathscr{C})| \simeq |wS^{1,1}(S^{1,1}_{2n+1}\mathscr{C})| \times |wS^{1,1}(S^{2,1}_{n}\mathscr{C})|$$

and

$$|wS^{2,1}(S^{2,1}_{n+2}\mathscr{C})| \simeq |wS^{2,1}(S^{1,1}_{2n+1}\mathscr{C})| \times |wS^{2,1}(S^{2,1}_{n}\mathscr{C})|.$$

4 The delooping theorems

4.1 Technical preliminaries

Let $Sh_l \colon \Delta \to \Delta$ be the functor $[n] \mapsto [0] \amalg [n]$ and Sh_r be $[n] \mapsto [n] \amalg [0]$. We write Sh_{rl} for $Sh_r \circ Sh_l$. Note that Sh_{rl} commutes with the opposite-functor $\alpha \mapsto \bar{\alpha}$ on Δ and therefore extends to a functor $Sh_R \colon \Delta R \to \Delta R$. There is a natural transformation $m \colon Id_{\Delta R} \to Sh_R$ which is given by

$$m_n = d_{n+2} \circ d_0 \colon [n] \to [0] \amalg [n] \amalg [0] = [n+2].$$

The map $i_n \colon [1] \to [n+2]$ which sends 0 to 0 and 1 to $n+2$ extends to a natural transformation $i \colon [1] \to Sh_R$ where [1] here means the constant functor to [1] on ΔR.

For a real simplicial object $X \colon \Delta R \to \mathscr{A}$ we write $P_R X$ for the composite $X \circ Sh_R$. Then, by the discussion above there are natural maps of real simplicial objects

$$X_1 \xleftarrow{i_X} P_R X \xrightarrow{m_X} X.$$

The map i_X does not have a section in real simplicial sets but when we pass to subdivisions we note that $SdP_R X = P_l SdX$ and so here the sequence of natural transformations on Δ (see e.g.[Wal85, 1.5.1])

$$[n] \mapsto ([n+1] \to [0] \to [n+1])$$

where the last map sends 0 to 0 induces a sequence of maps

$$SdP_R X \xrightarrow{i_X} X_1 \xrightarrow{s_X} SdP_R X$$

where the last map s_X gives a section to $Sd(i_X)$.

Lemma 4.1: *There is a natural C_2-equivariant homotopy $\Delta^1 \times |P_R X|_R \to |P_R X|_R$ from the identity to the realization of the composite $|s_X \circ Sd(i_X)|_R$, where Δ^1 is the geometric 1-simplex with trivial C_2-action. In particular, the map $i_X \colon P_R X \to X_1$ is a weak homotopy equivalence of real simplicial sets.*

Proof. The map is the geometric realization of Waldhausen's simplicial homotopy in [Wal85, 1.5.1]. It is easily seen to be equivariant. □

The same argument proves the analog of the lemma for real simplicial spaces, real simplicial simplicial sets and so on.

Lemma 4.2: *The space $|wS^{1,1}\mathscr{C}|^{C_2}_R$ is an H-commutative H-group and the map $|wS^{1,1}\mathscr{C}|^{C_2}_R \to |wS^{1,1}\mathscr{D}|^{C_2}_R$ induced by a pointed exact functor $\mathscr{C} \to \mathscr{D}$ is an H-map.*

Proof. Since $|wS^{1,1}\mathscr{C}|_R^{C_2} \cong |Sym(SdS^{1,1}\mathscr{C}, w)|$ we will work with the latter space. The orthogonal sum $\perp$ makes $|Sym(SdS^{1,1}\mathscr{C}, w)|$ into an H-commutative H-space. The monoid $\pi_0|Sym(SdS^{1,1}\mathscr{C}, w)|$ is in fact a group (it is the Witt group $W_0(\mathscr{C})$) so it follows that $|Sym(SdS^{1,1}\mathscr{C}, w)|$ has a homotopy inverse (see e.g. [MP12, Lemma 9.2.3]), hence it is an H-group.

Exact functors preserve $\perp$ up to isomorphism and hence induce H-maps on $|Sym(SdS^{1,1}-, w)|$-spaces. $\square$

The next theorem of Bousfield and Friedlander uses the rather technical π_*-Kan condition [BF78, B.3] which we will not recall here. Instead we will recall two conditions on a bisimplicial set which both ensure that it satisfies the π_*-Kan condition. For a bisimplicial set X write X_n for the simplicial set $([p] \mapsto X_{n,p})$. We then have a simplicial object $([n] \mapsto X_n)$ in the category of simplicial sets. If either the simplicial sets X_n are connected for all n, or the simplicial object $([n] \mapsto X_n)$ takes values in H-groups and H-maps, then X satisfies the π_*-Kan condition. The following theorem is formulated for bisimplicial sets but easily generalizes to sufficiently nice simplicial spaces, in particular those obtained by geometrically realizing one variable of a bisimplicial set.

Theorem 4.3 (Bousfield-Friedlander)**:** *[BF78, B.4] Let*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

be a commuting square of bisimplicial sets such that for each $n \geq 0$ the square

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & Y_n \arrow[d] \\ Z_n \arrow[r] & W_n \end{tikzcd}$$

obtained by evaluating in the first variable is homotopy cartesian. Then, if Y and W satisfy the π_*-Kan condition and the map of simplicial sets $([n] \mapsto \pi_0 Y_n) \to ([n] \mapsto \pi_0 W_n)$ is a Kan fibration the square

$$\begin{tikzcd} dX \arrow[r] \arrow[d] & dY \arrow[d] \\ dZ \arrow[r] & dW \end{tikzcd}$$

of diagonals is homotopy cartesian.

In order to prove that certain maps are weak C_2-homotopy equivalences we will make repeated use of the following setup (cf. [Dot12, §6.2] and [DM14]): A C_2-square of pointed spaces consists of a diagram

$$\begin{tikzcd} X \arrow[r, "p_r"] \arrow[d, "p_l"'] & Y_r \arrow[d, "q_r"] \\ Y_l \arrow[r, "q_l"'] & Z \end{tikzcd}$$

along with pointed C_2 actions on X and Z, with σ_X and σ_Z representing the actions of the non-trivial element of C_2, and mutually inverse maps $f\colon Y_l \to Y_r$ and $g\colon Y_r \to Y_l$ satisfying the equivariance conditions $p_l \circ \sigma_X = g \circ p_r$, $p_r \circ \sigma_X = f \circ p_l$, $\sigma_Z \circ q_l = q_r \circ g$ and $\sigma_Z \circ q_r = q_l \circ f$. The homotopy pullback

$$Y_l \times_Z^h Y_r = \{(y, \gamma, y') \in Y_l \times Z^I \times Y_r \mid \gamma(0) = q_l(y) \text{ and } \gamma(1) = q_r(y)\}$$

has a natural C_2-action given by $(y, \gamma, y') \mapsto (g(y'), \sigma_Z \circ \bar{\gamma}, f(y))$ where $\bar{\gamma}(t) = \gamma(1-t)$ and Z^I is the space of unpointed maps from the unit interval I to Z. The usual map $h\colon X \to Y_l \times_Z^h Y_r$ is C_2-equivariant.

Lemma 4.4: *In a C_2-square as above assume that both the underlying square of pointed spaces and the square*

$$\begin{tikzcd} X^{C_2} \arrow[r, "p_r \circ \iota_X"] \arrow[d, "(q_r \circ p_r)^{C_2}"'] & Y_r \arrow[d, "q_r"] \\ Z^{C_2} \arrow[r, "\iota_Z"'] & Z \end{tikzcd}$$

are homotopy cartesian, where ι_X and ι_Z are the inclusions of the fixed point spaces. Then the map $h\colon X \to Y_l \times^h_Z Y_r$ is a weak C_2-equivalence.

If, moreover, the spaces Y_l and Y_r are contractible, there is an induced a weak C_2-equivalence

$$X \to \Omega^{1,1} Z.$$

Proof. For the first part we must show that h induces weak equivalences on underlying spaces and on fixed points. On underlying spaces h is a weak equivalence, since the underlying square is homotopy cartesian. For the fixed points note that h^{C_2} factors as

$$X^{C_2} \to Z^{C_2} \times^h_Z Y_r \xrightarrow{\cong} (Y_l \times^h_Z Y_r)^{C_2}.$$

The latter map is the natural homeomorphism given by $(z, \gamma, y) \mapsto (g(y), (\sigma_Z \circ \bar{\gamma}) * \gamma, y)$, where $*$ means concatenation of paths. The former map is a weak equivalence by assumption, hence h^{C_2} is a weak equivalence.

For the second part let $H_l\colon I \times Y_l \to Y_l$ be a homotopy from the constant map to the basepoint to the identity. Then $H_r(y,t) = f(H_l(g(y), 1-t))$ is a homotopy from the identity to the constant map to the basepoint on Y_r. The map $Y_l \times^h_X Y_r \to \Omega^{1,1} X$ given by $(y, \gamma, y') \mapsto H_l(-, y) * \gamma * H_r(-, y')$ is an equivariant homotopy equivalence and precomposing this with h gives the desired weak equivalence. □

4.2 Delooping with respect to $S^{1,1}$

Theorem 4.5: *There are natural weak equivalences of pointed C_2-spaces*

$$|wS^{2,1}\mathscr{C}| \xrightarrow{\simeq} \Omega^{1,1}|wS^{2,1}S^{1,1}\mathscr{C}|$$

and

$$|wS^{1,1}\mathscr{C}| \xrightarrow{\simeq} \Omega^{1,1}|wS^{1,1}S^{1,1}\mathscr{C}|.$$

Proof. For both cases we will use the commuting square of simplicial exact categories

$$\begin{tikzcd} P_R S^{1,1}\mathscr{C} \arrow[r] \arrow[d] & P_r S\mathscr{C} \arrow[d] \\ P_l S\mathscr{C} \arrow[r] & S^{1,1}\mathscr{C}. \end{tikzcd}$$

There are dualities on the lower right and upper left objects, but not on the two remaining objects. instead, the dualities $(S_{n+1}\mathscr{C})^{op} \to S_{1+n}\mathscr{C}$ combine to give a map $((P_r S)^{op}\mathscr{C})^{op} \to P_l S\mathscr{C}$ which commutes with the dualities on $P_R S^{1,1}\mathscr{C}$ and $S^{1,1}\mathscr{C}$. Similarly there is a map $((P_l S)^{op}\mathscr{C})^{op} \to P_r S\mathscr{C}$ and its opposite is inverse to the previous map.

Now let $F(-)$ denote either $|wS^{1,1}(-)|$ or $|wS^{2,1}(-)|$. If $\mathscr{C}$ is a simplicial exact category we write $F(\mathscr{C})$ for the geometric realization $|[n] \mapsto F(\mathscr{C}_n)|$. The dualities give the diagram

$$\begin{tikzcd} F(P_R S^{1,1}\mathscr{C}) \arrow[r] \arrow[d] & F(P_r S\mathscr{C}) \arrow[d] \\ F(P_l S\mathscr{C}) \arrow[r] & F(S^{1,1}\mathscr{C}) \end{tikzcd}$$

the structure of a C_2-square. We will show that the conditions of Lemma 4.4 are satisfied so that we get a weak C_2-equivalence

$$f\colon F(P_R S^{1,1}\mathscr{C}) \xrightarrow{\simeq} \Omega^{1,1} F(S^{1,1}\mathscr{C}).$$

We first show that the underlying square is homotopy cartesian. It is obtained by geometrically realizing the square of simplicial spaces given in level $n \geq 0$ by

$$\begin{tikzcd} F(S_{1+n+1}\mathscr{C}) \arrow[r] \arrow[d] & F(S_{n+1}\mathscr{C}) \arrow[d] \\ F(S_{1+n}\mathscr{C}) \arrow[r] & F(S_n\mathscr{C}) \end{tikzcd}.$$

By the usual additivity theorem [Wal85, 1.4.2] this is weakly equivalent to the homotopy cartesian square

$$\begin{tikzcd} F(\mathscr{C})^{1\times n\times 1} \arrow[r] \arrow[d] & F(\mathscr{C})^{n\times 1} \arrow[d] \\ F(\mathscr{C})^{1\times n} \arrow[r] & F(\mathscr{C})^{\times n} \end{tikzcd}$$

and is therefore homotopy cartesian. Theorem 4.3 now applies, since the simplicial spaces are all levelwise connected.

Next we must show that the square

$$\begin{tikzcd} F(P_R S^{1,1}\mathscr{C})^{C_2} \arrow[r] \arrow[d] & F(P_l S\mathscr{C}) \arrow[d] \\ F(S^{1,1}\mathscr{C})^{C_2} \arrow[r] & F(S^{1,1}\mathscr{C}) \end{tikzcd}$$

of pointed spaces is homotopy cartesian. In order to analyze the C_2-fixed points we will subdivide both in the simplicial direction of the S-construction and in the nerve direction. We write $F_{Sd}(-)$ for $|SdwSdS^{1,1}(-)|$ or $|SdwSdS^{2,1}(-)|$. The square above is obtained by geometrically realizing the square of simplicial spaces given in level $n \geq 0$ by

$$\begin{tikzcd} F_{Sd}(S^{1,1}_{2n+3}\mathscr{C})^{C_2} \arrow[r] \arrow[d] & F_{Sd}(S_{2n+2}\mathscr{C}) \arrow[d] \\ F_{Sd}(S^{1,1}_{2n+1}\mathscr{C})^{C_2} \arrow[r] & F_{Sd}(S^{1,1}_{2n+1}\mathscr{C}). \end{tikzcd}$$

The Additivity Theorem 3.8 implies that this square is weakly equivalent to the homotopy cartesian square

$$\begin{tikzcd} F_{Sd}(\mathscr{C})^{\times n} \times F_{Sd}(\mathscr{C}) \times F_{Sd}(\mathscr{C})^{C_2} \arrow[r] \arrow[d, "pr\times id"] & F_{Sd}(\mathscr{C})^{\times 2n+1} \times F_{Sd}(\mathscr{C}) \arrow[d, "pr"] \\ F_{Sd}(\mathscr{C})^{\times n} \times F_{Sd}(\mathscr{C})^{C_2} \arrow[r] & F_{Sd}(\mathscr{C})^{\times 2n+1} \end{tikzcd}$$

and is therefore homotopy cartesian. Again, Theorem 4.3 applies, since the right hand spaces are levelwise connected.

The spaces $F(P_l S\mathscr{C})$ and $F(P_r S\mathscr{C})$ are contractible by [Wal85, 1.5.1], so all the conditions of the lemma are satisfied. By the analog of Lemma 4.1 for real simplicial *spaces* there is natural weak equivalence

$$F(\mathscr{C}) \xrightarrow{\cong} F(S^{1,1}_1\mathscr{C}) \xrightarrow{\simeq} F(P_R S^{1,1}\mathscr{C}).$$

Composing this with the map to the homotopy limit gives a weak C_2-equivalence

$$F(\mathscr{C}) \xrightarrow{\simeq} \Omega^{1,1} F(S^{1,1}\mathscr{C})$$

as desired.

□

4.3 Delooping with respect to $S^{2,1}$

By Theorem 3.9, the sequence

$$S^{1,1}_{2n+1}\mathscr{C} \to S^{2,1}_{n+2}\mathscr{C} \to S^{2,1}_n\mathscr{C}$$

becomes a split fiber sequence after applying $|wS^{1,1}(-)|_R$ or $|wS^{2,1}(-)|_R$. It is the n-th level of a sequence

$$Sd_2 S^{1,1}\mathscr{C} \to P_R S^{2,1}\mathscr{C} \to S^{2,1}\mathscr{C}$$

of simplicial categories with strict duality. Here Sd_2 is the subdivision induced by the functor $[n] \mapsto [n] \amalg [n]$ on Δ, introduced by Bökstedt, Hsiang and Madsen in [BHM93].

Proposition 4.6: *The sequence*

$$Sd_2 S^{1,1}\mathscr{C} \to P_R S^{2,1}\mathscr{C} \to S^{2,1}\mathscr{C}$$

becomes a fiber sequence after applying $|wS^{1,1}(-)|_R$ *or* $|wS^{2,1}(-)|_R$.

Proof. The case $|wS^{2,1}(-)|_R$ will be treated first. For each $n \geq 0$ the sequence

$$|wS^{2,1}S^{1,1}_{2n+1}\mathscr{C}|_R \to |wS^{2,1}S^{2,1}_{n+2}\mathscr{C}|_R \to |wS^{2,1}S^{2,1}_n\mathscr{C}|_R$$

of pointed C_2-spaces is a fiber sequence by the Additivity Theorem 3.8. Both the underlying simplicial spaces and the fixed point simplicial spaces are level-wise connected, so it follows from Theorem 4.3 that the sequence

$$|wS^{2,1}Sd_2S^{1,1}\mathscr{C}|_R \to |wS^{2,1}P_RS^{2,1}\mathscr{C}|_R \to |wS^{2,1}S^{2,1}\mathscr{C}|_R,$$

obtained by geometric realization, is a fiber sequence of C_2-spaces.

We now turn to the case $|wS^{1,1}(-)|_R$. On underlying spaces the argument is the same as for $S^{2,1}$. On the other hand the fixed point space $|wS^{1,1}\mathscr{C}|_R^{C_2}$ is usually not connected, indeed $\pi_0|wS^{1,1}\mathscr{C}|^{C_2}$ is the classical Witt group $W_0(\mathscr{C})$ of symmetric forms in $\mathscr{C}$. By Lemma 4.2 the functor $\mathscr{C} \mapsto |wS^{1,1}\mathscr{C}|^{C_2}$ takes values in H-groups and H-maps. Hence, by Theorem 4.3 the level-wise fiber sequences

$$|wS^{1,1}S^{1,1}_{4n+3}\mathscr{C}|^{C_2} \to |wS^{1,1}S^{2,1}_{2n+3}\mathscr{C}|^{C_2} \to |wS^{1,1}S^{2,1}_{2n+1}\mathscr{C}|^{C_2}$$

realize to a fiber sequence if the map of simplicial sets

$$([n] \mapsto \pi_0|wS^{1,1}S^{2,1}_{2n+3}\mathscr{C}|^{C_2}) \to ([n] \mapsto \pi_0|wS^{1,1}S^{2,1}_{2n+1}\mathscr{C}|^{C_2})$$

is a Kan fibration. This is shown in Lemma 4.7 below. □

Lemma 4.7: *The map of simplicial sets*

$$([n] \mapsto \pi_0|wS^{1,1}S^{2,1}_{2n+3}\mathscr{C}|^{C_2}) \to ([n] \mapsto \pi_0|wS^{1,1}S^{2,1}_{2n+1}\mathscr{C}|^{C_2})$$

is isomorphic to the map

$$EW_0(\mathscr{C}) \to BW_0(\mathscr{C}).$$

In particular, it is a Kan fibration.

Proof. By the Additivity Theorem 3.9, there is a weak equivalence of C_2-spaces

$$|wS^{1,1}S^{2,1}_{2k+1}| \xrightarrow{\simeq} |wS^{1,1}S^{2,1}_{2k-1}| \times |wS^{1,1}H(\mathscr{C})|^{\times k} \times |wS^{1,1}\mathscr{C}|,$$

so by induction on k we get the decomposition

$$|wS^{1,1}S^{2,1}_{2k+1}| \simeq \prod_{i=0}^{k} \left(|wS^{1,1}H\left(\mathscr{C}\right)|^{\times i} \times |wS^{1,1}\mathscr{C}|\right)$$

Taking C_2-fixed points on both sides we obtain a homotopy equivalence

$$|wS^{1,1}S^{2,1}_{2k+1}|^{C_2} \simeq \prod_{i=0}^{k} \left(|wS\left(\mathscr{C}\right)|^{\times i} \times |wS^{1,1}\mathscr{C}|^{C_2}\right),$$

since $|wS^{1,1}H\left(\mathscr{C}\right)|^{C_2} \simeq |wS\mathscr{C}|$. The space $|wS\mathscr{C}|$ is connected, so applying π_0 gives

$$\pi_0(|wS^{1,1}S^{2,1}_{2k+1}|^{C_2}) \cong \pi_0((|wS^{1,1}\mathscr{C}|^{C_2})^{\times k}) \cong W_0(\mathscr{C})^{\times k}.$$

Under this isomorphism the simplicial structure maps of $([n] \mapsto \pi_0|wS^{1,1}S^{2,1}_{2n+3}\mathscr{C}|^{C_2})$ and $(([n] \mapsto \pi_0|wS^{1,1}S^{2,1}_{2n+1}\mathscr{C}|^{C_2})$ correspond to the structure maps of $EW_0(\mathscr{C}) = B(*, W_0(\mathscr{C}), W_0(\mathscr{C}))$ and $BW_0(\mathscr{C}) = B(*, W_0(\mathscr{C}), *)$, respectively, and that the map between them is the usual projection map. This map is well known to be a Kan fibration. $\square$

Note that there is natural C_2-homeomorphism $|wS^{2,1}Sd_2S^{1,1}\mathscr{C}|_R \cong |wS^{2,1}S^{1,1}\mathscr{C}|_R$ and similarly for $|wS^{1,1}Sd_2S^{1,1}\mathscr{C}|_R$. The contracting homotopy on the middle term of the fiber sequence

$$|wS^{2,1}S^{1,1}\mathscr{C}|_R \to |wS^{2,1}P_RS^{2,1}\mathscr{C}|_R \to |wS^{2,1}S^{2,1}\mathscr{C}|_R$$

induces a weak C_2-equivalence $|wS^{2,1}S^{1,1}\mathscr{C}|_R \xrightarrow{\simeq} \Omega^{1,0}|wS^{2,1}S^{2,1}\mathscr{C}|_R$. Similarly, for the $S^{1,1}$-construction there is an induced weak C_2-equivalence $|wS^{1,1}S^{1,1}\mathscr{C}|_R \xrightarrow{\simeq} \Omega^{1,0}|wS^{1,1}S^{2,1}\mathscr{C}|_R$. Composing with the weak equilences of Theorem 4.5 gives weak equivalences

$$|wS^{2,1}\mathscr{C}|_R \xrightarrow{\simeq} \Omega^{2,1}|wS^{2,1}S^{2,1}\mathscr{C}|_R \quad \text{and} \quad |wS^{1,1}\mathscr{C}|_R \xrightarrow{\simeq} \Omega^{2,1}|wS^{1,1}S^{2,1}\mathscr{C}|_R.$$

In terms of KR-spectra this means the following:

Corollary 4.8: *The spectrum $\underline{KR}(\mathscr{C})$ is positively fibrant, i.e., the adjoint structure maps*

$$|wS^{2,1(n)}\mathscr{C}|_R \to \Omega^{2,1}|wS^{2,1(n+1)}\mathscr{C}|_R$$

are weak equivalences for $n > 0$.

Armed with the above deloopings we can now prove the following equivalence of S-constructions, proved in [HM] by different methods:

Theorem 4.9: *The sequence*

$$|wS^{1,1}\mathscr{C}|_R \to |wP_RS^{2,1}\mathscr{C}|_R \to |wS^{2,1}\mathscr{C}|_R$$

is a fiber sequence of C_2-spaces. The contracting homotopy on $|wP_RS^{2,1}\mathscr{C}|_R$ induces a weak C_2-equivalence

$$|wS^{1,1}\mathscr{C}|_R \xrightarrow{\simeq} \Omega^{1,0}|wS^{2,1}\mathscr{C}|_R.$$

Proof. The sequence

$$|wS^{1,1}\mathscr{C}|_R \to |wP_RS^{2,1}\mathscr{C}|_R \to |wS^{2,1}\mathscr{C}|_R,$$

fits in a diagram

$$\begin{tikzcd}
{|wS^{1,1}\mathscr{C}|_R} \arrow[r] \arrow[d] & {|wP_RS^{2,1}\mathscr{C}|_R} \arrow[r] \arrow[d] & {|wS^{2,1}\mathscr{C}|_R} \arrow[d] \\
{\Omega^{1,1}|wS^{1,1}S^{1,1}\mathscr{C}|_R} \arrow[r] & {\Omega^{1,1}|wP_RS^{2,1}S^{1,1}\mathscr{C}|_R} \arrow[r] & {\Omega^{1,1}|wS^{2,1}S^{1,1}\mathscr{C}|_R}
\end{tikzcd}$$

where the vertical maps are from the proof of Theorem 4.5. The diagram commutes since the vertical maps are induced by contracting homotopies of path constructions which are functorial. By Theorem 4.5 the left and right hand vertical maps are weak equivalences and since the middle spaces are both contractible the middle map is a weak equivalence. The lower sequence is obtained by applying $\Omega^{1,1}(-)$ to the first fiber sequence in Proposition 4.6, and is therefore a fiber sequence. The desired weak equivalence is induced by the contracting homotopy on $|wP_RS^{2,1}\mathscr{C}|$. $\square$

Theorem 4.10: *(i) For every category $\mathscr{C}$ with weak equivalences and duality there is a weak C_2-equivalence*

$$KR(\mathscr{C})^{C_2} \simeq GW(\mathscr{C}).$$

(ii) Let $F\colon \mathscr{B} \to \mathscr{C}$ be a duality preserving exact functor between exact categories with duality weak equivalences. If the induced maps

$$K(\mathscr{B}) \to K(\mathscr{C}) \text{ and } GW(\mathscr{B}) \to GW(\mathscr{C})$$

are weak equivalences, then the map

$$F_*\colon KR(\mathscr{B}) \to KR(\mathscr{C})$$

is a weak equivalence of C_2-spaces.

5 Consequences of the delooping theorems

In this section we will apply the results of the previous section to reformulate two classical results in algebraic K-theory and Grothendieck-Witt theory in terms of Real algebraic K-theory. The original K-theoretic results are due to Quillen and Grayson in the papers [Qui73] and [Gra76] and the Grothendieck-Witt versions are due to Schlichting and can both be found in [Sch10a]. We will not require the dualities to be strict in this section; instead, for an exact category $\mathscr{C}$ with weak equivalences and (not necessarily strict) duality we redefine $KR(\mathscr{C}) := KR(\hat{\mathscr{C}})$.

5.1 Cofinality

In order to state the cofinality theorem for Real algebraic K-theory we must first discuss C_2-Mackey functors.

A C_2-Mackey functor M consists of the following data:

- a pair of abelian groups $M(e)$ and $M(C_2)$.
- a C_2-action on $M(e)$, represented by a map $t\colon M(e) \to M(e)$ with $t^2 = id_{M(e)}$.
- maps $i_*\colon M(e) \to M(C_2)$ and $i^*\colon M(C_2) \to M(e)$ satisfying the relations $t \circ i^* = i^*$, $i_* \circ t_* = i_*$ and $i^* \circ i_* = id_{M(e)} + t$.

We will display this information in diagrammatical form

$$\begin{tikzcd} M(e) \arrow[loop left, "t"] \arrow[r, shift left, "i_*"] & M(C_2) \arrow[l, shift left, "i^*"] \end{tikzcd}$$

Mackey functors are important for us because the homotopy groups of a symmetric ρ_{C_2}-spectrum X naturally form a C_2-Mackey functor. We write $\underline{\pi}_n X$ for the C_2-Mackey functor

$$\begin{tikzcd} {[C_{2+} \wedge S^n, X]_{C_2}} \arrow[loop left, "t"] \arrow[r, shift left, "res"] & {[S^{n,0}, X]_{C_2},} \arrow[l, shift left, "tr"] \end{tikzcd}$$

where $[-,-]_{C_2}$ denotes C_2-stable homotopy classes of pointed maps and t is induced by the C_2-action on X. The restriction map res and transfer map tr are induced by the crush map $C_{2+} \wedge S^0 \to S^0$ and the stable Thom collapse map $S^0 \to C_{2+} \wedge S^0$ (which only exists in the stable homotopy category), respectively. Since $\Omega^\infty_{C_2} \underline{KR}(\mathscr{C}) \simeq \Omega^{1,1}|wS^{1,1}\mathscr{C}|_R$ it follows that the C_2-Mackey functor $\underline{\underline{KR}}_n(\mathscr{C}) = \underline{\pi}_n \underline{KR}(\mathscr{C})$ is given by

$$\begin{tikzcd} K_n(\mathscr{C}) \arrow[loop left, "D"] \arrow[r, shift left, "H"] & GW_n(\mathscr{C}). \arrow[l, shift left, "U"] \end{tikzcd}$$

Here D denotes the duality action, U is induced by the composite map

$$GW(\mathscr{C}) \xrightarrow{\Delta} GW(H(\mathscr{C})) \xrightarrow{\cong} K(\mathscr{C})$$

which corresponds in π_0 to forgetting forms. The map H comes from the map $\Omega|wS\mathscr{C}| \to (\Omega^{1,1}|wS^{1,1}\mathscr{C}|)^{C_2}$ induced by the equivariant pinch map $S^{1,1} \to C_{2+} \wedge S^1$. It can be thought of as arising from the hyperbolic functor $H\colon Sdw\mathscr{C} \to Sym(\mathscr{C}, w)$ which sends an object $f\colon X \to Y$ of $Sdw\mathscr{C}$ to the symmetric form on $X \oplus DY$ given by the matrix $\begin{pmatrix} 0 & Df \\ \eta \circ f & 0 \end{pmatrix}$.

An exact inclusion $i\colon \mathscr{A} \hookrightarrow \mathscr{B}$ of exact categories, where weak equivalences are isomorphisms, is called cofinal if it satisfies the following conditions:

1. The functor i is full, i.e., for all objects A and A' of $\mathscr{A}$ the map $\mathscr{A}(A, A') \to \mathscr{B}(i(A), i(A'))$ is surjective.
2. The functor i preserves and detects admissible monomorphisms.
3. The category $\mathscr{A}$ is closed under extension in $\mathscr{B}$, i.e., if $A' \rightarrowtail B \twoheadrightarrow A''$ is an exact sequence in $\mathscr{B}$ such that A' and A'' are isomorphic to objects in the image of i then B is isomorphic to an object in the image of i.
4. For every object B of $\mathscr{B}$ there is an object B' such that $B \oplus B'$ is isomorphic to an object in $\mathscr{A}$.

Example 5.1. Let R be a ring with anti-involution and consider a duality structure on $P(R)$ as in Example 2.4. The subcategory $F(R)$ of finitely generated free R-modules inherits a duality from the one on $P(R)$ and the inclusion $F(R) \hookrightarrow P(R)$ is cofinal in the above sense.

Let $i\colon \mathscr{A} \hookrightarrow \mathscr{B}$ be a duality preserving cofinal inclusion of exact categories with duality such that the duality on $\mathscr{A}$ is obtained by restriction of the duality on $\mathscr{B}$ (i.e. $\xi = id$). Then we define the relative Grothendieck group by $K_0(B, A) = coker\,(i_*\colon K_0\,(\mathscr{A}) \to K_0\,(\mathscr{B}))$ and the relative Grothendieck-Witt group by $GW_0(\mathscr{B}, \mathscr{A}) = coker(i_*\colon GW_0(\mathscr{A}) \to GW_0(\mathscr{B}))$. These assemble to form the relative KR-Mackey-functor of $(\mathscr{B}, \mathscr{A})$ by

$$\underline{\underline{KR}}_0(\mathscr{B}, \mathscr{A}) = \begin{tikzcd} K_0(\mathscr{B}, \mathscr{A}) \arrow[loop left, "D"] \arrow[r, shift left, "H"] & GW_0(\mathscr{B}, \mathscr{A}), \arrow[l, shift left, "U"] \end{tikzcd}$$

where H and U here mean the induced maps on cokernels.

A symmetric ρ_{C_2}-spectrum X is an Eilenberg-MacLane spectrum for the Mackey functor M if $\underline{\pi}_n X$ is isomorphic to M when $n = 0$ and is 0 otherwise. Such Eilenberg-Maclane spectra are unique up to weak equivalence, and (by abuse of notation) we will write $H(M)$ for any Eilenberg-Maclane spectrum for M.

Theorem 5.2 (Cofinality): *A cofinal duality preserving inclusion $\mathscr{A} \hookrightarrow \mathscr{B}$ induces a cofiber sequence of symmetric ρ_{C_2}-spectra*

$$\underline{KR}(\mathscr{A}) \to \underline{KR}(\mathscr{B}) \to H(\underline{\underline{KR}}_0(\mathscr{B}, \mathscr{A})).$$

Proof. The K-theory version of the statement can be found e.g. in [Sta89, Theorem 2.1] and the Grothendieck-Witt version was shown by Schlichting [Sch10a, Theorem 5.1]. The result follows by Theorem 4.10. □

5.2 Dévissage

In this section we assume that weak equivalences are isomorphisms. If $i\colon N \rightarrowtail X$ is an admissible subobject we write $N^{\perp} = ker(DX \overset{Di}{\twoheadrightarrow} DN)$. Recall that if (X, φ) is a symmetric form in an exact category with duality then a totally isotropic subspace N of (X, φ) is an admissible subobject $i\colon N \rightarrowtail X$ such that the restriction $\varphi|_N = Di \circ \varphi \circ i$ is 0 and the induced map $N \to N^{\perp}$ is an admissible monomorphism.

Theorem 5.3 (Dévissage): *Let $\mathscr{A}$ be an abelian category with duality and let $\mathscr{B} \subset \mathscr{A}$ be a full exact subcategory which is closed under the duality and under taking subobjects and quotients in $\mathscr{A}$. Assume also that*

1. *every object A in $\mathscr{A}$ has a finite filtration*

$$0 = A_0 \rightarrowtail A_1 \rightarrowtail \cdots \rightarrowtail A_n = A$$

   such that the filtration quotients A_{i+1}/A_i, for $0 \leq i \leq n$, are in $\mathscr{B}$

2. *for every non-degenerate symmetric form (A, φ) in $\mathscr{A}$ there is a totally isotropic subspace $N \rightarrowtail A$ such that $N^{\perp}/N$ is in $\mathscr{B}$.*

Then the inclusion $\mathscr{B} \hookrightarrow \mathscr{A}$ induces a weak equivalences of pointed C_2-spaces

$$KR(\mathscr{B}) \xrightarrow{\simeq} KR(\mathscr{A}).$$

Proof. This follows immediately from the classical Dévissage theorem of Quillen [Qui73, Thm. 4] and Schlichting's Dévissage theorem for Grothendieck-Witt spaces [Sch10a, Thm. 6.1] by Theorem 4.10. □

Now let A be a Dedekind domain with trivial involution and fraction field K and let $\epsilon \in \{\pm 1\}$. As an application of the Dévissage Theorem 5.3 we will compute the KR-theory of torsion modules over A with "ϵ-twisted" Pontryagin duality generalizing the situation in Example 2.5.

Let $S \subset A \setminus \{0\}$ be a multiplicative subset and write $\mathscr{T}_S$ for the category of finitely generated S-torsion modules over A. More precisely it is the full subcategory of $Mod(A)$ consisting of finitely generated modules M such that for each $m \in M$ there is an $s \in S$ such that $ms = 0$, or equivalently, $S^{-1}M = 0$. We give $\mathscr{T}_S$ the duality

$$M^* = Hom_A(M, K/A)$$

with double duality isomorphism

$$m \mapsto (f \mapsto \epsilon f(m)).$$

Note that the exact sequence

$$0 \longrightarrow A \longrightarrow K \longrightarrow K/A \longrightarrow 0$$

gives an injective resolution of A, so that $M^* \cong \mathrm{Ext}^1_A(M,A)$. If $\mathfrak{p} \subset A$ is a non-zero prime ideal $\mathscr{T}_\mathfrak{p}$ will denote the category of finitely generated torsion $A_\mathfrak{p}$-modules. In general, for a family $\{\mathscr{A}_i\}_{i\in I}$ of additive categories we can form a new category $\oplus_{i\in I}\mathscr{A}_i$ whose objects are tuples $(A_i)_{i\in I}$ such that A_i is an object of $\mathscr{A}_i$ and $A_i = 0$ for all but finitely many i. The morphisms are given by $Hom((A_i),(B_i)) = \oplus_{i\in I} Hom(A_i,B_i)$ with component-wise composition. There is a natural isomorphism of simplicial sets $N(\oplus_{i\in I}\mathscr{A}_i) \xrightarrow{\cong} \prod'_{i\in I} N\mathscr{A}_i$ where $\prod'$ denotes the restricted product consisting of tuples where only finitely many entries are non-zero. The restricted product may be expressed as the colimit $\mathrm{colim}_{J_{fin}\subset I}\prod_{j\in J} N\mathscr{A}_j$ where J_{fin} ranges over all finite subsets of I. The direct sum $\oplus_{i\in I}\mathscr{A}_i$ has a similar colimit description, so since KR commutes with products and filtering colimits there is a pointed equivariant homeomorphism

$$KR(\oplus_{i\in I}\mathscr{A}_i) \xrightarrow{\cong} \textstyle\prod'_{i\in I} KR(A_i).$$

We write $\mathfrak{p}|S$ if $\mathfrak{p}S^{-1}A = S^{-1}A$ and say (by an abuse of language) that $\mathfrak{p}$ divides S. Since A is Dedekind each module M in $\mathscr{T}_S$ has finite support, i.e., $M_\mathfrak{p} \neq 0$ for only finitely many primes $\mathfrak{p}$. Hence there is a well-defined functor

$$F\colon \mathscr{T}_S \to \bigoplus_{\mathfrak{p}|S} \mathscr{T}_\mathfrak{p}$$

sending a module M to the tuple $(M_\mathfrak{p})_{\mathfrak{p}|S}$. This functor is an equivalence of categories. Moreover, since each M is finitely generated the natural maps

$$Hom_A(M, K/A) \to \bigoplus_{\mathfrak{p}|S} Hom_{A_\mathfrak{p}}(M_\mathfrak{p}, K/A_\mathfrak{p})$$

are isomorphisms, hence F is also duality preserving. We write $KR(\mathscr{T}_{S,\epsilon})$ for the Real K-theory of $(\mathscr{T}_S, Hom(-,K/A), \eta_\epsilon)$ and similarly $KR(\mathscr{T}_{\mathfrak{p},\epsilon})$ and $KR(k(\mathfrak{p})_\epsilon)$ in the cases $\mathscr{T}_\mathfrak{p}$ and $Vec(k(\mathfrak{p}))$. The functor F induces a weak equivalence of pointed C_2-spaces

$$KR(\mathscr{T}_{S,\epsilon}) \xrightarrow{\simeq} KR\big(\bigoplus_{\mathfrak{p}|S} \mathscr{T}_{\mathfrak{p},\epsilon}\big).$$

Proposition 5.4: *There is a weak equivalence of pointed C_2-spaces*

$$KR(\mathscr{T}_{S,\epsilon}) \simeq \prod_{\mathfrak{p}|S}{}' KR(k(\mathfrak{p})_\epsilon).$$

Proof. We will apply the Dévissage theorem to the categories $\mathscr{T}_\mathfrak{p}$. Any module M in $\mathscr{T}_\mathfrak{p}$ has finite $\mathfrak{p}$-adic filtration

$$0 = \mathfrak{p}^n M \subset \mathfrak{p}^{n-1}M \subset \cdots \subset \mathfrak{p}M \subset M$$

for some $n \geq 0$. The filtration quotients $\mathfrak{p}^k M/\mathfrak{p}^{k+1}M$ are isomorphic to finite direct sums of $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}$ and are therefore semi-simple. By [QSS79, Theorem 6.6] the inclusion of the full subcategory $\mathscr{S}_\mathfrak{p} \subset \mathscr{T}_\mathfrak{p}$ of semi-simple objects satisfies the conditions of the Dévissage Theorem 5.3, hence there is an induced weak equivalence of C_2-spaces

$$KR(\mathscr{S}_{\mathfrak{p},\epsilon}) \xrightarrow{\simeq} KR(\mathscr{T}_{\mathfrak{p},\epsilon}).$$

Let p be a uniformizer of the discrete valuation ring $A_\mathfrak{p}$. The residue field $k(\mathfrak{p}) = A/pA$ embeds into K/A by the map $i_\mathfrak{p}$ which is given by

$$i_\mathfrak{p}(a + pA) = \frac{a}{p} + A.$$

The induced map $(i_\mathfrak{p})_*\colon Hom_{A_\mathfrak{p}}(k(\mathfrak{p}), k(\mathfrak{p})) \to Hom_{A_\mathfrak{p}}(k(\mathfrak{p}), K/A)$ is an isomorphism. Now consider the inclusion of the category $Vec(k(\mathfrak{p})) := P(k(\mathfrak{p}))$ with the usual ϵ-twisted duality $Hom_{k(\mathfrak{p})}(-, k(\mathfrak{p}))$ into $\mathscr{S}_\mathfrak{p}$, see e.g. Example 2.4. The composite map

$$Hom_{k(\mathfrak{p})}(V, k(\mathfrak{p})) \xrightarrow{\cong} Hom_{A_\mathfrak{p}}(V, k(\mathfrak{p})) \xrightarrow{i_\mathfrak{p}} Hom_{A_\mathfrak{p}}(V, K/A))$$

is an isomorphism because V is a finite direct sum of copies of $k(\mathfrak{p})$. Together with the inclusion functor this map provides a duality preserving equivalence of exact categories with duality $Vec(k(\mathfrak{p})) \hookrightarrow \mathscr{S}_\mathfrak{p}$ and hence a weak equivalence of C_2-spaces

$$KR(Vec(k(\mathfrak{p}))_\epsilon) = KR(Vec(k(\mathfrak{p}))_\epsilon) \xrightarrow{\simeq} KR(\mathscr{S}_{\mathfrak{p},\epsilon})$$

The result now follows.

□

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Homotopy theory of G-diagrams and equivariant excision

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Abstract

Let G be a finite group acting on a small category I. We study functors $X\colon I \to \mathscr{C}$ equipped with families of compatible natural transformations that give a kind of generalized G-action on X. Such objects are called G-diagrams. When $\mathscr{C}$ is a sufficiently nice model category we define a model structure on the category of G-diagrams in $\mathscr{C}$. There are natural G-actions on Bousfield-Kan style homotopy limits and colimits of G-diagrams. We prove that weak equivalences between point-wise (co)fibrant G-diagrams induce weak G-equivalences on homotopy (co)limits. A case of particular interest is when the indexing category is a cube. We use homotopy limits and colimits over such diagrams to produce loop and suspension spaces with respect to permutation representations of G. We go on to develop a theory of enriched equivariant homotopy functors and give an equivariant "linearity" condition in terms of cubical G-diagrams. In the case of G-topological spaces we prove that this condition is equivalent to Blumberg's notion of G-linearity. In particular we show that the Wirthmüller isomorphism theorem is a direct consequence of the equivariant linearity of the identity functor on G-spectra.

Contents

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Introduction

The concept of G-diagram was introduced, under different names, in Villarroel-Flores's thesis [VF99] and independently in the paper [JS01] of Jackowski and Słomińska, and they were further studied in [VF04]. In the current literature the theory of G-diagrams has been only partially developed. It is limited, due to the fact that it is used for very specific applications, to properties of homotopy colimits of G-diagrams in the category of spaces or of simplicial sets (see e.g. [JS01] or [TW91]). The contribution of the present paper is a systematic treatment of G-diagrams in a nice (simplicial, cofibrantly generated, etc.) model category. An immediate advantage of this general theory is that it allows us to work in the category of genuine G-spectra. Additionally, it is the first treatment of homotopy limits of G-diagrams. As an application of this abstract framework, we set up a theory of equivariant enriched homotopy functors and formulate an "equivariant excision" condition in terms of cubical G-diagrams. This condition agrees with Goodwillie's notion of excision [Goo92] when G is the trivial group, and with Blumberg's definition from [Blu06] for the category of G-spaces.

Given a finite group G acting on a category I by functors $a(g)\colon I \to I$, a G-diagram in a category $\mathscr{C}$ is a functor $X\colon I \to \mathscr{C}$ together with natural transformations $g_X\colon X \to X \circ a(g)$ for every g in G, which are compatible with the group structure. A map of G-diagrams is a natural transformation between the underlying diagrams that commutes with the structure maps (see Definitions 1.1 and 1.2). We write $\mathscr{C}^I_a$ for the resulting category of G-diagrams. The category $\mathscr{C}^I_a$ is isomorphic to the category of diagrams in $\mathscr{C}$ indexed on the Grothendieck construction of the action functor $a\colon G \to Cat$ (see Lemma 1.9 and [JS01, 2]). If the category of G-objects $\mathscr{C}^G$ is a sufficiently nice model category, such as G-spaces with the fixed points model structure, or orthogonal G-spectra with the genuine G-stable model structure, we prove the following 2.6.

Theorem: *Let $\mathscr{C}$ be a G-model category (see 2.1). There is a cofibrantly generated $sSet^G$-enriched model structure on the category of G-diagrams $\mathscr{C}^I_a$ with weak equivalences (resp. fibrations) the maps of G-diagrams $f\colon X \to Y$ such that the value f_i at the object $i \in obI$ is a weak equivalence (resp. fibration) in the model category $\mathscr{C}^{G_i}$ of objects with an action of the stabilizer group G_i.*

The authors first became interested in G-diagrams while working on equivariant delooping results for so-called Real algebraic K-theory and Real topological Hochschild homology. A recurring example of a G-diagram in this work is the following:

Example. Let X be a pointed space with an action of C_2, the cyclic group of order two, with $\sigma\colon X \to X$ representing the action of the non-trivial group element. A diagram of pointed spaces

$$Y \xrightarrow{\ p\ } X \xleftarrow{\ q\ } Z \tag{1}$$

together with mutually inverse homeomorphisms $r\colon Y \to Z$ and $l\colon Z \to Y$ which cover σ, in the sense that $p \circ l = \sigma \circ q$ and $q \circ r = \sigma \circ p$, defines a C_2-diagram of pointed spaces. The pullback $Y \times_X Z$ inherits a natural C_2-action given by $(y, z) \mapsto (l(z), r(y))$, and similarly the homotopy pullback

$$Y \times^h_X Z = \{(y, \gamma, z) \in Y \times X^I \times Z \,|\, p(y) = \gamma(0) \text{ and } \gamma(1) = q(z)\}$$

inherits the action $(y, \gamma, z) \mapsto (l(z), \sigma \circ \bar{\gamma}, r(y))$, where $\bar{\gamma}(t) = \gamma(1 - t)$. The usual inclusion $Y \times_X Z \hookrightarrow Y \times^h_X Z$ is equivariant with respect to these actions. Let $\mathbb{R}^{1,1}$ denote the sign representation of C_2 on $\mathbb{R}$ and let $\Omega^{1,1}X$ be the space of pointed maps from the one point compactification $S^{\mathbb{R}^{1,1}}$ to X with C_2 acting by conjugation. If Y (and hence Z) is contractible, then a contracting homotopy induces a C_2-homotopy equivalence

$$Y \times^h_X Z \simeq \Omega^{1,1}X.$$

On underlying spaces this just an instance of the well-known homotopy equivalence

$$\Omega X \simeq \mathrm{holim}(* \to X \leftarrow *).$$

This example illustrates how limits and homotopy limits of punctured C_2-squares of spaces carry a C_2-action, and how these can be used to construct the loop space by the sign representation of C_2. More generally, when it makes sense to talk about the limit, colimit, homotopy limit or homotopy colimit of a G-diagram X in any ambient category $\mathscr{C}$, these constructions have natural G-actions induced by the structure maps g_X (see Corollary 1.5 and §1.2). Moreover, the usual comparison maps $\lim X \to \operatorname{holim} X$ and $\operatorname{hocolim} X \to \operatorname{colim} X$ are equivariant as we already observed for the C_2-diagram (1). In general most constructions involving (co)limits and (co)ends enrichments applied to G-diagrams produce G-objects and equivariant maps between them. The homotopy limits and colimits of G-diagrams are homotopy invariant in the following sense (see also Proposition 2.22):

Proposition: *The functors* $\operatorname{holim}\colon \mathscr{C}_a^I \to \mathscr{C}^G$ *and* $\operatorname{hocolim}\colon \mathscr{C}_a^I \to \mathscr{C}^G$ *preserve equivalences between fibrant diagrams and point-wise cofibrant diagrams respectively.*

We prove other fundamental properties of these equivariant homotopy limits and colimits functors, analogous to classical theorems from homotopy theory of diagrams:

- *2.25 Homotopy cofinality theorem for homotopy limits and colimits of G-diagrams, generalizing the results [TW91, 1] and [VF04, 6],*
- *2.26 A twisted Fubini theorem, showing that homotopy colimits of G-diagrams over a Grothendieck construction can be calculated "point-wise" (an equivariant analogue of [CS02, 26.5]). As an immediate corollary we obtain an equivariant analogue of Thomason's homotopy colimit theorem from [Tho79],*
- *2.28 An Elmendorf theorem, showing that for suitable ambient categories one can equivalently define the homotopy theory of G-diagrams by replacing G with the opposite of its orbit category (an equivariant analogue of the classical result of [Elm83]).*

As an application of this model categorical theory of G-diagrams, we define and study equivariant excision. Classically, a homotopy invariant functor between model categories is excisive if it sends homotopy cocartesian squares to homotopy cartesian squares (see [Goo92]). Blumberg shows in [Blu06] that this notion is not well behaved when the categories involved are categories of G-objects; enriched homotopy functors on the category of pointed G-spaces $Top_*^G \to Top_*^G$ that are classically linear (excisive and sending the point to a G-contractible space) are a model only for the category of naïve G-spectra. In order to model genuine G-spectra, one needs a property stronger than classical linearity. Blumberg achieves this by adding an extra condition to linearity; a compatibility condition with equivariant Spanier-Whitehead duality.

In the present paper we take a different approach to equivariant excision, following the idea that the relation between equivariant excision and excision should resemble the relation between genuine G-spectra and naïve G-spectra. Instead of adding an extra condition to classical excision, we replace squares by "equivariant cubes", similarly to the way one replaces integers with G-representations in defining G-spectra. For a finite G-set J we consider the poset category $\mathcal{P}(J)$ of subsets of J ordered by inclusion. This category inherits a G-action from the G-action on J.

Definition (G-excision)**.** A J-cube X in $\mathscr{C}$ is a G-diagram in $\mathscr{C}$ shaped over $\mathcal{P}(J)$, i.e. it is an object of $\mathscr{C}_a^{\mathcal{P}(J)}$. We say that X is homotopy cartesian if the canonical map

$$X_\emptyset \longrightarrow \operatorname*{holim}_{\mathcal{P}(J)\backslash\emptyset} X$$

is a weak equivalence in the model category of G-objects $\mathscr{C}^G$. Dually, it is homotopy cocartesian if the canonical map $\operatorname*{hocolim}_{\mathcal{P}(J)\backslash J} X \to X_J$ is an equivalence in $\mathscr{C}^G$. A suitably homotopy invariant functor $\Phi\colon \mathscr{C}^G \to \mathscr{D}^G$ is called G-excisive if it sends homotopy cocartesian G_+-cubes to homotopy cartesian G_+-cubes.

Here G_+ is the set G with an added disjoint base point, and G acts on it by left multiplication. It plays the role of a "regular" G-set, analogous to the regular representation of G in stable equivariant homotopy theory. The added basepoint has an important role, discussed in details in 3.12. We prove in 3.28 that this notion of G-excision is equivalent to Blumberg's definition from [Blu06] when $\mathscr{C}$ is the category of pointed spaces. The paper contains a series of fundamental properties of G-excision, that appropriately reflect the fundamental properties of excision to a genuine equivariant context. They can be summarized as follows:

- *3.11 A G-excisive functor $\mathscr{C}^G \to \mathscr{D}^G$ is classically excisive, that is, it sends homotopy cocartesian squares in $\mathscr{C}^G$ to homotopy cartesian squares in $\mathscr{D}^G$,*
- *3.20 A G-linear functor is also H-linear for every subgroup H of G,*
- *3.33 Every enriched G-linear homotopy functor Φ from finite G-CW-complexes to G-spectra is equivalent to one of the form $E_\Phi \wedge (-)$ for some G-spectrum E_Φ,*
- *3.32 The identity functor on G-spectra is G-excisive: For any finite G-set J, a J-cube of spectra is homotopy cartesian if and only if it is homotopy cocartesian,*
- *3.17 Any G-excisive reduced homotopy functor $\Phi\colon \mathscr{C}^G \to \mathscr{D}^G$ satisfies the Wirthmüller isomorphism theorem, that is, the canonical map $\Phi(G \otimes_H c) \to \hom_H(G, \Phi(c))$ is an equivalence in $\mathscr{D}^G$ for every subgroup H of G and H-object c of $\mathscr{C}^H$.*
- *3.25,3.26 If $\mathscr{D}^G$ is suitably presentable, a construction similar to Goodwillie's derivative of [Goo92] defines a universal G-excisive approximation to any homotopy functor $\mathscr{C}^G \to \mathscr{D}^G$.*

These properties have interesting consequences for the identity functor on G-spectra. The fact that it is G-excisive shows that the theory of equivariant cubes provides a good context in which the category of G-spectra is "G-stable". Moreover, Theorem 3.17 applied to the identity functor on G-spectra gives a new proof of the classical Wirthmüller isomorphism theorem. An analysis of the structure of the proofs of 3.17 and 3.32 gives the following argument: The identity on G-spectra is G-excisive as a direct consequence of the equivariant Freudenthal suspension theorem, by formally manipulating homotopy limits and colimits. Given an H-equivariant spectrum E, there is an explicit homotopy cocartesian $(G/H)_+$-cube of spectra WE with initial vertex $(WE)_\emptyset = G_+ \wedge_H E$, and with $\operatorname*{holim}_{\mathcal{P}(G/H_+)\setminus\emptyset} WE = F_H(G_+, E)$. By G-excision for the identity functor WE is homotopy cartesian, that is, the canonical map $G_+ \wedge_H E \to F_H(G_+, E)$ is a stable equivalence of G-spectra.

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1 Definitions and setup

1.1 Categories of G-diagrams

We first introduce some notation and conventions. If $\mathscr{C}$ is a (possibly large) category and I is a small category we write $\mathscr{C}^I$ for the usual category of functors from I to $\mathscr{C}$. By topological space we will mean compactly generated weak Hausdorff space and Top is the category of such spaces with continuous maps between them. We write $Map(X, Y)$ for the space of maps from X to Y endowed with the compact-open topology. The based variants of the above are Top_* and $Map_*(X, Y)$.

In the following $\mathscr{C}$ will be a category, G a finite group, and I a small category. By a slight abuse of notations we will also write G for the category with one object $*$ and one morphism $g\colon * \to *$ for each element $g \in G$, and with composition given by $g \circ h = gh$. The group G will act on I from the left and we will encode the action as a functor $a\colon G \to Cat$ sending $*$ to I. Most of the content of this section can be found in the work of Jackowski-Słomińska[JS01] or in Villarroel-Flores's paper [VF04].

Definition 1.1. (cf. [JS01, 2.2], [VF04, 3.1])Let $X\colon I \to \mathscr{C}$ be an I-shaped diagram in $\mathscr{C}$. A G-structure on X with respect to the action a is a collection of natural transformations $\{g_X\colon X \to X \circ a(g)\}$ such that

1. $e_X = id_X$
2. $(g_X)_{a(h)} \circ h_X = (gh)_X$ for all $g, h \in G$,

where $(g_X)_{a(h)}$ is the natural transformation obtained by restricting g_X along the functor $a(h)\colon I \to I$. An I-shaped diagram X with a G-structure will be called an I-shaped G-diagram in $\mathscr{C}$ with respect to the action a, or simply a G-diagram in $\mathscr{C}$ if I and a are understood.

In order to simplify the notation we will mostly write g in stead of $a(g)$ when this does not cause confusion. Accordingly, when X and Y are I-indexed G-diagrams we will write f_g for the restriction of a map $f\colon X \to Y$ along the functor $g = a(g)\colon I \to I$. In the later sections we will sometimes write g instead of g_X.

Definition 1.2. A map of G-diagrams $f\colon X \to Y$ is a natural transformation $f\colon X \to Y$ of underlying diagrams such that for each $g \in G$ the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g_X"'] & Y \arrow[d, "g_Y"] \\ X \circ g \arrow[r, "f_g"'] & Y \circ g \end{tikzcd}$$

commutes in $\mathscr{C}^I$.

The composite of two maps of G-diagrams is again a map of G-diagrams. For a fixed action a of the group G on I we write $\mathscr{C}^I_a$ for the category whose objects are the G-diagrams in $\mathscr{C}$ with respect to a and with morphisms the maps of G-diagrams.

Example 1.3. Let $[n]$ be the usual category with objects $0, 1, \dots, n$ and a morphism $i \to j$ if and only if $i \leq j$. For a small category I the nerve NI is the usual simplicial set with $NI_n = Fun([n], I)$. Taking over-categories gives a functor $N(I/-)\colon I \to sSet$. The G-action on I gives maps $N_{/i,g}\colon N(I/i) \to N(I/gi)$ for $g \in G$ and i an object of I, by mapping

$$(i_0 \to \dots \to i_n \to i) \overset{g}{\longmapsto} (gi_0 \to \dots \to gi_n \to gi)$$

These maps combine to give a G-diagram structure on $N(I/-)$. Similarly the functor $N(-/I)^{op}\colon I^{op} \to sSet$ with the maps $N_{i,g/}\colon N(i/I)^{op} \to N(gi/I)^{op}$ defines a G-diagram in $sSet$.

Let I and J be small categories with G-actions a and b respectively and let $F\colon I \to J$ be a functor. We say that F is G-equivariant if it commutes strictly with the G-actions, that is, if $F(gi) = gF(i)$ and $F(g\alpha) = gF(\alpha)$ for all objects i in I and morphisms α in I. If Y is a J-shaped G-diagram then the restriction $F^*Y = Y \circ F$ has a naturally induced G-structure with maps $g_{(F^*Y)} = F^*(g_Y)$.

Now assume that $\mathscr{C}$ is complete and cocomplete. Then the functor $F^*\colon \mathscr{C}^J \to \mathscr{C}^I$ has a left adjoint F_* and a right adjoint $F_!$ given by left and right Kan extension, respectively. We will now see that if X is an I-shaped G-diagram, then there are natural G-structures on F_*X and $F_!X$. We treat the left Kan extension first.

The value of the functor F_*X on an object j of J is given by the coequalizer

$$\coprod_{(i_0 \xrightarrow{\alpha} i_1, f\colon F(i_1) \to j)} X_{i_0} \overset{s}{\underset{t}{\rightrightarrows}} \coprod_{(i_0, f\colon F(i_0) \to j)} X_{i_0} \twoheadrightarrow F_*X_j,$$

where s projects onto the source of the indexing map α and t maps into the target of α by the map $X(\alpha)$. For an element $g \in G$ the natural transformation g_X induces a map of diagrams

$$\begin{tikzcd}
\coprod\limits_{(i_0 \xrightarrow{\alpha} i_1, f\colon F(i_1)\to j)} X_{i_0} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "\coprod g_{X_{i_0}}"] & \coprod\limits_{(i_0, f\colon F(i_0)\to j)} X_{i_0} \arrow[r, two heads] \arrow[d, "\coprod g_{X_{i_0}}"] & F_*X_j \arrow[d, dashed, "g_{F_*X_j}"] \\
\coprod\limits_{(i_0' \xrightarrow{\alpha'} i_1', f'\colon F(i_1')\to gj)} X_{i_0'} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \coprod\limits_{(i_0', f'\colon F(i_0')\to gj)} X_{i_0'} \arrow[r, two heads] & F_*X_{gj}
\end{tikzcd}$$

and the dotted arrow is the j-component of the natural transformation $g_{F_*X}\colon F_*X \to (F_*X)\circ g$. It is not hard to see that the set $\{g_{F_*X}\}_{g\in G}$ constitutes a G-structure on F_*X and that the underlying functor F_* takes maps of I-indexed G-diagrams to maps of J-indexed G-diagrams. Similarly, for the right Kan extension $F_!$ a dual construction with equalizers gives a G-structure $\{g_{F_!X}\}_{g\in G}$ on $F_!X$. We write simply F_*X and $F_!X$ for the G-diagrams obtained in this way.

Proposition 1.4: *The constructions F_*X and $F_!X$ define functors $F_*\colon \mathscr{C}_a^I \to \mathscr{C}_b^J$ and $F_!\colon \mathscr{C}_a^I \to \mathscr{C}_b^J$.*

A particularly interesting case of the above is when $J = *$ the category with one object and one morphism and trivial G-action. In this case the functors F_* and $F_!$ are more commonly known as colim_I and $\lim_I$, respectively.

Corollary 1.5: *Let X be an I-indexed G-diagram. Then the above constructions induce natural left G-actions on $\operatorname{colim}_I X$ and $\lim_I X$.*

Example 1.6. (Products and coproducts) Let I be a discrete category with G-action, i.e., a G-set and consider a G-diagram X in the category *Set* of sets. The coproduct $\coprod_I X$ is the set of pairs (i,x) where $x \in X_i$ and the action of $g \in G$ is given by

$$g(x,i) = (g_{X_i}(x), gi).$$

The product $\prod_I X$ is the set of functions $\mathbf{x}\colon I \to \bigcup_{i\in I} X_i$ such that $\mathbf{x}(i) \in X_i$ for all $i \in I$. The action of $g \in G$ on $\mathbf{x} \in \prod_I X$ is determined by the equation

$$(g\mathbf{x})(gi) = g_{X_i}(\mathbf{x}(i)).$$

This example generalizes to arbitrary categories with products and coproducts but the notation becomes more cumbersome when one can no longer speak about elements of objects.

We now give an alternative description of G-diagrams which is sometimes easier to work with.

Definition 1.7. Let $G \rtimes_a I$ be the following category:

- $obG \rtimes_a I = obI$
- A morphism $i \to j$ in $G \rtimes_a I$ is a pair $(g, \alpha\colon gi \to j)$ where $g \in G$.
- Composition is given by $(h, \beta\colon hj \to k) \circ (g, \alpha\colon gi \to j) = (gh, \beta \circ h\alpha\colon ghi \to k)$.

Remark 1.8. The category $G \rtimes_a I$ is the Grothendieck construction of the functor $a\colon G \to Cat$, sometimes denoted $G \int a$ (see e.g. [Tho79]).

A G-diagram X gives rise to a functor $X^{\rtimes_a}\colon G \rtimes_a I \to \mathscr{C}$ by setting

$$X_i^{\rtimes_a} = X_i$$

on objects, and defining

$$X^{\rtimes_a}(g, \alpha\colon gi \to j) = X(\alpha) \circ g_{X_i}$$

on morphisms. We leave it to the reader to check that this respects composition of maps.

Lemma 1.9: *The assignment $X \mapsto X^{\rtimes_a}$ is functorial and defines an isomorphism of categories*

$$\Phi \colon \mathscr{C}_a^I \xrightarrow{\cong} \mathscr{C}^{G \rtimes_a I}.$$

Proof. The functoriality is clear. We define a functor $\Phi' \colon \mathscr{C}^{G\rtimes_a I} \to \mathscr{C}_a^I$ which is inverse to Φ. For a diagram $Y \colon G \rtimes_a I \to \mathscr{C}$ define the underlying diagram of $\Phi'(Y)$ to be $(Y|_I)$, i.e., the restriction of Y along the canonical inclusion $\iota \colon I \hookrightarrow G \rtimes_a I$ given by $\iota(i) = i$ and $\iota(\alpha \colon i \to j) = (e, \alpha \colon i \to j)$. For an element $g \in G$ the natural transformation $g_{\Phi'(Y)}$ is defined at an object i by $Y(g, id \colon gi \to gi)$. Both naturality of the $g_{\Phi'(Y)}$'s and conditions 1) and 2) of Definition 1.1 follow from the functoriality of Y with respect to morphisms in $G \rtimes_a I$. For a natural transformation $f \colon Y \to Z$ in $\mathscr{C}^{G\rtimes_a I}$ we define $\Phi'(f) = f|_I$. It is now easy to check that the functors Φ and Φ' are mutually inverse. $\square$

Corollary 1.10: *Let $\mathscr{C}$ be a bicomplete category. Then $\mathscr{C}_a^I$ is also bicomplete.*

Proof. The diagram category $\mathscr{C}^{G\rtimes_a I}$ is bicomplete since $\mathscr{C}$ is. It follows from 1.9 that $\mathscr{C}_a^I$ is bicomplete. $\square$

1.2 Enrichments and homotopy (co)limits

If $\mathscr{C}$ is any category, then the category $\mathscr{C}^G$ is naturally enriched in left G-sets in the following way. For objects c, d of $\mathscr{C}^G$ let $\mathscr{C}(c, d)$ be the set of maps between the underlying objects in $\mathscr{C}$. Then G acts on $\mathscr{C}(c, d)$ by conjugation

$$g \cdot f = g_d \circ f \circ (g^{-1})_c$$

where $(g^{-1})_c$ and g_d represent the actions of g^{-1} and g on c and d respectively. The fixed points set $\mathscr{C}(c, d)^G$ is precisely the set of G-equivariant maps from c to d.

If I is small a category with an action a of G, then the category $\mathscr{C}_a^I$ of G-diagrams becomes enriched in left G-sets by taking $\underline{\mathscr{C}}_a^I(X, Y)$ to be the set $\mathscr{C}^I(X, Y)$ of maps of underlying diagrams $f \colon X \to Y$ with action given by

$$g \cdot f = (g_Y)_{g^{-1}} \circ f_{g^{-1}} \circ (g^{-1})_X.$$

If f is fixed under the action of G, then

$$f = g^{-1} f = ((g^{-1})_Y)_g \circ f_g \circ g_X = (g_Y)^{-1} \circ f_g \circ g_X.$$

In other words, f is fixed if and only if the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g_X"'] & Y \arrow[d, "g_Y"] \\ X \circ g \arrow[r, "f_g"'] & Y \circ g \end{tikzcd}$$

commutes for all $g \in G$. It follows that the fixed points $\underline{\mathscr{C}}_a^I(X, Y)^G$ are precisely the maps of G-diagrams $\mathscr{C}_a^I(X, Y)$. If $I = *$ then this statement reduces to the one above about maps in $\mathscr{C}^G$.

Proposition 1.11: *Let I and J be small categories with G-actions a and b, respectively and let $F \colon I \to J$ be an equivariant functor. Then, for X an I-indexed G-diagram and Y a J-indexed G-diagram the bijections*

$$\phi_{X,Y} \colon \underline{\mathscr{C}}_a^I(X, F^*Y) \xrightarrow{\cong} \underline{\mathscr{C}}_b^J(F_*X, Y)$$

and

$$\psi_{X,Y} \colon \underline{\mathscr{C}}_a^I(F^*Y, X) \xrightarrow{\cong} \underline{\mathscr{C}}_b^J(Y, F_!X)$$

induced by the adjunctions on underlying diagrams are G-equivariant.

Proof. We show that $\phi = \phi_{X,Y}$ is equivariant, the argument for $\psi_{X,Y}$ is similar.

Let $f\colon X \to F^*Y$ be a map of diagrams and $g \in G$. Then $\phi(g \cdot f)$ is the unique map $F_*X \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "g\cdot f"] \arrow[d, "\eta_X"'] & F^*Y \\ F^*F_*X \arrow[ur, dashed, "F^*(\phi(g\cdot f))"'] & \end{tikzcd} \tag{2}$$

commutes, where η_X is the unit of the (F_*, F^*)-adjunction at the object X. Consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "(g^{-1})_X"] \arrow[d, "\eta_X"'] & X \circ g^{-1} \arrow[r, "f_{g^{-1}}"] \arrow[d, "\eta_{X,g^{-1}}"'] & (F^*Y) \circ g^{-1} \arrow[r, "(F^*g_Y)_{g^{-1}}"] \arrow[d, "="] & F^*Y \\ F^*F_*X \arrow[r, "F^*((g^{-1})_{F_*X})"'] & (F^*F_*X) \circ g^{-1} \arrow[r, "F^*\phi(f)_{g^{-1}}"'] & (F^*Y) \circ g^{-1}. \arrow[ur, "(F^*g_Y)_{g^{-1}}"'] & \end{tikzcd}$$

The commutativity of the left hand square follows immediately from the definition of g_{F_*X} and middle square commutes by the definition of $\phi(f)$. Composing the maps in the top row gives $(F^*g_Y)_{g^{-1}} \circ f_{g^{-1}} \circ (g^{-1})_X = g \cdot f$ and composing along the bottom row from F^*F_*X to F^*Y gives

$$F^*((g_Y)_{g^{-1}} \circ \phi(f)_{g^{-1}} \circ (g^{-1})_{F_*X}) = F^*(g \cdot \phi(f)).$$

It follows that $F^*(g \cdot \phi(f))$ defines a lift in the diagram (2) so, by uniqueness of the lift, we conclude that $\phi(g \cdot f) = g \cdot \phi(f)$. $\square$

Taking fixed points in Proposition 1.11 we immediately get the following:

Corollary 1.12: *The functors F_* and $F_!$ are left and right adjoint, respectively, to the restriction functor $F^*\colon \mathscr{C}_b^J \to \mathscr{C}_a^I$. In particular the diagonal $\Delta_I = p^*\colon \mathscr{C}^G \to \mathscr{C}_a^I$ induced by the projection $p\colon I \to *$ has left and right adjoints $p_* = \operatorname{colim}_I$ and $p_! = \lim_I$, respectively.*

Let I be a category with G-action a and let G act diagonally on the product $I^{op} \times I$. Given a G-diagram $Z\colon I^{op} \times I \to \mathscr{C}$ recall that the *end* $\int_i Z_{i,i}$ of Z is the equalizer

$$\begin{tikzcd} \displaystyle\int_i Z_{i,i} \arrow[r, tail] & \displaystyle\prod_i Z_{i,i} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \displaystyle\prod_{\alpha\colon i \to j} Z_{j,i} \end{tikzcd}$$

where s and t act on the left and right, respectively by the map α. The end $\int_i Z_{i,i}$ inherits a left G-action by the maps

$$\begin{tikzcd} \displaystyle\int_i Z_{i,i} \arrow[r] \arrow[d, dashed, "g_{(\int Z)}"] & \displaystyle\prod_i Z_{i,i} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "\prod_i g_{Z_{(i,i)}}"] & \displaystyle\prod_{\alpha\colon i \to j} Z_{j,i} \arrow[d, "\prod_\alpha g_{Z_{(j,i)}}"] \\ \displaystyle\int_i Z_{i,i} \arrow[r] & \displaystyle\prod_i Z_{i,i} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \displaystyle\prod_{\alpha\colon i \to j} Z_{j,i} \end{tikzcd} \tag{$*$}$$

The *coend* $\int^i Z_{i,i}$ is the coequalizer

$$\begin{tikzcd} \displaystyle\coprod_{\alpha\colon i \to j} Z_{j,i} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \displaystyle\coprod_i Z_{i,i} \arrow[r, two heads] & \displaystyle\int_i Z_{i,i}. \end{tikzcd}$$

which inherits a G-action in a similar way.

Example 1.13. If $X, Y \colon I \to \mathscr{C}$ are diagrams in $\mathscr{C}$ then we can describe the set of maps (natural transformations) between them as the end

$$\mathscr{C}^I(X,Y) = \int_i \mathscr{C}(X_i, Y_i).$$

Similarly, for G-diagrams X, Y in $\mathscr{C}^I_a$ there is a natural isomorphism of G-sets

$$\underline{\mathscr{C}}^I_a(X,Y) \cong \int_i \mathscr{C}(X_i, Y_i)$$

with the G-action on the left hand as described above.

By a *simplicial category* we will mean a category $\mathscr{C}$ that is enriched, tensored and cotensored in simplicial sets, in the sense of e.g. [DS07, 2.2] or [GJ09, II,2.1]. This means that for any two objects c and d in $\mathscr{C}$ there is a simplicial set $Map_{\mathscr{C}}(c,d)$, and a natural bijection $\mathscr{C}(c,d) \cong Map_{\mathscr{C}}(c,d)_0$. Moreover, given a simplicial set K there are objects $K \otimes c$ and $map_{\mathscr{C}}(K, c)$ of $\mathscr{C}$. These satisfy some associativity constraints and naturality conditions making $Map_{\mathscr{C}}(-,-)$ and $map_{\mathscr{C}}(-,-)$ contravariant functors in the first variable and covariant in the second variable and $-\otimes-$ covariant in both variables. Finally for all c, d in $\mathscr{C}$ and K in $sSet$ there are natural isomorphisms in $sSet$

$$Map_{\mathscr{C}}(K \otimes c, d) \cong Map(K, Map_{\mathscr{C}}(c,d)) \cong Map_{\mathscr{C}}(c, map_{\mathscr{C}}(K,d)),$$

where Map with no subscript denotes the usual internal hom-object in $sSet$.

Using this structure we will now describe additional structure on the category $\mathscr{C}^I_a$ of I-indexed G-diagrams in a simplicial category $\mathscr{C}$. We begin with enrichment. We noted above that for a pair X, Y of G-diagrams in $\mathscr{C}$ the set $\mathscr{C}^I(X,Y)$ has a G-action induced by the G-structures on X and Y. This gives $\mathscr{C}^I_a$ the structure of a category enriched in left G-sets. The functor $i, j \mapsto Map_{\mathscr{C}}(X_i, Y_j)$ going from $I^{op} \times I$ to $sSet$ becomes a G-diagram by letting $g \in G$ act at i, j by

$$Map_{\mathscr{C}}(g^{-1}_{X_i}, g_{Y_j}) \colon Map_{\mathscr{C}}(X_i, Y_j) \to Map_{\mathscr{C}}(X_{gi}, Y_{gj}).$$

Definition 1.14. With X, Y as above, set

$$Map_{\mathscr{C}^I_a}(X,Y) = \int_i Map_{\mathscr{C}}(X_i, Y_i)$$

with the G-action as described in the diagram $(*)$.

In other words the mapping space $Map_{\mathscr{C}^I_a}(X,Y)$ is the equalizer

$$Map_{\mathscr{C}^I_a}(X,Y) \rightarrowtail \prod_i Map_{\mathscr{C}}(X_i, Y_i) \overset{s}{\underset{t}{\rightrightarrows}} \prod_{\alpha\colon i \to j} Map_{\mathscr{C}}(X_j, Y_i)$$

It is not hard to see that this defines an enrichment of $\mathscr{C}^I_a$ in $sSet^G$ and that for each $n \geq 0$ there is an isomorphism of G-sets

$$Map_{\mathscr{C}^I_a}(X,Y)_n \cong \underline{\mathscr{C}}^I_a(\Delta^n \otimes X, Y).$$

Definition 1.15. Let $K \colon I \to sSet$, $L \colon I^{op} \to sSet$, and $X \colon I \to \mathscr{C}$ be G-diagrams. We set

$$map^a_I(K,X) = \int_i map_{\mathscr{C}}(K_i, X_i) \tag{3}$$

$$L \otimes^a_I X = \int^i L_i \otimes X_i \tag{4}$$

and give both the G-actions from $(*)$.

When K and L are respectively the G-diagrams of simplicial sets $N(I/-)$ and $N(-/I)^{op}$ from 1.3, these constructions specify to the following.

Definition 1.16. For a G-diagram X in $\mathscr{C}$ the homotopy limit and homotopy colimit of X are respectively

$$\operatorname*{holim}_I X = map_I^a(N(I/-), X) \qquad \operatorname*{hocolim}_I X = N(-/I)^{op} \otimes_I^a X$$

This constructions define functors $\operatorname{holim}, \operatorname{hocolim} \colon \mathscr{C}_a^I \to \mathscr{C}^G$.

In the presence of a model structure the words homotopy limit and colimit will always refer to these particular construction and not, a priori, the derived functors of the limit and colimit respectively.

Note that there are maps of diagrams $N(-/I)^{op} \to *$ and $N(I/-) \to *$, where $*$ denotes a chosen one-point simplicial set in both cases. From the formulas above it is easy to see that there are natural isomorphisms $map_I^a(*, X) \cong \lim X$ and $X \otimes_I^a * \cong \operatorname{colim} X$. The maps to the terminal diagrams induce equivariant maps

$$\lim X \to \operatorname{holim} X \qquad \operatorname{hocolim} X \to \operatorname{colim} X$$

This paper is in part motivated by the question "when are these maps weak equivalences in $\mathscr{C}^G$?"

1.3 Examples of G-diagrams

In this section we will provide many of the motivating examples for the theory of G-diagrams. The diagrams will usually have values in the category Top_* of pointed spaces.

For the first two examples we need to fix some notation. Let Z be a pointed space with an action by the finite group G. If T is a finite left G-set, we write $\mathbb{R}[T]$ for the permutation representation with basis $\{e_t\}_{t\in T}$. The subspace of $\mathbb{R}[T]$ generated by the element $N_T = \sum_{t\in T} e_t$ is a one-dimensional trivial subrepresentation of $\mathbb{R}[T]$. We define $S^{\tilde{T}}$ to be the one-point compactification of the orthogonal complement of $\mathbb{R}\cdot N_T$ under the usual inner product. We write $\Omega^{\tilde{T}}Z$ for the G-space of continuous pointed maps $Map_*(S^{\tilde{T}}, Z)$ with the conjugation action of G and $\Sigma^{\tilde{T}}Z$ for the smash product $S^{\tilde{T}} \wedge Z$ with the diagonal G-action.

Example 1.17. The power set $\mathcal{P}(T)$ inherits a left G-action from the action on T. We think of the poset $\mathcal{P}(T)\setminus\emptyset$ as a category with G-action. Let $\omega^{\tilde{T}}Z$ be the $\mathcal{P}(T)\setminus\emptyset$-indexed G-diagram whose value on a subset $U\subseteq T$ is $*$ if $U\neq T$ and Z if $U=T$. The G-structure on $\omega^{\tilde{T}}Z$ is given by the action of G on Z at the fixed object T and by the unique maps $*\to *$ elsewhere in the diagram. We claim that there is a G-homeomorphism

$$\operatorname*{holim}_{\mathcal{P}(T)\setminus\emptyset} \omega^{\tilde{T}}Z \cong \Omega^{\tilde{T}}Z$$

which is natural in Z. To see this we begin by noticing that the realization of the category $|N(\mathcal{P}(T)\setminus\emptyset)$ is G-homeomorphic to the (barycentric subdivision of the) standard simplex $\Delta^{|T|-1}$ in the complement of $\mathbb{R}\cdot N_T$ in $\mathbb{R}[T]$. Since $\omega^{\tilde{T}}Z$ has all entries trivial except at the last vertex T we see that $\operatorname{holim}\omega^{\tilde{T}}Z$ is homeomorphic to the subspace in $Map(\Delta^{|T|-1}, Z)$ of maps whose restriction to the boundary is the constant map to the base-point of Z, that is $\Omega^{\tilde{T}}Z$. The naturality is clear, so this proves the claim.

Example 1.18. Similarly, we think of the poset $\mathcal{P}(T)\setminus T$ as a category with G-action and define the G-diagram $\sigma^{\tilde{T}}Z$ to have the value Z at the vertex $\emptyset$ and $*$ elsewhere. The G-diagram structure is induced by the G-action on Z and the unique maps $*\to *$. A similar argument to the one for $\omega^{\tilde{T}}Z$ shows that there is a natural G-homeomorphism

$$\operatorname*{hocolim}_{\mathcal{P}(T)\setminus T} \sigma^{\tilde{T}}Z \cong \Sigma^{\tilde{T}}Z.$$

Example 1.19. More generally, for any pointed category $\mathscr{C}$ and G-object $c \in \mathscr{C}^G$ define the $\tilde{T}$-loop space and $\tilde{T}$-suspension of c respectively as the pullback and pushout in $\mathscr{C}^G$

$$\begin{tikzcd} \Omega^{\tilde{T}} c \arrow[r] \arrow[d] & map_{\mathscr{C}}(N\mathcal{P}(T)\backslash\emptyset, c) \arrow[d] \\ * \arrow[r] & map_{\mathscr{C}}(\partial N\mathcal{P}(T)\backslash\emptyset, c), \end{tikzcd} \qquad \begin{tikzcd} (\partial N\mathcal{P}(T)^{op}\backslash T) \otimes c \arrow[r] \arrow[d] & * \arrow[d] \\ (N\mathcal{P}(T)^{op}\backslash T) \otimes c \arrow[r] & \Sigma^{\tilde{T}} c. \end{tikzcd}$$

In the case of a pointed G-space or G-spectra we recover the usual equivariant loop and suspension spaces. These constructions define an adjoint pair of functors $(\Sigma^{\tilde{T}}, \Omega^{\tilde{T}})$ on $\mathscr{C}^G$, by the sequence of natural bijections

$$\mathscr{C}^G(\Sigma^{\tilde{T}} c, d) \cong \mathscr{C}^{\mathcal{P}(2)\backslash 2}\big((N\mathcal{P}(T)^{op}\backslash T) \otimes c \leftarrow (\partial N\mathcal{P}(T)^{op}\backslash T) \otimes c \to * \otimes c, \Delta d\big) \cong$$
$$\mathscr{C}^{\mathcal{P}(2)\backslash\emptyset}\big(\Delta c, map_{\mathscr{C}}(N\mathcal{P}(T)\backslash\emptyset, d) \to map_{\mathscr{C}}(\partial N\mathcal{P}(T)\backslash\emptyset, d) \leftarrow map_{\mathscr{C}}(*, d)\big) \cong \mathscr{C}^G(c, \Omega^{\tilde{T}} d).$$

Here we used that $* \otimes c = *$ and $map_{\mathscr{C}}(*, d) = *$, as $\mathscr{C}$ is pointed. Similarly to the previous examples there are natural isomorphisms in $\mathscr{C}^G$

$$\operatorname*{holim}_{\mathcal{P}(T)\backslash\emptyset} \omega^{\tilde{T}} c \cong \Omega^{\tilde{T}} c \quad \text{and} \quad \operatorname*{hocolim}_{\mathcal{P}(T)\backslash T} \sigma^{\tilde{T}} c \cong \Sigma^{\tilde{T}} c.$$

Example 1.20. We already saw that for a category I with G-action the functor $N(I/-)\colon I \to sSet$ has an obvious G-structure. For a functor $F\colon I \to J$ and an object j of J one can form the over-category F/j and the assignment $j \mapsto N(F/j)$ defines a functor $N(F/-)\colon J \to sSet$. If F is an equivariant functor between categories with G-action there are functors $F/j \to F/(gj)$ induced by the G-actions, and after applying the nerve these give a G-structure on the diagram $N(F/-)$. In fact, $N(F/-)$ with this G-structure is the left Kan extension $F_*N(I/-)$ of $N(I/-)$ along F. This will be important later when we discuss homotopy cofinality and cofibrancy of G-diagrams.

Example 1.21. Let $X\colon I \to \mathscr{C}$ be a diagram in a simplicial category $\mathscr{C}$. Define the diagram qX by $qX_i = \operatorname{hocolim}_{I/i} u_i^* X$ where $u_i\colon I/i \to I$ is the functor that forgets the map to i. A map $\alpha\colon i \to j$ in I induces a functor $I/i \to I/j$ and hence a map $qX_i \to qX_j$. The natural map from the homotopy colimit to the colimit induces maps

$$qX_i = \operatorname*{hocolim}_{I/i} u_i^* X \to \operatorname*{colim}_{I/i} u_i^* X \xrightarrow{\cong} X_i,$$

which combine to a map of diagrams $\rho_X\colon qX \to X$. If X is a G-diagram then the functor $I/i \to I/gi$ induced by multiplication by $g \in G$ induces a map $qX_i \to qX_{gi}$ and together these maps constitute a G-structure on qX. It is a classical fact that the objects $\operatorname{colim}_I qX$ and $\operatorname{hocolim}_I X$ are isomorphic and in ii of Proposition 2.16 we prove that this isomorphism is G-equivariant when X is a G-diagram.

2 G-diagrams and model structures

This section provides a framework in which the equivariant constructions of homotopy limits and colimits defined earlier in the paper have homotopical sense, and are well behaved. The first step in developing this framework is to give the ambient category $\mathscr{C}$ enough structure to be able to define a model structure on the category of G-diagrams in $\mathscr{C}$. It turns out that having a model structure on the category $\mathscr{C}^G$ of G-objects in $\mathscr{C}$ is not enough, but one needs to have homotopical information for all the subgroups of G. The good context for a genuine equivariant homotopy theory seems to be that of an "equivariant model category".

2.1 Equivariant model categories

Let $\mathscr{C}$ be a complete and cocomplete category, G a finite group and $H, H' \leq G$ a pair of subgroups. A finite set K with commuting left H'-action and right H-action induces a pair of adjoint functors

$$K \otimes_H (-)\colon \mathscr{C}^H \rightleftarrows \mathscr{C}^{H'} \colon \hom_{H'}(K,-)$$

The left adjoint is defined as

$$K \otimes_H c = \operatorname{colim}\left(H \xrightarrow{\coprod_K c} \mathscr{C} \right)$$

where $\coprod_K c$ is the H-equivariant colimit of the constant H-diagram Δc on the discrete H-category K^δ, (see Example 1.6), and the H'-action is induced by the H'-action on K. Dually, define

$$\hom_{H'}(K,d) = \lim\left(H' \xrightarrow{\prod_K c} \mathscr{C} \right)$$

with left H-action defined by right action on K. These functors are adjoint via the sequence of natural isomorphisms

$$\begin{aligned}
&\mathscr{C}^{H'}(K \otimes_H c, d) \cong \mathscr{C}(K \otimes_H c, d)^{H'} \cong \mathscr{C}^H(\textstyle\coprod_K c, d)^{H'} \cong \\
&\mathscr{C}^K_a(\Delta_K c, \Delta_K d)^{H'} \cong \mathscr{C}^H(c, \textstyle\prod_K d)^{H'} \cong \mathscr{C}^H(c, \lim_{H'}(\prod_K d)) = \\
&\mathscr{C}^H(c, \hom_{H'}(K,d))
\end{aligned}$$

In the following we will always use the fixed point model structure on $sSet^G$ (see e.g. [Shi03, 1.2]) unless otherwise is stated.

Definition 2.1. A G-model category is a cofibrantly generated simplicial model category $\mathscr{C}$, together with the data of a cofibrantly generated model structure on $\mathscr{C}^H$ for every subgroup $H \leq G$, satisfying

1. The model structure on $\mathscr{C}^H$ together with the $sSet^H$-enrichment, tensored and cotensored structures induced from $\mathscr{C}$ forms a cofibrantly generated $sSet^H$-enriched model structure on $\mathscr{C}^H$,
2. For every pair of subgroups $H, H' \leq G$, and finite set K with commuting free left H'-action and free right H-action the adjunction

$$K \otimes_H (-)\colon \mathscr{C}^H \rightleftarrows \mathscr{C}^{H'} \colon \hom_{H'}(K,-)$$

   is a Quillen adjunction.

Remark 2.2. For $H' \leq H$ and $K = H$ with actions given by left H' and right H multiplications, the functor

$$H \otimes_H (-)\colon \mathscr{C}^H \longrightarrow \mathscr{C}^{H'}$$

is isomorphic to the functor $\operatorname{res}^H_{H'}$ that restricts the action. Similarly for $K = H$ with left H multiplication and right H' multiplication the functor

$$\hom_H(H,-)\colon \mathscr{C}^H \longrightarrow \mathscr{C}^{H'}$$

is also isomorphic to the functor $\operatorname{res}^H_{H'}$. It follows from the second condition that $\operatorname{res}^H_{H'}$ is both a left and a right Quillen functor, and therefore it preserves cofibrations, acyclic cofibrations, fibrations, acyclic fibrations and equivalences between cofibrant or fibrant objects.

Example 2.3. Let $\mathscr{C}$ be a cofibrantly generated $sSet$-enriched model category. The collection of projective model structures (naïve) on $\mathscr{C}^H$ for $H \leq G$ defines a G-model structure on $\mathscr{C}^G$. To see this, just notice that if H'-acts freely on K, a choice of section for the quotient map $K \to H' \backslash K$ induces a natural isomorphism

$$\operatorname{res}^H_e \hom_{H'}(K,c) \cong \prod_{H' \backslash K} c$$

where $\operatorname{res}^H_e \colon \mathscr{C}^H \to \mathscr{C}$ is the forgetful functor. Therefore $\hom_{H'}(K,-)$ preserves fibrations and acyclic fibrations.

Example 2.4. Let $\mathscr{C}$ be a cofibrantly generated $sSet$-enriched model category, and fix a pair of finite groups $H \leq G$. For all subgroup $L \leq H$, the L-fixed points functor $(-)^L \colon \mathscr{C}^H \to \mathscr{C}$ is defined as the composite

$$\mathscr{C}^H \xrightarrow{\mathrm{res}^H_L} \mathscr{C}^L \xrightarrow{\lim} \mathscr{C}$$

If these functors are cellular in the sense of [GM13], the category $\mathscr{C}^H$ inherits a $sSet^H$-enriched model structure where weak equivalences and fibrations are the maps that are sent by $(-)^L$ respectively to weak equivalences and fibrations in $\mathscr{C}$, for every subgroup $L \leq H$ (cf. [MM02, 2.8],[GM13],[Ste10]). This construction specifies to the standard fixed points model structure on (pointed) spaces with H-action.

The collection of model categories $\mathscr{C}^H$, for H running over the subgroups of G, assemble into a G-model category. Let us see that the left adjoint $K \otimes_H (-)$ is a left Quillen functor. The generating cofibrations of $\mathscr{C}^H$ are by definition the images of the generating cofibrations of $\mathscr{C}$ by the functors

$$J \otimes (-) \colon \mathscr{C} \longrightarrow \mathscr{C}^H$$

where J ranges over finite sets with left H-action. Similarly for generating acyclic cofibrations. There is a natural isomorphism

$$K \otimes_H (J \otimes (-)) \cong (K \times_H J) \otimes (-)$$

and the right hand functor preserves cofibrations and acyclic cofibrations by assumption. Thus $K \otimes_H (-)$ preserves generating (acyclic) cofibrations. Since it is a left adjoint it preserves colimits, and therefore all (acyclic) cofibrations (see e.g. [Hir03, 11.2]).

Example 2.5. Let $\mathscr{C} = \mathrm{Sp}^O$ be the category of orthogonal spectra and G a finite group. The category $(\mathrm{Sp}^O)^G$ of G-objects in Sp^O is naturally equivalent to the category of orthogonal G-spectra $\mathscr{J}_G^{\mathscr{V}}\mathscr{S}$ of [MM02] indexed on a universe $\mathscr{V}$ for finite dimensional G-representations (cf.[MM02, V.1], [Sch13, 2.7]). Given any subgroup $H \leq G$, we endow $(\mathrm{Sp}^O)^H$ with the model structure induced by the stable model structure on $\mathscr{J}_H^{i^*\mathscr{V}}\mathscr{S}$ of [MM02] under the equivalence of categories $(\mathrm{Sp}^O)^H \simeq \mathscr{J}_H^{i^*\mathscr{V}}\mathscr{S}$. Here $i \colon H \to G$ denotes the inclusion, and $i^*\mathscr{V}$ is the universe of representations of H that are restrictions of representations of G in $\mathscr{V}$. The adjunctions

$$K \otimes_H (-) \colon (\mathrm{Sp}^O)^H \rightleftarrows (\mathrm{Sp}^O)^{H'} \colon \hom_{H'}(K, -)$$

are the standard induction-coinduction adjunctions, and they are Quillen adjunctions by [MM02, V-2.3]. The collection of model categories $\{(\mathrm{Sp}^O)^H\}_{H \leq G}$ then forms a G-model category.

2.2 The "G-projective" model structure on G-diagrams

Let G be a finite group, $\mathscr{C}$ a category, and I a small category with G-action a. Given a G-diagram X in $\mathscr{C}^I_a$ and an object $i \in I$, the vertex $X_i \in \mathscr{C}$ inherits from the G-structure on X an action by the stabilizer group $G_i \leq G$ of the object i. This gives an evaluation functor $\mathrm{ev}_i \colon \mathscr{C}^I_a \to \mathscr{C}^{G_i}$ for every object i.

Theorem 2.6: *Let $\mathscr{C}$ be a G-model category (see 2.1). There is a cofibrantly generated $sSet^G$-enriched model structure on the category of G-diagrams $\mathscr{C}^I_a$ with*

1. *weak equivalences the maps of G-diagrams $f \colon X \to Y$ whose evaluations $\mathrm{ev}_i f$ are weak equivalences in $\mathscr{C}^{G_i}$ for every $i \in I$,*
2. *fibrations the maps of G-diagrams $f \colon X \to Y$ whose evaluations $\mathrm{ev}_i f$ are fibrations in $\mathscr{C}^{G_i}$ for every $i \in I$,*
3. *generating cofibrations and acyclic cofibrations*

$$F\mathcal{I} = \bigcup_{i \in I} F_i \mathcal{I}_i \qquad \text{and} \qquad F\mathcal{J} = \bigcup_{i \in I} F_i \mathcal{J}_i$$

where $\mathcal{I}_i$ and $\mathcal{J}_i$ are respectively generating cofibrations and generating acyclic cofibrations of $\mathscr{C}^{G_i}$, and $F_i \colon \mathscr{C}^{G_i} \to \mathscr{C}^I_a$ is the left adjoint to the evaluation functor ev_i.

Remark 2.7. Under the isomorphism $\mathscr{C}_a^I \cong \mathscr{C}^{G\rtimes_a I}$ of Lemma 1.9 the evaluation functor ev_i corresponds to restriction along the functor $\iota_i\colon G_i \to G \rtimes_a I$ that sends the unique object to i and a morphism g to $(g, \mathrm{id}_i\colon gi = i \to i)$. Since $\mathscr{C}$ has all colimits a left adjoint for ev_i exists. Also notice that the model structure on $\mathscr{C}_a^I$ above does not correspond to the projective model structure on $\mathscr{C}^{G\rtimes_a I}$.

Before proving the theorem we need to identify the left adjoints of the evaluation functors. For fixed objects $i, j \in I$ let K_{ji} be the morphisms set

$$K_{ji} = \hom_{G\rtimes_a I}(i,j) = \{(g \in G, \alpha\colon gi \to j)\}$$

The stabilizer group G_j acts freely on the left on K_{ji} by left multiplication on G and by the category action on the morphism component. The group G_i acts freely on the right on K_{ji} by right multiplication on the G-component. For every $c \in \mathscr{C}^{G_i}$ define a diagram $F_i c\colon I \to \mathscr{C}$ by sending an object $j \in I$ to

$$(F_i c)_j = K_{ji} \otimes_{G_i} c$$

A morphism $\beta\colon j \to j'$ in I induces a map $(F_i c)_j \to (F_i c)_{j'}$ via the G_i-equivariant map $\beta_*\colon K_{ji} \to K_{j'i}$

$$\beta_*(g, \alpha\colon gi \to j) = (g, \beta \circ \alpha)$$

The G_i-equivariant maps $g\colon K_{ji} \to K_{(gj)i}$

$$g(g', \alpha\colon g'i \to j) = (gg', g\alpha\colon gg'i \to gj)$$

define a G-structure on $F_i c$. The construction is clearly functorial in c, defining a functor $F_i\colon \mathscr{C}^{G_i} \to \mathscr{C}_a^I$.

Lemma 2.8: *The functor $F_i\colon \mathscr{C}^{G_i} \to \mathscr{C}_a^I$ is left adjoint to the evaluation functor $\mathrm{ev}_i\colon \mathscr{C}_a^I \to \mathscr{C}^{G_i}$.*

Proof. We prove that under the isomorphism $\mathscr{C}_a^I \cong \mathscr{C}^{G\rtimes_a I}$ of Lemma 1.9 the functor F_i corresponds to the left Kan extension along the inclusion $\iota_i\colon G_i \to G \rtimes_a I$. For an object $j \in I$, the category ι_i/j is the disjoint union of categories

$$\iota_i/j = \coprod_{\substack{z\in G/G_i\\ zi\to j}} Ez$$

where Ez is the translation category of the right G_i-set z, with one object for every element of the orbit z, and a unique morphism $h\colon g \to g'$ whenever $g' = gh^{-1}$ for some $h \in G_i$. An object $c \in \mathscr{C}^{G_i}$ induces a diagram $Ec\colon Ez \to G_i \xrightarrow{c} \mathscr{C}$, where the first functor collapses all the objects to the unique object of G_i, and sends the unique morphism $g \to gh^{-1}$ to h. The left Kan extension along ι_i at c is by definition the diagram $L_i c$ with j-vertex

$$(L_i c)_j = \coprod_{\substack{z\in G/G_i\\ zi\to j}} \operatorname*{colim}_{Ez} Ec$$

Notice that the indexing set of the coproduct is precisely the orbit set K_{ji}/G_i. There is a canonical map of diagrams $F_i c \to L_j c$, which at a vertex j is induced by

$$\coprod_{K_{ji}} c \longrightarrow \coprod_{K_{ji}/G_i} \operatorname*{colim}_{Ez} Ec = (L_i c)_j$$

which on the (g,α)-component is the canonical map $c = (Ec)_g \to \mathrm{colim}_{E[g]}\, Ec$ to the $[g,\alpha]$-coproduct component. This map respects the G_j-structure, which on $L_i c$ acts on the indexing sets K_{ji}/G_i. To show that it is an isomorphism, choose a section $s\colon G/G_i \to G$ for the projection map. This gives a map

$$(L_i c)_j = \coprod_{K_{ji}/G_i} \operatorname*{colim}_{Ez} Ec \longrightarrow \coprod_{K_{ji}} c \longrightarrow K_{ji} \otimes_{G_i} c = (F_i c)_j$$

that on the (z,α)-component is the map induced by $s(z)^{-1}g\colon (Ez)_g = c \to c$ to the $(s(z),\alpha)$-component. □

Proof of 2.6. Weak equivalences and fibrations in $\mathscr{C}_a^I$ are by definition the morphisms that are sent to weak equivalences and fibrations, respectively, by the functor

$$\prod_{i\in I} \mathrm{ev}_i \colon \mathscr{C}_a^I \longrightarrow \prod_{i\in I} \mathscr{C}^{G_i}$$

It follows from Lemma 2.8 that the coproduct of the functors F_i defines a left adjoint

$$F\colon \prod_{i\in I} \mathscr{C}^{G_i} \xrightarrow{\prod F_i} \prod_{i\in I} \mathscr{C}_a^I \xrightarrow{\coprod} \mathscr{C}_a^I$$

for the product of the evaluation functors. The collections

$$\mathcal{I} = \bigcup_{i\in I}(\mathcal{I}_i \times \prod_{j\neq i} \mathrm{id}_{\emptyset_j}) \quad \text{and} \quad \mathcal{J} = \bigcup_{i\in I}(\mathcal{J}_i \times \prod_{j\neq i} \mathrm{id}_{\emptyset_j})$$

generate respectively the cofibrations and the acyclic cofibrations of $\prod \mathscr{C}^{G_i}$ (see e.g. [Hir03, 11.1.10]), where $\emptyset_j$ is the initial object of $\mathscr{C}^{G_i}$. Moreover their images by F are precisely the families $F\mathcal{I}$ and $F\mathcal{I}$ from the statement. Following [Hir03, 11.3.1] and [Ste10, D.21], we prove that

i) $\prod \mathrm{ev}_j$ takes relative $F\mathcal{I}$-cell complexes to cofibrations: Let λ be a non-zero ordinal and $X\colon \lambda \to \mathscr{C}_a^I$ a functor such that for all morphism $\beta \to \beta'$ in λ the map $X_\beta \to X_{\beta'}$ is a pushout of a map in $F\mathcal{I}$. We need to show that for every $j \in I$ the map

$$\mathrm{ev}_j X_0 \longrightarrow \mathrm{ev}_j \operatorname*{colim}_\lambda X = \operatorname*{colim}_\lambda \mathrm{ev}_j \circ X$$

is a cofibration in $\mathscr{C}^{G_i}$. Since ev_j commutes with colimits, each map $\mathrm{ev}_j X_\beta \to \mathrm{ev}_j X_{\beta'}$ is the pushout of a map in $\mathrm{ev}_j F\mathcal{I}$. Thus we need to show that every map in $\mathrm{ev}_j F\mathcal{I}$ is a cofibration of $\mathscr{C}^{G_j}$. By definition of $\mathcal{I}$, this is the same as showing that for all $i, j \in I$ every generating cofibration of $\mathcal{I}_i$ is sent by $\mathrm{ev}_j F_i$ to a cofibration of $\mathscr{C}^{G_j}$. The composite functor $\mathrm{ev}_j F_i$ is by definition

$$\mathrm{ev}_j F_i = K_{ji} \otimes_{G_i} (-)\colon \mathscr{C}^{G_i} \longrightarrow \mathscr{C}^{G_j}$$

which sends generating cofibrations to cofibrations as part of the axioms of a G-model category (see 2.1).

ii) $\prod \mathrm{ev}_j$ takes relative $F\mathcal{J}$-cell complexes to acyclic cofibrations: the argument is similar to the one above.

Moreover $\prod \mathrm{ev}_j$ preserves colimits. By [Hir03, 11.3.1] and [Ste10, D.21], the families $F\mathcal{I}$ and $F\mathcal{J}$ are respectively a class of generating cofibrations and acyclic cofibrations for the $sSet^G$-enriched model structure on $\mathscr{C}_a^I$ with the fibrations and weak equivalences of the statement. $\square$

Remark 2.9. Recall the isomorphism $\mathscr{C}_a^I \cong \mathscr{C}^{G\rtimes_a I}$ of Lemma 1.9. The model structure on $\mathscr{C}_a^I$ does not correspond to the projective model structure on $\mathscr{C}^{G\rtimes_a I}$. However, every fibration (resp. weak equivalence) in $\mathscr{C}_a^I$ is in particular a fibration (resp. weak equivalence) in $\mathscr{C}^{G\rtimes_a I}$. This means that the cofibrations of $\mathscr{C}^{G\rtimes_a I}$ are also cofibrations in $\mathscr{C}_a^I$. In particular, a sufficient condition for an object of $\mathscr{C}_a^I$ to be cofibrant is to be cofibrant in the projective model structure of $\mathscr{C}^{G\rtimes_a I}$.

Proposition 2.10: *If $X \in \mathscr{C}_a^I$ is cofibrant, each vertex X_i is cofibrant in $\mathscr{C}^{G_i}$.*

Proof. An argument dual to the proof of Lemma 2.8 shows that the right adjoint R_i to the evaluation functor $\mathrm{ev}_i\colon \mathscr{C}_a^I \to \mathscr{C}^{G_i}$ has j-vertex

$$\mathrm{ev}_j R_i = \hom_{G_i}(K_{ji}^*, -)$$

where K_{ji}^* is the set K_{ji} with left G_i-action $g \cdot k := k \cdot g^{-1}$ and right G_j-action $k \cdot g := g^{-1} \cdot k$. Hence $\mathrm{ev}_j R_i$ is a right Quillen functor by the axioms of a G-model category. Since the fibrations and the equivalences on $\mathscr{C}_a^I$ are point-wise, $R_i \colon \mathscr{C}^{G_i} \to \mathscr{C}_a^I$ is also a right Quillen functor. It follows that ev_i is a left Quillen functor, and in particular it preserves cofibrant objects. □

Definition 2.11. Let $\mathscr{C}$ and $\mathscr{D}$ be G-model categories. A G-Quillen adjunction (resp. equivalence) is an enriched adjunction $\mathscr{C} \rightleftarrows \mathscr{D}$ such that the induced adjunction $\mathscr{C}^H \rightleftarrows \mathscr{D}^H$ is a Quillen adjunction (resp. equivalence) for every subgroup $H \leq G$.

Example 2.12. The Quillen equivalence $|-| \colon sSet \rightleftarrows Top \colon Sing$ (see [GJ09, I]) is a G-Quillen equivalence for any finite group G.

Corollary 2.13: *A G-Quillen equivalence $L \colon \mathscr{C} \rightleftarrows \mathscr{D} \colon R$ induces a Quillen equivalence*

$$L \colon \mathscr{C}_a^I \rightleftarrows \mathscr{D}_a^I \colon R.$$

Proof. The adjunction $L \colon \mathscr{C}_a^I \rightleftarrows \mathscr{D}_a^I \colon R$ is a Quillen adjunction since the right adjoint preserves fibrations and acyclic fibrations, as they are defined point-wise. Let $X \in \mathscr{C}_a^I$ be cofibrant and $Y \in \mathscr{D}_a^I$ fibrant. A map $X \to R(Y)$ is an equivalence if and only if its adjoint $L(X) \to Y$ is, since by Proposition 2.10 X is point-wise cofibrant. □

2.3 Cofibrant replacement of G-diagrams

When $\mathscr{C}$ is a cofibrantly generated simplicial model category and I is a small category a standard way to replace a diagram $X \colon I \to \mathscr{C}$ by a cofibrant diagram is by the construction of Example 1.21. Namely, one defines qX by $qX_i = \mathrm{hocolim}_{I/i}(u_i^* X)$ where $u_i \colon I/i \to I$ is the functor that forgets the map to i. Then qX is cofibrant in the projective model structure on $\mathscr{C}^I$ and the natural map $\rho_X \colon qX \to X$ is a weak equivalence if X has cofibrant values in $\mathscr{C}$. In this section we will generalize this to G-diagrams as follows:

Theorem 2.14: *If X is a G-diagram such that for all i in I the value X_i is cofibrant in $\mathscr{C}^{G_i}$, then the map $\rho_X \colon qX \to X$ is a cofibrant replacement of G-diagrams in the sense that qX is cofibrant and ρ_X is a weak equivalence.*

The proof is technical and will occupy the rest of this section. We begin by fixing some notation. Let I be a small category with an action a of G. Write I^δ for the discrete category with the same objects as I but no non-identity morphisms. The inclusion $I^\delta \hookrightarrow I$ is equivariant and induces a restriction functor $r \colon \mathscr{C}_a^I \to \mathscr{C}_a^{I^\delta}$ with left adjoint r_*. We abbreviate $r(X)$ as X^δ. Note that the functor r preserves fibrations and weak equivalences and hence is a right Quillen functor. It follows that the left adjoint r_* is a left Quillen functor. We say that an I-indexed G-diagram X is *point-wise cofibrant* if for each object i in I the value X_i is cofibrant in $\mathscr{C}^{G_i}$.

Lemma 2.15: *i) If Y is an I^δ-indexed G-diagram which is point-wise cofibrant, then Y is cofibrant in $\mathscr{C}_a^{I^\delta}$.*

ii) In particular, if X is a point-wise cofibrant I-indexed G-diagram then $r_* X^\delta$ is cofibrant in $\mathscr{C}_a^I$.

Proof. To see that part i) holds, consider a square

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & Z \arrow[d, two heads, "f", "\sim"'] \\ Y \arrow[r] & W \end{tikzcd} \tag{5}$$

in $\mathscr{C}_a^{I^\delta}$, where the right hand vertical map is a trivial fibration and $\emptyset$ denotes the initial object. The map f being a trivial fibration means exactly that each component $f_i\colon Z_i \to W_i$ is a trivial fibration in $\mathscr{C}^{G_i}$. Choose a representative i of each G-orbit in obI. Each resulting square

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, tail] & Z_i \arrow[d, two heads, "f_i", "\sim"'] \\
Y_i \arrow[r] \arrow[ur, dashed, "\lambda_i"] & W_i
\end{tikzcd}$$

has a lift λ_i since Y_i is cofibrant and f_i is a trivial fibration in $\mathscr{C}^{G_i}$. For $g \in G$ define $\lambda_{gi} = g_{Z_i} \circ \lambda_i \circ g_{Y_i}^{-1}$. Then, if $gi = i$ the G_i-equivariance of the map λ_i says precisely that $\lambda_i = g_{Z_i} \circ \lambda_i \circ g_{Y_i}^{-1} = \lambda_{gi}$, so for all i and all $g \in G$ the map λ_{gi} is well-defined. It is now easy to see that the λ_{gi}'s assemble to a map of G-diagrams giving a lift in the square (5).

Part ii) follows immediately from part i) and the fact that r_* is a left Quillen functor and hence preserves cofibrancy of objects. $\square$

The adjunction (r_*, r) induces a comonad r_*r on $\mathscr{C}_a^I$ in the usual way. For a G-diagram X the value $(r_*r)X$ on i is

$$(r_*r)X_i = \coprod_{\alpha\colon j\to i} X_j.$$

The counit $\varepsilon\colon (r_*r)X \to X$ maps the X_j-component in the coproduct indexed by $\alpha\colon j \to i$ to X_i by the map $X(\alpha)$. The comultiplication $c\colon (r_*r)X \to (r_*rr_*r)X$ has as i-component the map

$$\coprod_{\alpha\colon j\to i} X_j \to \coprod_{\alpha\colon j\to i}\left(\coprod_{\alpha'\colon k\to j} X_k\right)$$

that maps the X_j-summand indexed by $\alpha\colon j \to i$ by the identity to the X_j-summand indexed by id_j in the α-summand of the target.

Let X be a G-diagram indexed on I. The bar construction on the comonad r_*r gives a simplicial G-diagram $B(r_*r)X$ with $B_n(r_*r)X = (r_*r)^{n+1}X$ so that

$$B_n(r_*r)X_i = \coprod_{\alpha_0\colon i_0\to i}\coprod_{\alpha_1\colon i_1\to i_0}\cdots\coprod_{\alpha_n\colon i_n\to i_{n-1}} X_{i_n} \cong \coprod_{i_n\to\cdots\to i_0\to i} X_{i_n}.$$

Note that for varying n the indexing G_i-simplicial set can be identified with $N(I/i)^{op}$. For

$$\sigma = i_n \xrightarrow{\alpha_n} \cdots \xrightarrow{\alpha_1} i_0 \xrightarrow{\alpha_0} i$$

in $N_n(I/i)^{op}$ the face map d_{n-k} for $k < 0$ composes the maps α_k and α_{k-1} and d_0 maps X_{i_n} to the $X_{i_{n-1}}$ indexed by $d_0(\sigma) \in N_{n-1}(I/i)^{op}$ by the map $X(\alpha_n)$. The degeneracy map s_n inserts an identity in the $(n-l)$-spot. Note that

$$\operatorname*{colim}_I r_*rX = \operatorname*{colim}_{I^\delta} rX = \coprod_i X_i,$$

so that $\operatorname{colim}_I B_n(r_*r)X \cong \coprod_{\sigma\in N_n(I^{op})} X_{\sigma(n)}$ and $\operatorname{colim}_I B(r_*r)X$ is isomorphic to the usual simplicial replacement $\coprod_* X$ of Bousfield and Kan [BK72] with G-action induced by the G-structure on X.

Proposition 2.16: *Let X be an I-indexed G-diagram. Then there are natural isomorphisms in $\mathscr{C}^G$*

i) $N(-/I)^{op} \otimes_I^a X \cong |\coprod_* X|$

ii) $|\coprod_* X| \cong \operatorname{colim}_I qX$.

Proof. To see i we first decompose the tensor product as an iterated coend (cf. [Rie13, §6.6])

$$N(-/I)^{op} \otimes^a_I X = \int^i N(i/I)^{op} \otimes X_i \cong \int^i \left(\int^{[n]} \Delta^n \times N_n\,(i/I)^{op} \right) \otimes X_i.$$

Here and in the rest of the proof we leave it to the reader to check that this is compatible with the G-structures on the diagrams. Rearranging the parentheses and switching the order of the coends gives the isomorphic object

$$\int^{[n]} \int^i \Delta^n \otimes (N_n\,(i/I)^{op} \otimes X_i) \cong \int^{[n]} \Delta^n \otimes \left(\int^i \coprod_{i \to i_n \to \cdots \to i_0} X_i \right).$$

Now we analyze the latter $\int^i$-factor. It is a coend of the G-diagram $I^{op} \times I \to \mathscr{C}$ given by

$$(i,j) \mapsto \coprod_{i \to i_n \to \cdots \to i_0} X_j.$$

This is isomorphic to the diagram

$$(i,j) \mapsto \coprod_{i_n \to \cdots \to i_0} I(i, i_n) \otimes X_j$$

and we note that since coends commute with colimits there is an isomorphism

$$\int^i \coprod_{i_n \to \cdots \to i_0} I(i, i_n) \otimes X_i \cong \coprod_{i_n \to \cdots \to i_0} \int^i I(i, i_n) \otimes X_i.$$

Here we must be careful since the representable functor $I(-, i_n)$ is not itself a G-diagram, but the coproduct $\coprod_{\sigma \in N_n(I^{op})} I(-, \sigma(n))$ of representable functors is. Finally, we observe that $\int^i I(i, i_n) \otimes X_i \cong X_{i_n}$ so that

$$\int^{[n]} \Delta^n \otimes \left(\int^i \coprod_{i \to i_n \to \cdots \to i_0} X_i \right) \cong \int^{[n]} \Delta^n \otimes \left(\coprod_{i_n \to \cdots \to i_0} X_{i_n} \right) = |\textstyle\coprod_* X|$$

To get the isomorphism in ii) we recall the isomorphism $\operatorname{colim}_I B(r_* r)X \cong \coprod_* X$. Since realization commutes with colimits, there are natural isomorphisms

$$|\textstyle\coprod_* X| \cong |\operatorname*{colim}_I B(r_* r)X| \cong \operatorname*{colim}_I |B(r_* r)X|.$$

Evaluating at i gives

$$|B(r_* r)X|_i = \left| [n] \mapsto \coprod_{i_n \to \cdots \to i_0 \to i} X_{i_n} \right| \cong \operatorname*{hocolim}_{I/i}(u_i^* X)$$

where the last isomorphism is an instance of i) for the G_i-diagram $u_i^* X \colon I/i \to \mathscr{C}$. This gives an isomorphism

$$\operatorname*{colim}_I |B(r_* r)X| \cong \operatorname*{colim}_I qX.$$

□

Lemma 2.17: *If X is a point-wise cofibrant G-diagram, then the simplicial object $B(r_* r)X$ is Reedy cofibrant in $(\mathscr{C}^I_a)^{\Delta^{op}}$.*

Proof. Let $L = L_n B(r_* r X)$ be the n-th latching object of $B(r_* r X)$. The natural map

$$L_n B(r_* r X) \to B_n(r_* r X) = B$$

is at each i in I the inclusion of the summands indexed by the degenerate n-simplices in $N_n(I/i)^{op}$ into the coproduct over all n-simplices. Thus B decomposes as a coproduct $B = L \amalg N$ where the value of N at i is the coproduct indexed over all the *non*-degenerate simplices of the nerve. The decomposition is clearly compatible with the G-diagram structure on each factor. The diagram N is obtained by applying r_* to a point-wise cofibrant I^δ-indexed G-diagram and is therefore cofibrant. It follows that the map $L \to B$ is a cofibration. □

Corollary 2.18: *If X is a point-wise cofibrant G-diagram, then qX is cofibrant.*

Proof. We know from the proof of Proposition 2.16 that qX is the realization of the simplicial object $B(r_* r)X$ which is Reedy cofibrant by Lemma 2.17. Since realization takes Reedy cofibrant objects to cofibrant objects [GJ09, VII,3.6] it follows that qX is cofibrant. □

Example 2.19. Let $*_I$ be the I-indexed G-diagram with value the terminal object $*$ of $sSet$. Then $q(*_I)_i = \operatorname{hocolim}_{I/i}(*_{I/i}) \cong N(I/i)^{op}$, so that $q(*_I) \cong N(I/-)^{op}$ and similarly $q(*_{I^{op}}) \cong N(-/I)$. By Corollary 2.18 it follows that the diagrams $N(I/-)$ and $N(-/I)^{op}$ are cofibrant as G-diagrams since $*$ is cofibrant in $sSet^{G_i}$ for all i in I and taking opposite simplicial sets preserves cofibrations. Further, let I and J be categories with respective G-actions a and b, and $F\colon I \to J$ an equivariant functor. Since the left Kan extension F_* preserves cofibrancy the diagrams $N(F/-) \cong F_* N(I/-)$ and $N(-/F)^{op} \cong F_* N(-/I)^{op}$ are also cofibrant in $sSet^J_b$.

Proof of Theorem 2.14. It only remains to see that the map ρ_X is a weak equivalence. For this we must show that for each i the map $\rho_{X_i}\colon \operatorname{hocolim}_{I/i} u_i^* X \to X_i$ is a weak equivalence in $\mathscr{C}^{G_i}$. The functor $\iota\colon * \to I/i$ sending the unique object to the terminal object is homotopy cofinal in the sense of Definition 2.24, so by Theorem 2.25 the map $X_i = \operatorname{hocolim}_* \iota^* u_i^* X \to \operatorname{hocolim}_{I/i} u_i^* X$ is a weak equivalence. Since it is also section to the map ρ_{X_i} it follows by the two out of three property that ρ_{X_i} is a weak equivalence as well. □

2.4 Homotopy invariance of map, tensor and of homotopy (co)limits

In this section $\mathscr{C}$ is a G-model category in the sense of definition 2.1, and a is a G-action on a small category I.

Proposition 2.20: *Let $X \in \mathscr{C}^I_a$ be a G-diagram in $\mathscr{C}$. If X is fibrant, the functor*

$$map^a_I(-, X)\colon (sSet^I_a)^{op} \longrightarrow \mathscr{C}^G$$

preserves equivalences of cofibrant objects (in $sSet^I_a$). Dually, if X is point-wise cofibrant, the functor

$$(-) \otimes^a_I X\colon sSet^{I^{op}}_a \longrightarrow \mathscr{C}^G$$

preserves equivalences of cofibrant objects.

Proof. We prove the statement for map^a_I, the proof for $\otimes^a_I$ is similar. Let $K \to L$ be an equivalence of cofibrant diagrams in $sSet^I_a$. By Ken Brown's Lemma we can assume that $K \to L$ is a cofibration (cf. [Hir03, 7.7.1]). To show that the induced map is an equivalence, we need to solve the lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & map^a_I(L,X) \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & map^a_I(K,X) \end{tikzcd}$$

for every cofibration $A \to B$ in $\mathscr{C}^G$. Let $Map_{\mathscr{C}}(B,X)$ be the G-diagram in $sSet$ given by $i \mapsto Map_{\mathscr{C}}(B,X_i)$ and where the G-structure is given by the maps $Map_{\mathscr{C}}(g^{-1},g_{X_i})\colon Map_{\mathscr{C}}(B,X_i) \to Map_{\mathscr{C}}(B,X_{gi})$. The adjunction isomorphism

$$\underline{\mathscr{C}}^G(B, map_I^a(L,X)) \cong \underline{sSet}_a^I(L, Map_{\mathscr{C}}(B,X)).$$

is equivariant, and therefore the lifting problem above is equivalent to the lifting problem in $sSet_a^I$

$$\begin{tikzcd} K \arrow[r] \arrow[d, tail, "\simeq"'] & Map_{\mathscr{C}}(B,X) \arrow[d] \\ L \arrow[r] \arrow[ur, dashed] & Map_{\mathscr{C}}(A,X) \end{tikzcd}$$

This can be solved if $Map_{\mathscr{C}}(B,X) \to Map_{\mathscr{C}}(A,X)$ is a fibration in $sSet_a^I$, i.e., if for every object $i \in I$ the map $Map_{\mathscr{C}}(B,X_i) \to Map_{\mathscr{C}}(A,X_i)$ is a fibration of simplicial G_i-sets. By assumption X_i is fibrant in $\mathscr{C}^{G_i}$ and $A \to B$ restricts to a cofibration in $\mathscr{C}^{G_i}$, so by axiom $SM7$ for the $sSet^{G_i}$-enriched model category $\mathscr{C}^{G_i}$ the map is a fibration. □

Proposition 2.21: *If K is a cofibrant diagram in $sSet_a^I$, the functor*

$$map_I^a(K,-)\colon \mathscr{C}_a^I \longrightarrow \mathscr{C}^G$$

preserves equivalences of fibrant objects. Dually if K is cofibrant in $sSet_a^{I^{op}}$, the functor

$$K \otimes_I^a (-)\colon \mathscr{C}_a^I \longrightarrow \mathscr{C}^G$$

preserves equivalences of point-wise cofibrant objects.

Proof. The proof is the same as for the non-equivariant case of [Hir03, 18.4], using the equivariant adjunctions as in the proof of 2.20. □

The following result generalizes Villarroel's result [VF04, 6.1]:

Corollary 2.22: *The functors* $\operatorname{holim}\colon \mathscr{C}_a^I \to \mathscr{C}^G$ *and* $\operatorname{hocolim}\colon \mathscr{C}_a^I \to \mathscr{C}^G$ *preserve equivalences between fibrant G-diagrams and point-wise cofibrant G-diagrams respectively.*

Proof. Recall that homotopy limits and homotopy colimits are defined by cotensoring with $N(I/-)$ and tensoring with $N(-/I)^{op}$, respectively. By Proposition 2.21 it is enough to show that $N(I/-)$ is cofibrant in $sSet_a^I$ and $N(-/I)^{op}$ is cofibrant in $sSet_a^{I^{op}}$. This was shown in Example 2.19. □

For an equivariant functor $F\colon I \to J$ between categories with G-actions a and b respectively define the *homotopy left Kan extension* of a G-diagram X in $\mathscr{C}_a^I$ by

$$(\operatorname{ho} F_* X)_j = \operatorname{hocolim}(F/j \to I \xrightarrow{X} \mathscr{C})$$

with the induced G-structure. The usual homotopy colimit $\operatorname{hocolim}_I$ is the homotopy left Kan extension along the functor $I \to *$. Using the simplicial resolution $B(r^*r)X$ of Section 2.3 it is not hard to see that there is a natural isomorphism $\operatorname{ho} F_* X \cong F_*(qX)$.

Lemma 2.23: *(Transitivity of homotopy left Kan extensions) Let $F\colon I \to J$ and $F'\colon J \to K$ be equivariant functors between small categories with G-actions a, b and c, respectively. If X is a pointwise cofibrant object in $\mathscr{C}_a^I$ then the natural map*

$$\operatorname{ho} F'_*(\operatorname{ho} F_* X) \to \operatorname{ho}(F' \circ F)_* X$$

is a weak equivalence in $\mathscr{C}_c^K$. *In particular, if* $K = *$ *then there is a weak equivalence*

$$\operatorname*{hocolim}_J(\mathrm{ho}\, F_* X) \xrightarrow{\sim} \operatorname*{hocolim}_I X.$$

Proof. Since X is pointwise cofibrant the diagram qX is cofibrant and so $\mathrm{ho}\, F_* X \cong F_* qX$ is cofibrant as well, since F_* preserves cofibrancy. The functor F'_* preserves weak equivalences between cofibrant objects, so the natural map $F'_*(q\, \mathrm{ho}\, F_* X) \to F'_*(\mathrm{ho}\, F_* X)$ is a weak equivalence. The map in the lemma is the composite of the natural maps

$$\mathrm{ho}\, F'_*(\mathrm{ho}\, F_* X) \xrightarrow{\cong} F'_*(q\, \mathrm{ho}\, F_* X) \xrightarrow{\sim} F'_*(\mathrm{ho}\, F_* X) \xrightarrow{\cong} F'_*(F_* qX) \xrightarrow{\cong} \mathrm{ho}(F' \circ F)_* X,$$

where the second map is a weak equivalence by the discussion above. □

2.5 Equivariant cofinality

Let I and J be categories with respective G-actions a and b, $F \colon I \to J$ an equivariant functor, and $X \colon J \to \mathscr{C}$ a G-diagram.

We want to know when the canonical maps

$$\operatorname*{hocolim}_I F^* X \longrightarrow \operatorname*{hocolim}_J X \qquad \text{and} \qquad \operatorname*{holim}_J X \longrightarrow \operatorname*{holim}_I F^* X$$

are equivalences in $\mathscr{C}^G$. As in the non-equivariant setting, the categories F/j and j/F play a role in answering this question. For every object $j \in J$ these categories inherit a canonical action by the stabilizers group $G_j \leq G$ of j.

Definition 2.24. The functor $F \colon I \to J$ is left (resp. right) cofinal if for every $j \in J$ the nerve of the category F/j (resp. j/F) is weakly G_j-contractible.

Notice that for $H \leq G_i$, the H-fixed points of the nerve of F/j are isomorphic to the nerve of $(F/j)^H$. Therefore F is left cofinal if and only if the fixed categories $(F/j)^H$ are contractible for all $H \leq G_i$, and similarly for right cofinality.

The following cofinality theorem is a generalization of [TW91, 1] and [VF04, 6.3].

Theorem 2.25: *Let* $\mathscr{C}$ *be a* G*-model category,* $F \colon I \to J$ *be an equivariant functor, and* $X \in \mathscr{C}_b^J$ *a* G*-diagram in* $\mathscr{C}$*. If* F *is left cofinal and* X *is fibrant, the canonical map*

$$\operatorname*{holim}_J X \longrightarrow \operatorname*{holim}_I F^* X$$

is an equivalence in $\mathscr{C}^G$*. Dually, if* F *is right cofinal and* X *is point-wise cofibrant, the map*

$$\operatorname*{hocolim}_I F^* X \longrightarrow \operatorname*{hocolim}_J X$$

is an equivalence in $\mathscr{C}^G$*.*

Proof. We prove the part of the statement about left cofinality. The map $\mathrm{holim}_J X \to \mathrm{holim}_I F^* X$ factors as

$$map_J^b(NJ/(-), X) \xrightarrow{\cong} map_J^b(NF/(-), X) \to map_I^a(NI/(-), F^* X).$$

The first map is a cotensor version of the (F_*, F^*)-adjunction isomorphism. It is equivariant and it is showed to be an isomorphism in [Hir03, 19.6.6]. The second map is induced by the projection map $NF/(-) \to NJ/(-)$ which is an equivalence in $sSet_b^J$, since for all $H \leq G$ and all object $j \in J^H$ both categories F/j^H and J/j^H are contractible (J/j^H has a final object). Moreover, the G-diagrams $NJ/(-)$ and $NF/(-)$ are cofibrant in $sSet_a^J$, by Example 2.19. Therefore the induced map on mapping objects is an equivalence by the homotopy invariance of map_J^b of Proposition 2.20.

□

As an application of cofinality we prove a “twisted Fubini theorem” for homotopy colimits, describing the homotopy colimit of a G-diagram indexed over a Grothendieck construction. The classical version can be found in [CS02, 26.5]. Let I be a category with G-action and $\Psi \in Cat_a^I$ a G-diagram of small categories. The Grothendieck construction $I \wr \Psi$ of the underlying diagram of categories inherits a G-action, defined on objects by

$$g \cdot \big(i, c \in Ob\Psi(i)\big) = \big(gi, g_* c \in \Psi(gi)\big)$$

and sending a morphism $(\alpha\colon i \to j, \gamma\colon \Psi(\alpha)(c) \to d)$ from (i,c) to (j,d) to the morphism

$$g \cdot \big(\alpha, \gamma\big) = \big(g\alpha\colon gi \to gj, \Psi(g\alpha)(gc) = g\Psi(\alpha) \xrightarrow{g\gamma} gd\big)$$

Now let $X \in \mathscr{C}_a^{I\wr\Psi}$ be a G-diagram in a G-model category $\mathscr{C}$. This induces a G-diagram $I \to \mathscr{C}$ defined at an object i of I by $\operatorname{hocolim}_{\Psi(i)} X|_{\Psi(i)}$, where X is restricted along the canonical inclusion $\iota_i\colon \Psi(i) \to I \wr \Psi$. The G-structure is given by the maps

$$\operatorname*{hocolim}_{\Psi(i)} X|_{\Psi(i)} \xrightarrow{g} \operatorname*{hocolim}_{\Psi(i)} X|_{\Psi(gi)} \circ g \xrightarrow{g_*} \operatorname*{hocolim}_{\Psi(gi)} X|_{\Psi(gi)}$$

where the first map is induced by the natural transformation of $\Psi(i)$-diagrams $X|_{\Psi(i)} \to X|_{\Psi(gi)} \circ g$ provided by the G-structure on X, and the second map is the canonical map induced by the functor on indexing categories $g\colon \Psi(i) \to \Psi(gi)$.

Corollary 2.26: *For every point-wise cofibrant G-diagram $X \in \mathscr{C}_a^{I\wr\Psi}$ there is a natural equivariant weak equivalence*

$$\eta\colon \operatorname*{hocolim}_I \operatorname*{hocolim}_{\Psi(-)} X|_{\Psi(-)} \xrightarrow{\simeq} \operatorname*{hocolim}_{I\wr\Psi} X.$$

Remark 2.27. When $\mathscr{C}$ is the G-model category of spaces with the fixed point model structures and $X\colon I \wr \Psi \to Top$ is the constant one point diagram the corollary gives a G-equivalence

$$|N(I \wr \Psi)| \xrightarrow{\simeq} \operatorname*{hocolim}_{i\in I} |N\Psi(i)|$$

analogous to Thomason's theorem [Tho79]. Our proof is modeled on Thomason's proof.

Proof of 2.26. Let $p\colon I\wr\Psi \to I$ be the canonical projection. We start by defining a zig-zag of equivalences

$$\operatorname*{hocolim}_I \operatorname*{hocolim}_{\Psi(-)} X|_{\Psi(-)} \xleftarrow{\lambda_1} \operatorname*{hocolim}_I \operatorname{ho} p_* X \xrightarrow{\lambda_2} \operatorname*{hocolim}_{I\wr\Psi} X,$$

where $\operatorname{ho} p_*$ denotes homotopy left Kan extension, and λ_2 is the equivalence of transitivity of homotopy left Kan extensions 2.23.

For an object i of I define the functor $F_i\colon p/i \to \Psi(i)$ by $F_i(j, c, f\colon j \to i) = \Psi(f)(c)$ on objects and on morphisms from $(j, c, f_0\colon j \to i)$ to $(k, d, f_1\colon k \to i)$ by

$$F_i\big(h\colon j \to k, \alpha\colon \Psi(h)(c) \to d\big) = \Psi(f_1)(\alpha)\colon \Psi(f_0)(c) \to \Psi(f_1)(d)$$

The canonical functor $p/i \to I \wr \Psi$ used to define the homotopy left Kan extension $(\operatorname{ho} p_* X)_i$ factors as $p/i \xrightarrow{F_i} \Psi(i) \xrightarrow{\iota_i} I \wr \Psi$. This factorization induces a map $\gamma_i\colon (\operatorname{ho} p_* X)_i \to \operatorname{hocolim}_{\Psi(i)} X|_{\Psi(i)}$ which is natural in i and is compatible with the G-structures and hence defines a map of I-indexed G-diagrams $\gamma\colon \operatorname{ho} p_* X \to \operatorname{hocolim}_{\Psi(-)} X|_{\Psi(-)}$. This induces the map

$$\lambda_1\colon \operatorname*{hocolim}_I \operatorname{ho} p_* X \longrightarrow \operatorname*{hocolim}_I \operatorname*{hocolim}_{\Psi(-)} X|_{\Psi(-)}$$

in the zig-zag. Let us see that this is an equivalence. For an object c of $\Psi(i)$ the right fiber c/F_i has a $(G_i)_c$-invariant initial object and is therefore contractible. It follows by cofinality 2.25 that the maps

γ_i are weak G_i-equivalences. By homotopy invariance of homotopy colimits the induced map λ_1 is a G-equivalence.

It remains to introduce the map $\eta\colon \operatorname{hocolim}_{I\wr\Psi} X \to \operatorname{hocolim}_I \operatorname{hocolim}_{\Psi(-)} X|_{\Psi(-)}$ from the statement, and compare it with the zig-zag. It is defined using the simplicial replacements from §2.3. The iterated homotopy colimit $\operatorname{hocolim}_I \operatorname{hocolim}_{\Psi(-)} X_{\Psi(-)}$ is isomorphic to the realization of the simplicial $\mathscr{C}^G$-object

$$[p] \mapsto \coprod_{k_p\to\cdots\to k_0,\, i_p\xrightarrow{f_p}\cdots\xrightarrow{f_1} i_0} X_{(i_p,k_p)},$$

where the indexing strings of maps are in $N_p\Psi(i_p)^{op}$ and N_pI^{op}, respectively. The map η in level p maps a summand $X_{(i_p,k_p)}$ by the identity map to the summand of

$$\coprod_{\sigma\in N_p(I\wr\Psi)^{op}} X_{\sigma(p)},$$

indexed by the p-simplex $(i_p,k_p)\to(i_{p-1},\Psi(f_p)(k_{p-1}))\to\cdots\to(i_0,\Psi(f_p\cdots f_1)(k_0))$ of $N(I\wr\Psi)^{op}$. Just as in Thomason's original proof there is a simplicial homotopy from $\eta\circ\lambda_2$ to λ_1 and it follows that η is a weak equivalence (see in particular [Tho79, Lemma 1.2.5]).

□

2.6 The Elmendorf theorem for G-diagrams

Let $\mathscr{C}$ be a cofibrantly generated model category with cellular fixed points, in the sense of [GM13]. Then the category $\mathscr{C}^G$ of G-object admits the fixed point model structure, where weak equivalences and fibrations are the equivariant maps whose H-fixed points are weak equivalences and fibrations in $\mathscr{C}$, respectively, for every subgroup $H\leq G$. Let $\mathcal{O}_G$ be the orbit category of G, with quotient sets G/H as objects and equivariant maps as morphisms. Elmendorf's theorem (see [Ste10], [Elm83]) describes a Quillen equivalence

$$L\colon \mathscr{C}^{\mathcal{O}_G^{op}} \rightleftarrows \mathscr{C}^G\colon R$$

where the diagram category $\mathscr{C}^{\mathcal{O}_G^{op}}$ has the projective model structure. In this section we prove an analogous result, giving a Quillen equivalence between the category of G-diagrams in $\mathscr{C}$ and a category of diagrams with the projective model structure.

Let I be a small category with an action a of G. For convenience we will consider the category of G-diagrams in $\mathscr{C}$ as the category $\mathscr{C}^{G\rtimes_a I}$ of diagrams indexed over the Grothendieck construction of the action (see 1.9). The functor $a\colon G\to Cat$ induces a functor $\overline{a}\colon \mathcal{O}_G^{op}\to Cat$ that sends G/H to the category I^H of objects and morphisms of I fixed by the H-action. We denote its Grothendieck construction by $\mathcal{O}_G^{op}\rtimes_{\overline{a}} I$. The inclusion functor $G\to\mathcal{O}_G^{op}$ that sends the unique object to $G/1$ induces a functor $G\rtimes_a I\to\mathcal{O}_G^{op}\rtimes_{\overline{a}} I$, which itself induces a restriction functor

$$L\colon \mathscr{C}^{\mathcal{O}_G^{op}\rtimes_{\overline{a}} I}\longrightarrow \mathscr{C}^{G\rtimes_a I}$$

Recall from 2.4 that if the fixed point functors of $\mathscr{C}$ are cellular, the fixed point model structures on $\mathscr{C}^H$, for $H\leq G$, assemble into a G-model category.

Theorem 2.28: *Let $\mathscr{C}$ be a category such that the fixed points functors for the subgroups of G are cellular. The functor $L\colon \mathscr{C}^{\mathcal{O}_G^{op}\rtimes_{\overline{a}} I}\to\mathscr{C}^{G\rtimes_a I}$ is the left adjoint of a Quillen equivalence*

$$L\colon \mathscr{C}^{\mathcal{O}_G^{op}\rtimes_{\overline{a}} I}\rightleftarrows \mathscr{C}^{G\rtimes_a I}\colon R$$

where $\mathscr{C}^{G\rtimes_a I}$ has the model structure of 2.6 and $\mathscr{C}^{\mathcal{O}_G^{op}\rtimes_{\overline{a}} I}$ has the projective model structure.

Proof. The right adjoint sends a G-diagram X in $\mathscr{C}^{G\rtimes_a I}\cong \mathscr{C}^I_a$ to the diagram $R(X)\colon \mathcal{O}_G^{op}\rtimes_{\overline{a}} I\to \mathscr{C}$ that sends an object $(G/H, i\in I^H)$ to

$$R(X)(G/H,i\in I^H)=X_i^H$$

In order to define $R(X)$ on morphisms, recall that the set of equivariant maps $G/K\to G/H$ is in natural bijection with $(G/H)^K$. A morphism in $\mathcal{O}_G^{op}$ from $(G/H,i)$ to $(G/K,j)$ is a pair $(z\in (G/H)^K, (\alpha\colon zi\to j)\in I^K)$, which is sent to the composite

$$X_i^H \xrightarrow{z} X_{zi}^K \xrightarrow{\alpha_*^K} X_j^K$$

A morphism $f\colon X\to Y$ in $\mathscr{C}^I_a$ is sent to the natural transformation with value $X_i^H\xrightarrow{f_i^H} Y_i^H$ at the object $(G/H,i\in I^H)$. It is straightforward to see that R is a right adjoint for L. The counit $LRX\to X$ is an isomorphism, and the unit at a diagram Z of $\mathscr{C}^{\mathcal{O}_G^{op}\rtimes_{\overline{a}} I}$ is the natural transformation

$$\eta_Z\colon Z(G/H,i)\longrightarrow RL(Z)(G/H,i)=Z(G/1,i)^H$$

induced by the morphism $(H\in (G/H)^1,\mathrm{id}_i)\colon (G/H,i)\to (G/1,i)$ of $\mathcal{O}_G^{op}\rtimes_{\overline{a}} I$. By definition of the fixed point model structure and of the model structure on $\mathscr{C}^{G\rtimes_a I}$, the right adjoint R preserves and detects equivalences and fibrations. Thus the adjunction (L,R) is a Quillen pair.

Since R preserves and detects equivalences, (L,R) is a Quillen equivalence precisely if the unit $\eta_Z\colon Z\to RL(Z)$ is an equivalence for all cofibrant objects Z in $\mathscr{C}^{\mathcal{O}_G^{op}\rtimes_{\overline{a}} I}$. We prove this following the argument of [Ste10]. By cellularity of the fixed point functors RL preserves pushouts along generating cofibrations and directed colimits along point-wise cofibrations. Thus it is enough to show that η_Z is an isomorphism when Z is a generating cofibrant object, that is, an object of the form

$$Z=\hom_{\mathcal{O}_G^{op}\rtimes_{\overline{a}} I}((G/H,i),-)\otimes c$$

for fixed objects $(G/H,i)$ of $\mathcal{O}_G^{op}\rtimes_{\overline{a}} I$ and c of $\mathscr{C}$ cofibrant. For such a Z, the unit at an object $(G/K,j)$ is the top horizontal map of the commutative diagram

$$\begin{tikzcd}
\hom_{\mathcal{O}_G^{op}\rtimes_{\overline{a}} I}((G/H,i),(G/K,j))\otimes c \arrow[r, "\eta"] \arrow[d, "\cong"'] & (\hom_{\mathcal{O}_G^{op}\rtimes_{\overline{a}} I}((G/H,i),(G/e,j))\otimes c)^K \arrow[d, "\cong"] \\
\{(z\in (G/H)^K,\alpha\colon (zi\to j)\in I^K)\}\otimes c \arrow[r] \arrow[d, equal] & \big(\{(z\in G/H,\alpha\colon (zi\to j)\in I)\}\otimes c\big)^K \arrow[d, equal] \\
\Lambda_{ij}^K\otimes c \arrow[r] & (\Lambda_{ij}\otimes c)^K
\end{tikzcd}$$

where Λ_{ij} is the set of pairs $(z\in G/H,\alpha\in zi\to j)$ with K acting by left multiplication on G/H and by the category action on the map to j (notice that j belongs to I^K). The bottom horizontal map is an isomorphism by the cellularity conditions on the K-fixed points functor. □

For the G-model category of spaces, the Elmendorf theorem gives a description of the fixed points of the homotopy limit of a G-diagram as a space of natural transformations of diagrams.

Corollary 2.29: *For every G-diagram of spaces X in Top^I_a, there is a natural homeomorphism of spaces*

$$(\operatorname*{holim}_I X)^G\cong Map_{Top^{\mathcal{O}_G^{op}\rtimes_{\overline{a}} I}}\big(R(BI/(-)),R(X)\big)$$

where $R(X)\colon \mathcal{O}_G^{op}\rtimes_{\overline{a}} I\to Top$ has vertices $R(X)_{(G/H,i)}=X_i^H$.

Proof. The space $(\mathrm{holim}_I X)^G$ is by definition the mapping space from $BI/(-)$ to X in Top^I_a. As the counit of the adjunction of the Elmendorf theorem is an isomorphism, there is a sequence of natural homeomorphisms

$$Map_{Top^I_a}\big(BI/(-), X\big) \cong Map_{Top^I_a}\big(LR(BI/(-)), X\big) \cong Map_{Top^{\mathcal{O}_G^{op}\rtimes_{\overline{a}} I}}\big(R(BI/(-)), R(X)\big)$$

□

3 Equivariant excision

We use the homotopy theory of G-diagrams developed earlier in the paper to set up a theory of G-excisive homotopy functors.

Classical excision is formulated using cartesian and cocartesian squares, and captures the behavior of homology theories. Blumberg points out in [Blu06] that in the equivariant setting, squares of G-objects are not enough to capture the behavior of equivariant homology theories. In the rest of the paper we explain how to replace squares by cubical G-diagrams to fund a good theory of equivariant excision. We point out that this has already been achieved in [Blu06] in the category of based G-spaces. We prove in 3.28 that our approach and Blumberg's are equivalent in this category.

3.1 Equivariant cubes and G-excision

If J is a finite G-set, the poset category of subsets of J ordered by inclusion $\mathcal{P}(J)$ has a canonical G-action, where a group element $g \in G$ sends a subset $U \subset J$ to the set

$$g \cdot U = \{g \cdot u \mid u \in U\}$$

Let $\mathscr{C}$ be a G-model category (cf. 2.1).

Definition 3.1. The category of J-cubes in $\mathscr{C}$ is the category of G-diagrams $\mathscr{C}_a^{\mathcal{P}(J)}$ for the action a on $\mathcal{P}(J)$ described above.

In order to define a homotopy invariant notion of (co)cartesian cubes, we need to make our homotopy (co)limits homotopy invariant. Given a cube $X \in \mathscr{C}_a^{\mathcal{P}(J)}$ let FX denote a fibrant J-cube together with an equivalence $X \overset{\simeq}{\to} FX$. Similarly let $QX \overset{\simeq}{\to} X$ denote an equivalence with QX point-wise cofibrant, that is, with QX_U cofibrant in $\mathscr{C}^{G_U}$ for every $U \in \mathcal{P}(J)$.

Remark 3.2. To find a replacement FX one can simply use the fibrant replacement in the model category $\mathscr{C}_a^{\mathcal{P}(J)}$. Similarly, a cofibrant replacement QX in $\mathscr{C}_a^{\mathcal{P}(J)}$ is in particular point-wise cofibrant by 2.10. However, for a given cube one can often find a more explicit point-wise cofibrant replacement that is not necessarily cofibrant in $\mathscr{C}_a^{\mathcal{P}(J)}$ (see e.g. 3.4 and 3.5 below). For example, if a functorial cofibrant replacement Q in $\mathscr{C}$ lifts to a cofibrant replacement in $\mathscr{C}^H$ for every $H \leq G$, the diagram QX is point-wise cofibrant.

For an object i of I fixed by the G-action, let $I\backslash i$ be the full subcategory of I with objects different from i. The action on I restricts to $I\backslash i$, and the inclusion functor $\iota_i \colon I\backslash i \to I$ is equivariant.

Definition 3.3. Let $\mathscr{C}$ be a G-model category and J a finite G-set. A J-cube $X \in \mathscr{C}_a^{\mathcal{P}(J)}$ is homotopy cocartesian if the canonical map

$$\operatorname*{hocolim}_{\mathcal{P}(J)\backslash J} \iota_J^* QX \longrightarrow QX_J \overset{\simeq}{\to} X_J$$

is an equivalence in $\mathscr{C}^G$. Dually, $X \in \mathscr{C}_a^{\mathcal{P}(J)}$ is homotopy cartesian if the canonical map

$$X_\emptyset \overset{\simeq}{\to} FX_\emptyset \longrightarrow \operatorname*{holim}_{\mathcal{P}(J)\backslash \emptyset} \iota_\emptyset^* FX$$

is an equivalence in $\mathscr{C}^G$.

Example 3.4. Let J be a finite G-set, and J_+ be the G-set J with a disjoint fixed base point. For a cofibrant object $c \in \mathscr{C}^G$ define a J_+-cube $S^J c$ with verticies

$$(S^J c)_U = \begin{cases} c & ,U = \emptyset \\ C^U c & ,U \lneq J_+ \\ \Sigma^J c & ,U = J_+ \end{cases}$$

Here $\Sigma^J c = \Sigma^{\tilde{J}_+} c$ is the suspension by the permutation representation of J defined in 1.19, and $C^U c$ denotes the U-iterated cone

$$C^U c = \underset{\mathcal{P}(U)}{\mathrm{hocolim}} \left(S \longmapsto \begin{cases} c & \text{if } S = \emptyset \\ * & \text{otherwise} \end{cases} \right) \simeq *$$

Since c is cofibrant, $S^J c$ is point-wise cofibrant. Let us prove that it is homotopy cocartesian. Its restriction to $\mathcal{P}(J_+)\backslash J_+$ is the cofibrant replacement q of Theorem 2.14 for the diagram $\sigma^J c\colon \mathcal{P}(J_+)\backslash J_+ \to \mathscr{C}$ with $(\sigma^J c)_\emptyset = c$ and the terminal object at the other vertices. Since homotopy colimits and colimits agree on cofibrant objects (by the homotopy invariance of $\otimes_I^q$), the canonical map from the homotopy colimit factors as the equivalence

$$\underset{\mathcal{P}(J_+)\backslash J_+}{\mathrm{hocolim}}\, S^J c = \underset{\mathcal{P}(J_+)\backslash J_+}{\mathrm{hocolim}}\, q(\sigma^J c) \xrightarrow{\simeq} \underset{\mathcal{P}(J_+)\backslash J_+}{\mathrm{colim}}\, q(\sigma^J c) \cong \underset{\mathcal{P}(J_+)\backslash J_+}{\mathrm{hocolim}}\, \sigma^J c = \Sigma^J c$$

Example 3.5. Suppose that $\mathscr{C}$ has a zero object $*$ and denote the coproduct by $\bigvee$. Let c be a cofibrant object of $\mathscr{C}^G$ and J a finite G-set. Define a J-cube $W^J c$ with vertices

$$(W^J c)_U = \begin{cases} \bigvee_J c & ,U = \emptyset \\ c & ,|U| = 1 \\ * & ,|U| \geq 2 \end{cases}$$

with initial map $(W^J c)_\emptyset = \bigvee_J c \to c = (W^J c)_{\{j\}}$ the pinch map that collapses every wedge component different from j. This has a G-structure defined by the action on $\bigvee_J c$ on the initial vertex, and by the action maps $g\colon (W^J c)_{\{j\}} = c \to c = (W^J c)_{\{gj\}}$. The cube $W^J c$ is homotopy cocartesian, that is, its homotopy colimit over $\mathcal{P}(J)\backslash J$ is equivalent in $\mathscr{C}^G$ to the zero object. To see this, we replace $W^J c$ by the equivalent cube

$$(\overline{W}c)_U = \begin{cases} \bigvee\limits_J c & ,U = \emptyset \\ c \bigvee\limits_{J\backslash j} Cc & ,U = \{j\} \\ \bigvee\limits_J Cc & ,|U| \geq 2 \end{cases}$$

where Cc is the one-fold cone $Cc = \mathrm{hocolim}(c \to *)$ and the non-identity maps of the diagram are all induced by cone inclusions $c \to Cc$. The G-structure is defined similarly as before, by permuting the wedge components. The cube $\overline{W}c$ is cofibrant, since the latching maps are all cofibrations (see A.6). As homotopy colimits preserve equivalences of point-wise cofibrant diagrams we get

$$\underset{\mathcal{P}(J)\backslash J}{\mathrm{hocolim}}\, W^J c \xleftarrow{\simeq} \underset{\mathcal{P}(J)\backslash J}{\mathrm{hocolim}}\, \overline{W}c \xrightarrow{\simeq} \underset{\mathcal{P}(J)\backslash J}{\mathrm{colim}}\, \overline{W}c \cong \bigvee_J Cc$$

This is contractible since $\bigvee_J$ is a left Quillen functor and therefore preserves equivalences of cofibrant objects.

We use homotopy cartesian and cocartesian G_+-cubes to express equivariant excision for functors between G-model categories $\mathscr{C}$ and $\mathscr{D}$. We shall consider functors for which we can express compatibility conditions with the model structures on $\mathscr{C}^H$ and $\mathscr{D}^H$ for every subgroup $H \leq G$. These are functors $\Phi\colon \mathscr{C} \to \mathscr{D}^G$. Such a functor Φ induces a functor $\Phi_*\colon \mathscr{C}_a^I \to \mathscr{D}_a^I$ for any category with G-action I. The G-structure on $\Phi_*(X) = \Phi \circ X$ is defined by the maps

$$\Phi(X_i) \xrightarrow{g} \Phi(X_i) \xrightarrow{\Phi(g)} \Phi(X_{gi})$$

Since each map $\Phi(g)$ is G-equivariant $\Phi(g)g = g\Phi(g)$. For $I = *$ the trivial category this functor is the classical extension $\Phi_*\colon \mathscr{C}^G \to \mathscr{D}^G$. Similarly, the functor $\Phi\colon \mathscr{C} \to \mathscr{D}^H$ obtained by restricting the G-action to $H \leq G$, extends to a functor $\Phi_*\colon \mathscr{C}^H \to \mathscr{D}^H$.

Definition 3.6. We call $\Phi\colon \mathscr{C} \to \mathscr{D}^G$ a homotopy functor if for every subgroup $H \leq G$ the extended functor $\Phi_*\colon \mathscr{C}^H \to \mathscr{D}^H$ preserves equivalences of cofibrant objects. In particular the induced functor $\Phi_*\colon \mathscr{C}_a^I \to \mathscr{D}_a^I$ preserves equivalences of point-wise cofibrant G-diagrams.

Remark 3.7. The following are all examples of functors $\mathscr{C}^G \to \mathscr{D}^G$ that are extensions of homotopy functors $\mathscr{C} \to \mathscr{D}^G$.

- The identity functor $\mathscr{C}^G \to \mathscr{C}^G$,
- For a fixed pointed G-space K, the functors $K \wedge (-), Map_*(K, -)\colon Top_*^G \to Top_*^G$,
- For a fixed orthogonal G-spectrum E the functor $E \wedge (-)\colon Top_*^G \to (\mathrm{Sp}^O)^G$.

An example of a functor $\mathscr{C}^G \to \mathscr{D}^G$ that is not the extension of a functor $\mathscr{C} \to \mathscr{D}^G$ is the functor $(-)/G\colon Top^G \to Top^G$ that sends a G-space to its orbit space with trivial G-action.

Definition 3.8. Let $\mathscr{C}$ and $\mathscr{D}$ be G-model categories. A homotopy functor $\Phi\colon \mathscr{C} \to \mathscr{D}^G$ is called G-excisive if the induced functor $\Phi_*\colon \mathscr{C}_a^{\mathcal{P}(G_+)} \to \mathscr{D}_a^{\mathcal{P}(G_+)}$ sends homotopy cocartesian G_+-cubes to homotopy cartesian G_+-cubes. If $\mathscr{C}$ and $\mathscr{D}$ are pointed, Φ is called G-linear if it is G-excisive and $\Phi(*)$ is equivalent to the zero object in $\mathscr{D}^G$.

The choice of indexing the cubes on the G-set G_+ seems arbitrary at first sight. We justify and explain this choice, including the extra basepoint added to G, in 3.10 and 3.12 below.

Example 3.9. The following are examples of G-linear homotopy functors, as we will see later in the paper.

- Let M be an abelian group with additive G-action. Consider the homotopy functor $M(-)\colon sSet_* \to sSet_*^G$ that sends a simplicial set Z to
$$M(Z)_n = \bigoplus_{z \in Z_n} Mz/M*$$
where G acts diagonally on the direct summands. We show in 3.30 that this functor is G-linear, and explain how this is related to the equivariant Eilenberg-MacLane spectrum HM being a fibrant orthogonal G-spectrum. The homotopy groups of the extension of $M(-)$ to $sSet_*^G$ are Bredon cohomology of the Mackey functor $H \mapsto M^H$.
- For a fixed orthogonal G-spectrum E in $(\mathrm{Sp}^O)^G$, the homotopy functor $E \wedge (-)\colon Top_* \to (\mathrm{Sp}^O)^G$ is G-linear (see 3.33). The stable homotopy groups of the extension of $E \wedge (-)$ to pointed G-spaces is the equivariant cohomology theory associated to E.
- The inclusion of spectra with trivial G-action $\mathrm{Sp}^O \to (\mathrm{Sp}^O)^G$ (which extends to the identity on G-spectra) is G-linear (see 3.32).

The next result shows that our choice of indexing the cubes on the G-set G_+ in the definition of G-excision plays a minor role, and we could equivalently have indexed the cubes on transitive G-sets with disjoint basepoints.

Proposition 3.10: *A homotopy functor $\Phi\colon \mathscr{C} \to \mathscr{D}^G$ is G-excisive if and only if the induced functor $\Phi_*\colon \mathscr{C}_a^{\mathcal{P}(G/H_+)} \to \mathscr{D}_a^{\mathcal{P}(G/H_+)}$ sends homotopy cocartesian G/H_+-cubes to homotopy cartesian G/H_+-cubes, for every subgroup $H \leq G$.*

Remark 3.11. Setting $H = G$ in 3.10 we see that $\Phi_*\colon \mathscr{C}_a^{\mathcal{P}(1_+)} \to \mathscr{D}_a^{\mathcal{P}(1_+)}$ sends cocartesian squares in $\mathscr{C}^G$ to cartesian squares in $\mathscr{D}^G$. That is, if Φ is G-excisive then the induced functor $\Phi_*\colon \mathscr{C}^G \to \mathscr{D}^G$ is excisive in the classical sense.

Proof of 3.10. The "if"-part of the statement is trivial. For the "only if"-part, let H be a subgroup of G and consider the projection map $p\colon G_+ \to G/H_+$. As part of a broader discussion on how to calculate homotopy limits and colimits of punctured cubes, we show in A.1 and A.3 that the induced restriction functor $p^*\colon \mathscr{C}_a^{\mathcal{P}(G/H_+)} \to \mathscr{C}_a^{\mathcal{P}(G_+)}$ preserves homotopy cocartesian cubes and detects homotopy cartesian cubes. Therefore, given a homotopy cocartesian cube X in $\mathscr{C}_a^{\mathcal{P}(G/H_+)}$, the cube p^*X in $\mathscr{C}_a^{\mathcal{P}(G_+)}$ is homotopy cocartesian, and by G-excision of Φ the cube $\Phi_*(p^*X) = p^*\Phi_*(X)$ is homotopy cartesian in $\mathscr{D}_a^{\mathcal{P}(G_+)}$. As p^* detects homotopy cocartesian cubes, $\Phi_*(X)$ is homotopy cartesian in $\mathscr{D}_a^{\mathcal{P}(G/H_+)}$. □

Remark 3.12. The basepoint added to G in the definition of G-excision 3.8 has the role of combining in a single condition the behavior of $\Phi\colon \mathscr{C} \to \mathscr{D}^G$ on squares and on G-cubes. We already saw (3.11) that if Φ is G-excisive it sends homotopy cocartesian squares to homotopy cartesian squares. It turns out that $\Phi_*\colon \mathscr{C}_a^{\mathcal{P}(G/H)} \to \mathscr{D}_a^{\mathcal{P}(G/H)}$ also turns homotopy cocartesian G/H-cubes into homotopy cartesian ones. This can be proved by extending a G/H-cube to a G/H_+-cube by means of the functor $p\colon \mathcal{P}(G/H_+) \to \mathcal{P}(G/H)$ that intersects a subset with G/H, with a proof analogous to 3.10. Conversely, similar techniques show that if $\Phi\colon \mathscr{C} \to \mathscr{D}^G$ turns homotopy cocartesian squares and G-cubes into homotopy cartesian ones, it is G-excisive.

Remark 3.13. G-linearity is hereditary with respect to taking subgroups, under a mild assumption on the G-model category $\mathscr{D}$. That is to say, if Φ is G-linear it is also H-linear for every subgroup H of G. The proof we suggest requires a surprizing amount of machinery and it is given in 3.20 as a corollary of a higher Wirthmüller isomorphism theorem. It is still unknow to the authors if in the unpointed case G-excision satisfies a similar property.

Proposition 3.14: *Let $\mathscr{C}$ and $\mathscr{D}$ be pointed G-model categories, and $\Phi\colon \mathscr{C} \to \mathscr{D}^G$ be a G-linear homotopy functor. For any finite G-set J and any cofibrant G-object $c \in \mathscr{C}^G$ the canonical map*

$$\Phi(\bigvee_J c) \longrightarrow \prod_J F\Phi(c)$$

is an equivalence in $\mathscr{D}^G$.

Proof. First assume that $J = 1_+$ with trivial G-action. The square Vc

$$\begin{tikzcd} c \vee c \arrow[r, "p_+"] \arrow[d, "p_1"'] & c \arrow[d] \\ c \arrow[r] & * \end{tikzcd}$$

in $\mathscr{C}^G$ is homotopy cocartesian (cf. 3.5). By 3.11 its image $\Phi(Vc)$ is homotopy cartesian, that is, the map

$$\Phi(c \vee c) \xrightarrow{\simeq} F\Phi(c \vee c) \to \operatorname*{holim}_{\mathcal{P}(1_+)\backslash\emptyset} F\Phi(Vc) \cong F\Phi(c) \times F\Phi(c)$$

is a weak equivalence in $\mathscr{D}^G$, with diagonal action on the target. By induction, the map of the statement is an equivalence for every J with trivial G-action. Given a finite G-set J, decompose it as disjoint union of transitive G-sets $J = \coprod_{z\in G\backslash J} z$. The map of the statement decomposes as

$$\Phi(\bigvee_J c) = \Phi(\bigvee_{z\in G\backslash J}\bigvee_z c) \xrightarrow{\simeq} \prod_{z\in G\backslash J} F\Phi(\bigvee_z c) \longrightarrow \prod_{z\in G\backslash J}\prod_z F\Phi(c) = \prod_J F\Phi(c)$$

with the first map an equivalence as the action on the quotient $G\backslash J$ is trivial. Therefore it is enough to show that the map is an equivalence for $J = G/H$ a transitive G-set.

Consider the G/H_+-cube Wc with vertices

$$(Wc)_U = \begin{cases} \bigvee_{G/H} c & , U = \emptyset \\ c & , U = \{j \neq +\} \\ * & , \text{otherwise} \end{cases}$$

It is homotopy cocartesian by an argument completely similar to 3.5. By 3.10 the cube $\Phi(Wc)$ is homotopy cartesian, that is, the canonical map

$$\Phi(\bigvee_{G/H} c) \to \operatorname*{holim}_{\mathcal{P}(G/H_+)\backslash\emptyset} F\Phi(Wc) \cong \prod_{G/H} F\Phi(c)$$

is an equivalence in $\mathscr{D}^G$. $\square$

Remark 3.15. In this equivariant setting G_+-cubes (or equivalently J_+-cubes for J transitive) play the role that squares play in the classical theory. The equivariant analogue of n-cubes should be cubes indexed on G-sets with n distinct G-orbits and a disjoint basepoint. Following [Goo92], the behavior of Φ on these cubes should be related to higher order G-excision. This will be the subject of a later article.

3.2 The generalized Wirthmüller isomorphism theorem

Let $\mathscr{C}$ be a bicomplete category, and G a finite group. We recall from §2.1 that a finite set K with commuting left H'-action and right H-action induces an adjunction

$$K \otimes_H (-)\colon \mathscr{C}^H \rightleftarrows \mathscr{C}^{H'} \colon \hom_{H'}(K, -)$$

Let K^* be the set K with left H-action and right H'-action defined by $h \cdot k \cdot h' = (h')^{-1} \cdot k \cdot h^{-1}$. If $\mathscr{C}$ has a zero-object $*$ and if the actions on K are free, a functor $\Phi\colon \mathscr{C} \to \mathscr{D}^G$ induces a natural transformation

$$\eta\colon \Phi(K \otimes_H (-)) \longrightarrow \hom_H(K^*, \Phi(-))$$

of functors $\mathscr{C}^H \to \mathscr{D}^{H'}$. The map η_c is the image by the composition

$$\begin{gathered}\mathscr{C}^H((K^* \times_{H'} K) \otimes_H c, c) \xrightarrow{\Phi} \mathscr{D}^H(\Phi((K^* \times_{H'} K) \otimes_H c), \Phi(c)) \to \\ \mathscr{D}^H(K^* \otimes_{H'} \Phi(K \otimes_H c), \Phi(c)) \xrightarrow{\cong} \mathscr{D}^{H'}(\Phi(K \otimes_H c), \hom_H(K^*, \Phi(c)))\end{gathered}$$

of the map $\bigvee_{K^* \times_{H'} K} c \to c$ defined by $h\colon c \to c$ on a (k, k')-component with $k'h = k$, and by the trivial map $c \to * \to c$ otherwise. Notice that since the H-action is free there is at most one h for which $k'h = k$.

Example 3.16. Suppose that $K = G = H'$ with left G-multiplication and right H-multiplication. Sending an element to its inverse defines a H-G-equivariant isomorphism between G^* and G with left H-multiplication and right G-multiplication. We saw in 2.2 that the forgetful functor $\mathscr{C}^G \to \mathscr{C}^H$ is right adjoint to $G \otimes_H (-)$ and left adjoint to $\hom_H(G^*, -)$. The map η for the identity functor is the standard map

$$G \otimes_H (-) \longrightarrow \hom_H(G^*, -)$$

which in the case of spectra is the classical Wirthmüller isomorphism map. In 3.32 we apply 3.17 below to recover the Wirthmüller isomorphism theorem for G-spectra.

Theorem 3.17: *Let $\mathscr{C}$ and $\mathscr{D}$ be pointed G-model categories, and suppose that K admits an H'-H-equivariant map to G, this happens e.g. if $K = G$. For every G-linear homotopy functor $\Phi\colon \mathscr{C} \to \mathscr{D}^G$ and every object c in $\mathscr{C}^H$ the composite*

$$\Phi(K \otimes_H c) \xrightarrow{\eta} \hom_H(K^*, \Phi(c)) \longrightarrow \hom_H(K^*, F\Phi(c))$$

is an equivalence in $\mathscr{D}^{H'}$, where $\Phi(c) \overset{\simeq}{\rightarrowtail} F\Phi(c)$ is a fibrant replacement of $\Phi(c)$ in $\mathscr{D}^H$.

In particular, if the right Quillen functor $\hom_H(K^*, -)$ preserves all weak equivalences, the map $\eta\colon \Phi(K \otimes_H c) \to \hom_H(K^*, \Phi(c))$ is a weak equivalence for any $c \in \mathscr{C}^H$.

Proof. We express the map of the statement as a canonical map into the homotopy limit of a punctured cube, and we use the G-linearity of Φ to conclude that the map is an equivalence. For this we will compare the source and target of η with an indexed coproduct and product, respectively.

Choose a section $s_G\colon G/H \to G$ and an H'-H-equivariant map $\phi\colon K \to G$. These choices give a commutative diagram (of sets)

$$\begin{tikzcd} K \arrow[r, "\phi"] \arrow[d, "\pi_K"] & G \arrow[d, "\pi_G"'] \\ K/H \arrow[r, "\overline{\phi}"'] \arrow[u, bend left, "s_K"] & G/H \arrow[u, bend right, "s_G"'] \end{tikzcd}$$

where $s_K(kH) := k \cdot (\phi(k)^{-1} \cdot s_G\pi_G\phi(k))$ is a section for π_K, satisfying the relation $\phi s_K = s_G\overline{\phi}$. This gives a map $\gamma\colon H' \times K/H \to H$ defined by

$$\gamma(h', z) = s_G(h'\overline{\phi}(z))^{-1} \cdot h' \cdot s_G\overline{\phi}(z)$$

which we use to define two functors $\bigvee_{K/H}(-)\colon \mathscr{C}^H \to \mathscr{C}^{H'}$ and $\prod_{K/H}(-)\colon \mathscr{D}^H \to \mathscr{D}^{H'}$. These send objects c and d to the coproduct $\bigvee_{K/H} c$ and product $\prod_{K/H} d$, respectively, with H'-actions[1]

$$h' \cdot (z, x) = (h'z, \gamma(h', z) \cdot x) \qquad \text{and} \qquad (h' \cdot \underline{y})_z = \gamma(h', z) \cdot y_{(h')^{-1}z}, \text{ respectively.}$$

There is a commutative diagram of natural transformations

$$\begin{tikzcd} \Phi(\bigvee_{K/H} c) \arrow[r] \arrow[d, "\Phi(s_K \otimes \mathrm{id}_c)"', "\cong"] & \bigvee_{K/H} \Phi(c) \arrow[r] \arrow[d, "s_K \otimes \mathrm{id}_{\Phi(c)}"', "\cong"] & \prod_{K/H} \Phi(c) \arrow[d, "\cong"', "(-)\circ s_K"] \\ \eta\colon \Phi(K \otimes_H c) \arrow[r] & K \otimes_H \Phi(c) \arrow[r] & \hom_H(K^*, \Phi(c)) \end{tikzcd}$$

The top right horizontal map is the canonical map from the coproduct to the product. The first two vertical maps are induced by the composite

$$s_K \otimes \mathrm{id}\colon \bigvee_{K/H} c = K/H \otimes c \to K \otimes c \twoheadrightarrow K \otimes_H c.$$

It is an isomorphism with inverse $(k, x) \mapsto (\pi_K k, (s_G\pi_G\phi(k))^{-1}\phi(k) \cdot x)$. The right vertical map $(-) \circ s_K$ is defined dually and it is also an isomorphism. We can therefore equivalently study the top composition $\Phi(\bigvee_{K/H} c) \to \prod_{K/H} \Phi(c)$.

Consider the K/H_+-cube $Wc\colon \mathcal{P}(K/H_+) \to \mathscr{C}$ defined by

$$(Wc)_S = \begin{cases} \bigvee_{K/H} c & , S = \emptyset \\ c & , |S| = 1, S \neq \{+\} \\ * & , |S| \geq 2 \text{ or } S = \{+\} \end{cases}$$

[1]For convenience we only spell these actions out in the case that the objects of $\mathscr{C}$ have "elements".

with initial map $\bigvee_{K/H} c \to c = (Wc)_{\{z\}}$ the pinch map that collapses all the wedge components not indexed by $\{z\}$. The structure maps $c = (Wc)_z \to (Wc)_{h'z} = c$ are defined by action by $\gamma(h', z) \in H$.

The cube Wc is homotopy cocartesian. Indeed, if $Q_H c \xrightarrow{\simeq} c$ is a cofibrant replacement of c in $\mathscr{C}^H$, the cube $WQ_H c$ is point-wise cofibrant with homotopy colimit over $\mathcal{P}(K/H_+)\backslash K/H_+$ contractible (see 3.5). Let $\Phi(Wc) \xrightarrow{\simeq} F\Phi(Wc)$ be a fibrant replacement of $\Phi(Wc)$. By linearity of Φ, the canonical map

$$\Phi(\bigvee_{K/H} c) \xrightarrow{\simeq} \operatorname*{holim}_{\mathcal{P}(K/H_+)\backslash\emptyset} F\Phi(Wc) \cong \prod_{K/H} F\Phi(c)$$

is an equivalence in $\mathscr{D}^{H'}$. This proves the first part of the theorem.

Moreover, the map above fits into a commutative diagram

$$\begin{tikzcd} \Phi(\bigvee_{K/H} c) \arrow[r] \arrow[dr, "\simeq"'] & \prod_{K/H} \Phi(c) \arrow[d] \\ & \prod_{K/H} F\Phi(c) \end{tikzcd}$$

where the right vertical map is an equivalence if $\hom_H(K^*, -)$ (and therefore $\prod_{K/H}(-)$) preserves weak equivalences. □

Corollary 3.18: *If the trivial action inclusion functor $\mathscr{C} \to \mathscr{C}^G$ is G-linear, the left and right adjoints to the evaluation functor $\mathrm{ev}_i\colon \mathscr{C}_a^I \to \mathscr{C}^{G_i}$ are naturally equivalent on fibrant objects for every $i \in I$.*

Proof. We saw in 2.8 that the left adjoint $F_i\colon \mathscr{C}^{G_i} \to \mathscr{C}_a^I$ has j-vertex

$$(F_i c)_j = K_{ji} \otimes_{G_i} c$$

where $K_{ji} = \hom_{G \rtimes_a I}(i, j)$ projects G_j-G_i-equivariantly to G. Similarly the right adjoint has j-vertex

$$(R_i c)_j = \hom_{G_i}(K_{ji}^*, c)$$

and 3.17 provides a natural equivalence from F_i to R_i. □

We give a "higher version" of the Wirthmüller isomorphism theorem, that compares the left and the right adjoints of the functor on J-cubes that restricts the action to a subgroup H of G. Given a G-set J, let $J|_H$ be the H-set obtained by restricting the G-action to H. The poset category with H-action $\mathcal{P}(J|_H)$ is the category $\mathcal{P}(J)$ with the restricted action $a|_H$. There is a forgetful functor $\mathscr{C}_a^{\mathcal{P}(J)} \to \mathscr{C}_{a|_H}^{\mathcal{P}(J|_H)}$ that restricts the G-structure to a H-structure. It has both a left and a right adjoint, that we denote respectively L^J and R^J. This can easily be seen with the description of G-diagrams as diagrams on a Grothendieck construction of 1.9, as the restriction functor above corresponds to restriction along the inclusion $\iota\colon H \rtimes_{a|_H} \mathcal{P}(J|_H) \to G \rtimes_a \mathcal{P}(J)$. The following result specializes to theorem 3.17 for $K = G$ when J is the empty G-set.

Theorem 3.19: *For every G-linear homotopy functor $\Phi\colon \mathscr{C} \to \mathscr{D}^G$ and every $J|_H$-cube $X \in \mathscr{C}_{a|_H}^{\mathcal{P}(J|_H)}$, there is an equivalence of J-cubes*

$$\Phi L^J(X) \xrightarrow{\eta} R^J\Phi(X) \longrightarrow R^J F\Phi(X)$$

where $\Phi(X) \xrightarrow{\simeq} F\Phi(X)$ is a fibrant replacement of $\Phi(X)$.

Proof. Let us describe the left adjoint L^J explicitly, by calculating the left Kan extension of X along $\iota\colon H\rtimes_{a|_H}\mathcal{P}(J|_H)\to G\rtimes_a\mathcal{P}(J)$. By definition this has values

$$L^J(X)_U=\operatorname{colim}\big(\iota/_U\to H\rtimes_{a|_H}\mathcal{P}(J|_H)\xrightarrow{X}\mathscr{C}\big).$$

The over category $\iota/_U$ is the poset with objects $(g\in G, A\in\mathcal{P}(g^{-1}U))$, and a unique morphism $(g,A)\to(g',A')$ whenever $g(g')^{-1}$ belongs to H and $g(g')^{-1}A\subset A'$. This can be written as the disjoint union of categories

$$\iota/_U=\coprod_{z\in G/H}(Ez\wr\Psi_z)$$

where Ez is the translation category of the right H-set z (see 2.8) and $Ez\wr\Psi_z$ is the Grothendieck construction of the functor $\Psi_z\colon Ez\to Cat$ that sends $g\in G/H$ to the category $\mathcal{P}(g^{-1}U)$. Hence the left Kan extension $L^J(X)$ is naturally isomorphic to

$$L^J(X)_U\cong\bigvee_{z\in G/H}\operatorname*{colim}_{(g,A)\in Ez\wr\Psi_z}X_A\cong\bigvee_{z\in G/H}\operatorname*{colim}_{g\in Ez}\operatorname*{colim}_{A\in\mathcal{P}(g^{-1}U)}X_A\xrightarrow{\cong}\bigvee_{z\in G/H}\operatorname*{colim}_{g\in Ez}X_{g^{-1}U}$$

Here the first isomorphism is the Fubini theorem for colimits (see e.g. [CS02, 40.2], as it is an isomorphism it is enough to see that it is equivariant). The last map is an isomorphism is because $g^{-1}U$ is a terminal object in $\mathcal{P}(g^{-1}U)$. A choice of section $s\colon G/H\to G$ gives a further identification

$$L^J(X)_U\cong\bigvee_{z\in G/H}X_{s(z)^{-1}U}$$

Chasing through the isomorphisms one can see that the G-structure is given by the maps

$$g\colon X_{s(z)^{-1}U}\xrightarrow{s(gz)^{-1}gs(z)}X_{s(gz)^{-1}gU}$$

The same choice of section gives a similar identification for the right adjoint

$$R^J(X)_U\cong\prod_{z\in G/H}X_{s(z)^{-1}U}$$

A G/H_+-cube argument completely analogous to 3.17 shows that the inclusion of wedges into products induces a G-equivalence $\Phi L^J(X)\to R^JF\Phi(X)$ □

Corollary 3.20: *Let $\Phi\colon\mathscr{C}\to\mathscr{D}^G$ be a homotopy functor, and suppose that the functor* $\hom_H(G,-)\colon\mathscr{D}^H\to\mathscr{D}^G$ *detects equivalences of fibrant objects. If $\Phi_*\colon\mathscr{C}_a^{\mathcal{P}(J)}\to\mathscr{D}_a^{\mathcal{P}(J)}$ sends homotopy cocartesian cubes to homotopy cartesian cubes, so does $\Phi_*\colon\mathscr{C}_{a|_H}^{\mathcal{P}(J|_H)}\to\mathscr{D}_{a|_H}^{\mathcal{P}(J|_H)}$.*

It follows that if Φ is G-linear, it is also H-linear for every subgroup $H\le G$.

Proof. From the explicit descriptions of L^J and R^J of 3.19 one can see that L^J commutes with homotopy colimits and that R^J commutes with homotopy limits. In particular, if X is a homotopy cocartesian $J|_H$-cube, the J-cube $L^J(X)$ is also homotopy cocartesian. Hence by our assumption on Φ, the J-cube $\Phi_*L^J(X)$ is homotopy cartesian. The top horizontal map in the commutative diagram

$$\begin{tikzcd}
\Phi_*L^J(X)_\emptyset \arrow[r, "\simeq"] \arrow[d] & \operatorname*{holim}_{\mathcal{P}(J)\backslash\emptyset}F\Phi_*L^J(X) \arrow[d] \\
R^JF\Phi_*(X)_\emptyset \arrow[r] & \operatorname*{holim}_{\mathcal{P}(J)\backslash\emptyset}R^JF\Phi_*(X)
\end{tikzcd}$$

is therefore an equivalence. The vertical maps are also equivalences by the higher Wirthmüller isomorphism theorem 3.19. Thus the bottom horizontal map is also an equivalence, and it factors as

$$R^J F\Phi_*(X)_\emptyset \to R^\emptyset \operatorname*{holim}_{\mathcal{P}(J)\backslash\emptyset} F\Phi_*(X) \xrightarrow{\simeq} \operatorname*{holim}_{\mathcal{P}(J)\backslash\emptyset} R^J F\Phi_*(X)$$

The first map of the factorization is therefore also an equivalence, and by the explicit description of R^J in the proof of 3.19, it is just the canonical map

$$\hom_H(G, F\Phi_*(X)_\emptyset) \longrightarrow \hom_H(G, \operatorname*{holim}_{\mathcal{P}(J)\backslash\emptyset} F\Phi_*(X)).$$

Since $\hom_H(G,-)$ detects equivalences of fibrant objects, $\Phi_*(X)$ is homotopy cartesian.

For the second part of the statement, assume that Φ is G-linear and let X be a be a homotopy cocartesian H_+-cube. Consider the H-equivariant surjection $p\colon G_+|_H \to H_+$ which is the identity on H and that collapses the complement of H to the basepoint. It induces a functor $p^*\colon \mathscr{C}_a^{\mathcal{P}(H_+)} \to \mathscr{C}_a^{\mathcal{P}(G_+|_H)}$ which by A.3 preserves homotopy cocartesian cubes. Hence p^*X is a homotopy cocartesian $G_+|_H$-cube. By the first part of the corollary and G-linearity, $\Phi_*(p^*X) = p^*\Phi_*(X)$ is homotopy cartesian. By A.1, p^* detects homotopy cartesian cubes, hence $\Phi_*(X)$ is homotopy cartesian.

$\square$

3.3 G-linearity and adjoint assembly maps

Let $\mathscr{C}$ and $\mathscr{D}$ be pointed G-model categories, and $\Phi\colon \mathscr{C} \to \mathscr{D}^G$ a $sSet$-enriched reduced homotopy functor. Its extension $\Phi\colon \mathscr{C}^G \to \mathscr{D}^G$ is then enriched over G-$sSet$, and for any simplicial G-set K there is an assembly map

$$K \otimes \Phi(c) \longrightarrow \Phi(K \otimes c)$$

in $\mathscr{D}^G$. It is adjoint to the map of simplicial G-sets

$$K \longrightarrow Map_{\mathscr{C}}(c, K \otimes c) \xrightarrow{\Phi} Map_{\mathscr{D}}(\Phi(c), \Phi(K \otimes c))$$

where the first map is adjoint to the identity on $K \otimes c$. When $K = N(\mathcal{P}(J_+)\backslash\emptyset)$ this induces a map

$$\alpha\colon \Phi(c) \longrightarrow \Omega^J \Phi(\Sigma^J c)$$

called the adjoint assembly map (see 1.19 for the definitions of Ω^J and Σ^J in a general simplicial category). The aim of this section is to explore the relationship between G-linearity of Φ and the adjoint assembly map.

Remark 3.21. Given a cofibrant G-object c in $\mathscr{C}^G$ and a finite G-set J, recall the cofibrant J_+-cube

$$(S^J c)_U = \begin{cases} c & , U = \emptyset \\ C^U c & , U \lneq J_+ \\ \Sigma^J c & , U = J_+ \end{cases}$$

from 3.4. This induces a zig-zag

$$\Phi(c) \xrightarrow{\simeq} F\Phi(c) \to \operatorname*{holim}_{\mathcal{P}(J_+)\backslash\emptyset} F\Phi(S^J c) \xleftarrow{\simeq} \Omega^J F\Phi(\Sigma^J c)$$

where the last equivalence is induced by the equivalence of fibrant $\mathcal{P}(J_+)\backslash\emptyset$-diagrams

$$\omega^J\big(F\Phi(\Sigma^J c)\big) \xrightarrow{\simeq} F\Phi(S^J c)|_{\mathcal{P}(J_+)\backslash\emptyset}$$

for the G-diagram $\omega^J d$ from 1.19 associated to an object d of $\mathscr{D}^G$, with vertices $(\omega^J d)_{J_+} = d$ and $(\omega^J d)_U = *$ for $U \neq J_+$. The adjoint assembly map above fits into the commutative diagram

$$\begin{tikzcd}
\Phi(c) \arrow[r, "\simeq"] \arrow[dr, "\alpha"'] & F\Phi(c) \arrow[r] & \operatorname*{holim}_{\mathcal{P}(J_+)\backslash\emptyset} F\Phi(S^J c) \\
& \Omega^J\Phi(\Sigma^J c) \arrow[r] & \Omega^J F\Phi(\Sigma^J c). \arrow[u, "\simeq"']
\end{tikzcd}$$

Hence the map $\Phi(c) \to \operatorname*{holim}_{\mathcal{P}(J_+)\backslash\emptyset} F\Phi(S^J c)$ can be thought of as a model for the adjoint assembly map which can be defined without using that Φ is an enriched functor.

Proposition 3.22: *Let $\mathscr{C}$ and $\mathscr{D}$ be pointed G-model categories, and $\Phi\colon \mathscr{C} \to \mathscr{D}^G$ a sSet-enriched G-linear homotopy functor. For any finite G-set J and any cofibrant G-object $c \in \mathscr{C}^G$ the composite*

$$\Phi(c) \xrightarrow{\alpha} \Omega^J\Phi(\Sigma^J c) \longrightarrow \Omega^J F\Phi(\Sigma^J c)$$

is a weak equivalence in $\mathscr{D}^G$.

Proof. The decomposition of J as disjoint union of transitive G-sets $J_+ \cong (\coprod_{z\in G\backslash J} z)_+$ gives a factorization the map of the statement as an iterated construction

$$\Phi(c) \to \Omega^{z_1}F\Phi(\Sigma^{z_1}c) \to \cdots \to \Omega^{z_1}\ldots\Omega^{z_n}F\Phi(\Sigma^{z_1}\ldots\Sigma^{z_m}c)$$

The functor $\Sigma^z(-)$ preserves cocartesian cubes and Ω^z preserves fibrant objects, so using the natural weak equivalences $\Sigma^z\Sigma^w c \xrightarrow{\simeq} \Sigma^{z\amalg w}d$ for d cofibrant and $\Omega^{z\amalg w}d \xrightarrow{\simeq} \Omega^z\Omega^w d$ for d fibrant, it suffices to show that the map $\Phi(c) \to \Omega^{G/H}F\Phi(\Sigma^{G/H}c)$ is an equivalence for every transitive G-set G/H.

By 3.10, Φ sends the homotopy cocartesian G/H_+-cube $S^{G/H}c$ of 3.21 to a homotopy cartesian G/H_+-cube. That is, the second map in the zig-zag

$$\Phi(c) \xrightarrow{\simeq} F\Phi(c) \to \operatorname*{holim}_{\mathcal{P}(G/H_+)\backslash\emptyset} F\Phi(S^{G/H}c) \xleftarrow{\simeq} \Omega^{G/H}F\Phi(\Sigma^{G/H}c)$$

is an equivalence in $\mathscr{D}^G$. The statement now follows from the commutativity of the diagram in 3.21 above. □

We aim at proving a converse to 3.22. We remind the reader that a simplicial category $\mathscr{C}$ is locally finitely presentable if there is a set Θ of objects in $\mathscr{C}$ such that every object of $\mathscr{C}$ is isomorphic to a filtered colimit of objects in Θ, and for every $\theta \in \Theta$ the functor $Map_{\mathscr{C}}(\theta, -)\colon \mathscr{C} \to sSet$ preserves filtered colimits (see [AR94], [Kel82]). For example the categories of simplicial sets and of spectra (of simplicial sets) satisfy this condition. We will write Ω^ρ for Ω^G and Σ^ρ for Σ^G.

Theorem 3.23: *Let $\mathscr{C}$ and $\mathscr{D}$ be pointed G-model categories and suppose that the simplicial categories $\mathscr{D}^H$ are locally finitely presentable for every $H \leq G$. Let $\Phi\colon \mathscr{C} \to \mathscr{D}^G$ be a sSet-enriched reduced homotopy functor and let J be a finite G-set. If the canonical map*

$$\Phi(c) \longrightarrow \Omega^{J|_H}F\Phi(\Sigma^{J|_H}c)$$

is a weak equivalence in $\mathscr{D}^H$ for every cofibrant object $c \in \mathscr{C}^H$ and every subgroup $H \leq G$, then the induced functor $\Phi_*\colon \mathscr{C}_a^{\mathcal{P}(J_+)} \to \mathscr{D}_a^{\mathcal{P}(J_+)}$ sends homotopy cocartesian J_+-cubes to homotopy cartesian J_+-cubes.

In particular, if $\Phi(c) \xrightarrow{\simeq} \Omega^{\rho|_H}F\Phi(\Sigma^{\rho|_H}c)$ is an equivalence for every subgroup $H \leq G$ and every cofibrant H-object c, the functor Φ is G-linear.

The proof of this theorem is technical and it is given at the end of the section.

Remark 3.24. The theorem above holds also in the G-model categories of pointed spaces or orthogonal spectra, even though these are not locally finitely presentable. The presentability condition is used to commute a sequential homotopy colimit and a finite equivariant homotopy limit, as explained in A.8. These commute also in Top_* and Sp^O, for the following reason. They commute in $sSet_*$ as $sSet_*^H$ is locally finitely presentable. This property can be transported through the G-Quillen equivalence $|-|\colon sSet_* \rightleftarrows Top_*\colon Sing$, using that realization commutes with finite limits and $Sing$ with sequential colimits along cofibrations. It can be further deduced for Sp^O as limits and colimits are levelwise.

Corollary 3.25: *Under the hypotheses of 3.23, suppose additionally that the functor* $\hom_H(G,-)\colon \mathscr{D}^H \to \mathscr{D}^G$ *detects equivalences of fibrant objects for every subgroup H of G. Then the following are equivalent:*

1. *Φ is G-linear,*
2. *For every cofibrant object $c\in\mathscr{C}^H$ and every $H\leq G$, the canonical map $\Phi(c)\to\Omega^{\rho|_H}F\Phi(\Sigma^{\rho|_H}c)$ is an equivalence in $\mathscr{D}^H$,*
3. *For every finite G-set J the functor $\Phi_*\colon \mathscr{C}_a^{\mathcal{P}(J_+)}\to\mathscr{D}_a^{\mathcal{P}(J_+)}$ sends homotopy cocartesian J_+-cubes to homotopy cartesian J_+-cubes.*

Proof. (1) $\Rightarrow$ (2) By 3.20 the functor Φ is H-linear for every subgroup $H\leq G$. The implication then follows from 3.22 for the H-set $G_+|_H$.
(2) $\Rightarrow$ (3) By 3.23 it is enough to show that $\Phi(c)\to\Omega^{J|_H}F\Phi(\Sigma^{J|_H}c)$ is an equivalence for every finite G-set J. But Φ is G-linear by 3.23, and hence H-linear by 3.20. The adjoint assembly is then an equivalence by 3.22.
(3) $\Rightarrow$ (1) For $J=G$ the conclusion in (3) is the definition of G-linearity. $\square$

Remark 3.26. Define the G-derivative (at the zero object) of a reduced enriched homotopy functor $\Phi\colon\mathscr{C}\to\mathscr{D}^G$ to be the functor $D_*\Phi\colon\mathscr{C}\to\mathscr{D}^G$ defined by

$$D_*\Phi(c)=\operatorname{hocolim}\big(Q\Phi(c)\to Q\Omega^{\rho}F\Phi(\Sigma^{\rho}c)\to Q\Omega^{2\rho}F\Phi(\Sigma^{2\rho}c)\to\dots\big)$$

where $\Sigma^{n\rho}=\Sigma^{nG}$ is the suspension by the permutation representation of n-disjoint copies of G. As a direct consequence of point 2 of 3.25 the functor $D_*\Phi$ is G-linear, and it is equipped with a universal natural transformation $\Phi\to D_*\Phi$. The argument of [Goo03, 1.8] applies verbatim to our equivariant situation, showing that $\Phi\to D_*\Phi$ is essentially initial among maps from Φ to a G-excisive functor.

Proof of 3.23. We follow the strategy of the proofs of [Goo03, 1.8,1.9] and [Rez13] of showing that the adjoint assembly map evaluated at a cocartesian cube factors through a cartesian cube. It is convenient to introduce a new model for the loop space. For a cofibrant object $c\in\mathscr{C}^G$ we define

$$\overline{\Omega}^J F\Phi(\Sigma^J c):=\operatorname*{holim}_{\mathcal{P}(J_+)\backslash\emptyset}F\Phi(S^J c).$$

This object comes with a natural weak equivalence $\overline{\Omega}^J F\Phi(\Sigma^J c)\xleftarrow{\simeq}\Omega^J F\Phi(\Sigma^J c)$ (see 3.21). Let $X\colon\mathcal{P}(J_+)\to\mathscr{C}$ be a cofibrant J_+-cube. Define a G-diagram $K\colon\mathcal{P}(J_+)\times\mathcal{P}(J_+)\to\mathscr{C}$ by

$$K(U,T)=\operatorname*{hocolim}_{S\in\mathcal{P}(J_+)\backslash J_+}X_{(S\cap U)\cup T}$$

and define a J_+-cube $Y\colon\mathcal{P}(J_+)\to\mathscr{D}$ by

$$Y_T=\operatorname*{holim}_{\mathcal{P}(J_+)\backslash\emptyset}F\Phi(K(-,T))$$

The key of this proof is to define, for every $T \subset J_+$, a factorization, natural in T

$$\begin{tikzcd} \Phi(X_T) \arrow[rr] \arrow[dr, "\phi"'] & & \overline{\Omega}^J F\Phi(\Sigma^J X_T) \\ & Y_T \arrow[ur, "\psi"'] & \end{tikzcd}$$

and show that Y is homotopy cartesian when X is homotopy cocartesian. Writing $\Delta^{\tilde{U}}$ for $N\mathcal{P}(U)\backslash\emptyset$, the first map of the factorization has U-component

$$\phi_U \colon \Phi(X_T) \longrightarrow map(\Delta^{\tilde{U}}, F\Phi(K(U,T)))$$

adjoint to the composite

$$\Delta^{\tilde{U}} \otimes \Phi(X_T) \to \Phi(\Delta^{\tilde{U}} \otimes X_T) \to \Phi(K(U,T)) \to F\Phi(K(U,T))$$

where the second map is induced by $\Delta^{\tilde{U}} \otimes X_T \to \Delta^{\tilde{U}} \otimes X_{T\cup U} \to K(U,T)$. The map ψ is the homotopy limit over U of the map of diagrams $F\Phi(K(U,T)) \to F\Phi((S^J X_T)_U)$ induced by the map $K(U,T) \to (S^J X_T)_U$ defined as follows. For $U \neq J_+$, it is the composite

$$K(U,T) = \operatorname*{hocolim}_{S\in\mathcal{P}(J_+)\backslash J_+} X_{(S\cap U)\cup T} = \operatorname*{hocolim}_{S\in\mathcal{P}(J_+)\backslash J_+} (-\cap U)^* X_{S\cup T} \to \operatorname*{hocolim}_{S\in\mathcal{P}(U)} X_{S\cup T} \to C^U X_T$$

where the first arrow is the canonical map induced by the functor $(-\cap U)\colon \mathcal{P}(J_+)\backslash J_+ \to \mathcal{P}(U)$ and the second arrow is induced by collapsing all the non-initial verticies. For $U = J_+$, the map is

$$K(J_+,T) = \operatorname*{hocolim}_{S\in\mathcal{P}(J_+)\backslash J_+} X_{S\cup T} \to \operatorname*{hocolim}_{\mathcal{P}(J_+)\backslash J_+} \sigma^J X_T = \Sigma^J X_T$$

induced on homotopy colimits by the map of J_+-cubes given by the identity on the empty set vertex, and that collapses the other vertices to the point.

Now suppose that X is homotopy cocartesian, and let us see that Y is homotopy cartesian. There is a natural equivalence $K(U,T) \xrightarrow{\simeq} X_{U\cup T}$. Indeed, the maps $X_{(S\cap U)\cup T} \to X_{((S\cup\{t\})\cap U)\cup T}$ are the identity for all $t \in T$, and therefore $K(U,T) \xrightarrow{\simeq} X_{U\cup T}$ as long as $T \neq \emptyset$, by the lemma 3.27 below. For $T = \emptyset$ and $U \neq J_+$ there is a weak equivalence

$$K(U,\emptyset) = \operatorname*{hocolim}_{S\in\mathcal{P}(J_+)\backslash J_+} X_{S\cap U} \xrightarrow{\simeq} X_U$$

again by 3.27, as the maps $X_{S\cap U} \to X_{(S\cup\{v\})\cap U}$ are the identity for all $v \in J_+\backslash U$. Finally,

$$K(J_+,\emptyset) = \operatorname*{hocolim}_{S\in\mathcal{P}(J_+)\backslash J_+} X_S \xrightarrow{\simeq} X_{J_+}$$

since X is assumed to be homotopy cocartesian. This shows that

$$Y_T \xrightarrow{\simeq} \operatorname*{holim}_{U\in\mathcal{P}(J_+)\backslash\emptyset} F\Phi(X_{U\cup T})$$

For every fixed $U \neq \emptyset$, the cube $T \longmapsto F\Phi(X_{U\cup T})$ is homotopy cartesian by 3.27, as the maps $F\Phi(X_{U\cup T}) \to F\Phi(X_{U\cup T\cup\{u\}})$ are the identity for all $u \in U$. The cube Y is then a homotopy limit of cartesian cubes, and therefore also cartesian since homotopy limits commute with each other.

Iterating this construction and using that Σ^J and $\overline{\Omega}^J$ preserve homotopy cocartesian and -cartesian J_+-cubes, respectively, one gets a factorization of each map in the colimit system

$$\Phi(X) \xrightarrow{\simeq} \overline{\Omega}^J F\Phi(\Sigma^J X) \xrightarrow{\simeq} \overline{\Omega}^{2J} F\Phi(\Sigma^{2J} X) \xrightarrow{\simeq} \dots$$

through a homotopy cartesian J_+-cube $Y^{(n)}$. The maps in this system are weak equivalences, since all maps $\Phi(c) \to \Omega^{J|_H} F\Phi(\Sigma^{J|_H} c)$ are assumed to be weak equivalences. By (classical) cofinality for diagrams in $\mathscr{D}^G$ the homotopy colimit of the sequence above is equivalent to $\mathrm{hocolim}_n\, Y^{(n)}$. By A.8 in the appendix we know that under our presentability assumptions sequential homotopy colimits preserve homotopy cartesian J_+-cubes. Therefore $\Phi(X) \simeq \mathrm{hocolim}_n\, Q\overline{\Omega}^{nJ} F\Phi(\Sigma^{nJ} X)$ is homotopy cartesian. □

Lemma 3.27: *Let J be a finite G-set, $X\colon \mathcal{P}(J) \to \mathscr{C}$ a J-cube and $I \subset J$ a non-empty G-invariant subset such that the maps $X_S \to X_{S\cup i}$ are isomorphisms for all $S \subset J$ and $i \in I$. If X is a fibrant diagram, it is homotopy cartesian. Similarly, if X is point-wise cofibrant, it is homotopy cocartesian.*

Proof. Let $\mathcal{P}_I(J)$ be the subposet of $\mathcal{P}(J)\backslash\emptyset$ consisting of non-empty subsets of J that contain I and write ι for the inclusion map. The map $U \mapsto U \cup I$ defines a retraction $u_I\colon \mathcal{P}(J)\backslash\emptyset \to \mathcal{P}_I(J)$. The assumption on the maps $X_S \to X_{S\cup i}$ implies that the natural map $X \to u_I^*\iota^* X$ is an isomorphism. The composite of the maps

$$\operatorname*{holim}_{\mathcal{P}_I(J)} \iota^* X \xrightarrow{\simeq} \operatorname*{holim}_{\mathcal{P}(J)\backslash\emptyset} u_I^*\iota^* X \to \operatorname*{holim}_{\mathcal{P}_I(J)} \iota^* u_I^* \iota^* X$$

is the identity map and the left hand map is a weak equivalence since u_I is right G-cofinal. Hence the right-hand map is a weak equivalence. It fits into a commutative diagram

$$\begin{tikzcd} X_\emptyset \arrow[r] \arrow[d, "\cong"'] & \operatorname*{holim}_{\mathcal{P}(J)\backslash\emptyset} X \arrow[d, "\simeq"] \\ X_I \arrow[r, "\simeq"'] & \operatorname*{holim}_{\mathcal{P}_I(J)} X. \end{tikzcd}$$

The left vertical map is a G-map, which is an isomorphism by assumption and the bottom horizontal map is a G-equivalence since I is initial in $\mathcal{P}_I(J)$. Therefore the top map in the square is a weak equivalence and X is homotopy cartesian.

A completely analogous argument shows that X is homotopy cocartesian. □

3.4 G-linear functors on pointed G-spaces

In [Blu06] Blumberg defines a notion of G-linearity for endofunctors of the category of pointed G-spaces, for a compact Lie group G. When G is finite, we show that his definition and ours agree up to a suspension factor.

Before starting, let us remark that when working with spaces we can drop all the point-wise fibrant and cofibrant replacements from the last sections, as homotopy limits and homotopy colimits of G-diagrams of spaces are always homotopy invariant. For homotopy limits, it is just because every G-space is fibrant. For homotopy colimits, there is a natural homeomorphism

$$(\operatorname*{hocolim}_I X)^H \cong \operatorname*{hocolim}_{I^H} (\iota_H^* X)^H$$

for every G-diagram X in $(Top_*)^I_a$ and subgroup $H \leq G$. Here $\iota_H\colon I^H \to I$ is the inclusion of the subcategory of I of objects and morphisms fixed by the H-action. Therefore homotopy invariance of homotopy colimits of G-diagrams follows from homotopy invariance of classical homotopy colimits of spaces, proved in [DI04].

Proposition 3.28: *An enriched reduced homotopy functor $\Phi\colon Top_* \to Top_*^G$ is G-linear if and only if the following two conditions hold:*

a) The induced functor $\Phi_*\colon (Top_*^G)^{\mathcal{P}(1_+)} \to (Top_*^G)^{\mathcal{P}(1_+)}$ sends homotopy cocartesian squares of pointed G-spaces to homotopy cartesian ones.

b) *For all finite G-sets J the natural map*

$$\Phi(\bigvee_J Z) \to \prod_J \Phi(Z)$$

is an equivalence of pointed G-spaces.

Remark 3.29. The two conditions of 3.28 are essentially the definition of G-linearity in the case of a finite group G in [Blu06].

Proof. If Φ is G-linear, it sends homotopy cocartesian squares to homotopy cartesian squares by 3.11, and the map $\Phi(\bigvee_J Z) \to \prod_J \Phi(Z)$ is an equivalence by 3.14.

Conversely, Blumberg proves in [Blu06] that the two conditions above imply that the adjoint assembly map $\Phi(Z) \to \Omega^V \Phi(Z \wedge S^V)$ is a G-equivalence for every G-representation V. By 3.23 this implies G-linearity of Φ. □

Example 3.30. Let M be a commutative well-pointed topological monoid with additive G-action, and suppose that the fixed point monoids M^H are group-like for every subgroup H of G. The equivariant Dold-Thom construction $M(-)\colon Top_* \to Top_*^G$ sends a pointed space Z to the space $M(Z)$ of reduced configurations of points in Z with labels in M, with G acting on the labels. After extending $M(-)$ to Top_*^G the group acts both on the labels and on the space. If M is discrete the homotopy groups of $M(-)$ are Bredon cohomology of the Mackey functor $H \mapsto M^H$. For a pointed G-simplicial set K the simplicial Dold-Thom construction of 3.9 compares to the topological one by a natural G-homeomorphism $|M(K)| \cong M(|K|)$.

We prove that $M(-)\colon Top_* \to Top_*^G$ is G-linear by checking the two conditions from 3.28. Given a pointed G-space Z, the fixed points of the map $M(Z) \to \Omega M(Z \wedge S^1)$ compares by natural homeomorphisms to the adjoint assembly map

$$M(Z)^H \longrightarrow \Omega M(Z \wedge S^1)^H \cong \Omega(M(Z))(S^1)^H \cong \Omega M(Z)^H(S^1)$$

for the topological group-like monoid $M(Z)^H$. This is an equivalence by standard arguments, see [May75, 7.6]. This implies, by 3.23 for the trivial G-set $J = \{1\}$, that the functor $M(-)$ sends homotopy cocartesian squares of G-spaces to homotopy cartesian ones, proving the first property of 3.28. The second property easily follows, as the map $M(\bigvee_J Z) \to \prod_J M(Z)$ is an equivariant homeomorphism.

Notice that by G-linearity the map $M(Z) \to \Omega^J M(Z \wedge S^J)$ is a G-equivalence for every finite G-set J. This shows that the associated Eilenberg-MacLane G-spectrum $HM_n = M(S^n)$ is fibrant in $(\mathrm{Sp}^O)^G$.

3.5 G-linear functors to G-spectra

We show that the identity functor on G-spectra is G-linear, and deduce from this the classical Wirthmüller isomorphism theorem. We further classify all G-linear functors from finite pointed simplicial sets to G-spectra.

Let us start by clarifying that when working with spectra, as for spaces, we can forget all about the point-wise cofibrant and fibrant replacements from the previous sections, thanks to the following result.

Lemma 3.31: *Let G be a finite group and let a be an action of G on a small category I.*

- *The homotopy colimit functor* hocolim: $(\mathrm{Sp}^O)^I_a \to (\mathrm{Sp}^O)^G$ *preserves weak equivalences between any two diagrams (not necessarily of cofibrant objects).*
- *If I has finite dimensional nerve, the homotopy limit functor* holim: $(\mathrm{Sp}^O)^I_a \to (\mathrm{Sp}^O)^G$ *preserves weak equivalences between any two diagrams (not necessarily of fibrant objects).*

Proof. For any H-spectrum E there is a functorial cofibrant replacement $QE \to E$ where the map is a level equivalence. By 2.22 it is enough to show that homotopy colimits preserve level equivalences of maps of G-diagrams. Since homotopy colimits of spectra are defined level-wise, this follows from homotopy invariance of homotopy colimits for spaces (see §3.4).

For the statement about homotopy limits, take a G-diagram of spectra X. The positive equivariant homotopy groups of $\operatorname{holim}_I X$ are the homotopy groups of the G-space

$$\operatorname*{hocolim}_n \Omega^{n\rho}(\operatorname*{holim}_I X)(n\rho).$$

Here we use the notation $E(n\rho) = E_n \wedge_{O(n)} L(\mathbb{R}^{n|G|}, n\rho)^+$ for a G-spectrum E, where $L(\mathbb{R}^{n|G|}, n\rho)$ is the space of isomorphisms of vector spaces from $\mathbb{R}^{n|G|}$ to $n\rho$. There are natural weak equivalences

$$\begin{aligned}\operatorname{hocolim}_n \Omega^{n\rho}(\operatorname{holim}_I X)(n\rho) &\cong \operatorname{hocolim}_n \Omega^{n\rho} \operatorname{holim}_I(X(n\rho)) \cong \\ \operatorname{hocolim}_n \operatorname{holim}_I \Omega^{n\rho}(X(n\rho)) &\xrightarrow{\simeq} \operatorname{holim}_I \operatorname{hocolim}_n \Omega^{n\rho}(X(n\rho))\end{aligned}$$

where the last map is a weak equivalence by A.8 as sequencial homotopy colimits and finite homotopy limits of G-diagrams of spaces commute. Therefore, a weak equivalence of G-diagrams of spectra $f\colon X \to Y$ induces an isomorphism in positive homotopy groups of the homotopy limit precisely when the map $\operatorname{holim}_I \operatorname{hocolim}_n \Omega^{n\rho} f^{(n\rho)}$ is an equivalence of G-spaces. Since f is an equivalence of G-diagrams of spectra, the map $\operatorname{hocolim}_n \Omega^{n\rho} f_i^{(n\rho)}$ is an equivalence of G_i-spaces for all objects i of I. It follows by homotopy invariance 2.20 that the map of G-spaces $\operatorname{holim}_I \operatorname{hocolim}_n \Omega^{n\rho} f^{(n\rho)}$ is a weak equivalence since it is a homotopy limit of a weak equivalence of G-diagrams of spaces. A similar argument shows that $\operatorname{holim}_I f$ is an equivalence in negative homotopy groups.

□

Theorem 3.32: *Let J be a finite G-set and let a be the induced action of G on $\mathcal{P}(J_+)$. Any homotopy cocartesian J_+-cube X in $(\mathrm{Sp}^O)_a^{\mathcal{P}(J_+)}$ is homotopy cartesian. That is, the inclusion functor $\mathrm{Sp}^O \to (\mathrm{Sp}^O)^G$ is G-linear.*

In particular, this implies the Wirthmüller isomorphism theorem, stating that for any subgroup $H \leq G$ and H-spectrum $E \in (\mathrm{Sp}^O)^H$ the canonical map

$$\eta\colon G \otimes_H E = G_+ \wedge_H E \longrightarrow F_H(G_+, E) = \hom_H(G, E)$$

is a weak equivalence of G-spectra.

Proof. By the equivariant suspension theorem, the map $E \to \Omega^{\rho|_H}(E \wedge S^{\rho|_H})$ is a weak equivalence for any H-spectrum E. By 3.23 (see also 3.24) this is equivalent to G-linearity of the functor $\mathrm{Sp}^O \to (\mathrm{Sp}^O)^G$. The map $\eta\colon G \otimes_H E \longrightarrow \hom_H(G, E)$ is a weak equivalence by 3.17, as $\hom_H(G, -)\colon (\mathrm{Sp}^O)^H \to (\mathrm{Sp}^O)^G$ preserves weak equivalences. □

We end the section with a complete characterization of G-linear functors from the category $sSet_*^f$ of finite pointed simplicial sets to G-spectra.

Proposition 3.33: *Let $\Phi\colon sSet_*^f \to (\mathrm{Sp}^O)^G$ be a sSet-enriched reduced homotopy functor such that the spectrum $\Phi(S^0)$ is level-wise well-pointed. Then the following conditions are equivalent:*

1. *The functor Φ is G-linear.*
2. *The functor $\Phi_*\colon ((sSet_*^f)^G)^{\mathcal{P}(1_+)} \to ((\mathrm{Sp}^O)^G)^{\mathcal{P}(1_+)}$ sends homotopy cocartesian squares in $(sSet_*^f)^G$ to homotopy cartesian squares of G-spectra, and $\Phi(\bigvee_J K) \to \prod_J \Phi(K)$ is an equivalence for every finite pointed simplicial G-set K and finite G-set J.*
3. *For every $K \in (sSet_*^f)^G$ the assembly map*

$$\Phi(S^0) \wedge |K| \longrightarrow \Phi(K)$$

is an equivalence of G-spectra.

Proof. (1) ⇒ (2) This is true in general, by 3.11 and 3.14.
(2) ⇒ (3) This can be proven by induction on the skeleton of K. The wedges into products condition gives the equivalence for the 0-skeleton, and the induction step follows from the condition on squares. We refer to [Dot13] for the details.
(3) ⇒ (1) Since G-linearity is invariant under equivalences, we show that $E \wedge |-|$ is G-linear for any level-wise well-pointed G-spectrum E. If $X : \mathcal{P}(G_+) \to sSet_*^f$ is homotopy cocartesian, the cube of spectra $E \wedge |X|$ is also homotopy cocartesian. Indeed, after applying geometric fixed points F^H the map from the homotopy colimit to the value at G_+ factors as

$$F^H(\operatorname*{hocolim}_{\mathcal{P}(G_+)\backslash G_+} E \wedge |X|) \cong F^H(E \wedge \operatorname*{hocolim}_{\mathcal{P}(G_+)\backslash G_+} |X|) \cong F^H(E) \wedge (\operatorname*{hocolim}_{\mathcal{P}(G_+)\backslash G_+} |X|)^H \xrightarrow{\simeq}$$
$$\xrightarrow{\simeq} F^H(E) \wedge |X_{G_+}|^H \cong F^H(E \wedge |X_{G_+}|),$$

where the third map is a weak equivalence since X is homotopy cocartesian, and since smashing with a level-wise well-pointed spectrum preserves weak equivalences. By 3.32 the diagram $E \wedge X$ is also homotopy cartesian. □

A Appendix

A.1 Computing homotopy (co)limits of punctured cubes

We compare homotopy limits and colimits of punctured cubes of different sizes, specifically how functors between categories of cubes in $\mathscr{C}$ induced by maps $p : K \to J$ of finite G-sets behave on homotopy cartesian and cocartesian cubes.

Proposition A.1: *Let $p : K \to J$ be a surjective equivariant map of finite G-sets. Taking the image by p induces an equivariant functor $p_\emptyset : \mathcal{P}(K)\backslash\emptyset \to \mathcal{P}(J)\backslash\emptyset$, which is left G-cofinal. In particular, the induced functor $p^* : \mathscr{C}_a^{\mathcal{P}(J)} \to \mathscr{C}_a^{\mathcal{P}(K)}$ preserves and detects homotopy cartesian cubes.*

Proof. We show that for any subgroup $H \leq G$ and any non-empty object $U \in \mathcal{P}(J)^H$ the set $p^{-1}(U) \subset K$ is the final object of $(p_\emptyset/U)^H$. It is non-empty since p is assumed to be surjective, and clearly satisfies $pp^{-1}(U) = U \subset U$. It is final since objects $V \in (p_\emptyset/U)^H$ satisfy $p(V) \subset U$, and therefore

$$V \subset p^{-1}p(V) \subset p^{-1}(U)$$

This shows that $p_\emptyset$ is left G-cofinal. Now let $X : \mathcal{P}(J) \to \mathscr{C}$ be a J-cube, and $X \xrightarrow{\simeq} FX$ a fibrant replacement. There is a commutative diagram

$$\begin{tikzcd}
\operatorname*{holim}_{\mathcal{P}(J)\backslash\emptyset} \iota_\emptyset^* FX \arrow[d, "p_\emptyset^*"', "\simeq"] & FX_\emptyset = (p^*FX)_\emptyset \arrow[l] \arrow[d] \\
\operatorname*{holim}_{\mathcal{P}(K)\backslash\emptyset} p_\emptyset^*(\iota_\emptyset^* FX) \arrow[r, equal] & \operatorname*{holim}_{\mathcal{P}(K)\backslash\emptyset} \iota_\emptyset^* p^* FX
\end{tikzcd}$$

where the left vertical map is an equivalence by G-cofinality 2.25 and where $\iota_\emptyset : \mathcal{P}(J)\backslash\emptyset \to \mathcal{P}(J)$ is the canonical inclusion. Notice moreover that $p^*X \xrightarrow{\simeq} p^*FX$ is a fibrant replacement for p^*X, as for every subset $S \subset K$ there is an inclusion of the stabilizer groups $G_S \leq G_{p(S)}$, and the forgetful functor $\mathscr{C}^{G_{p(S)}} \to \mathscr{C}^{G_S}$ preserves fibrant objects and equivalences by assumption. From the diagram above we see that X is homotopy cartesian if and only if p^*X is. □

Looking for a similar statement for the behavior of p^* on cocartesian cubes we run into the problem that p does not restrict to a functor $\mathcal{P}(K)\backslash K \to \mathcal{P}(J)\backslash J$. There is a formally dual version of the proof of A.1 that uses the complement dualities on $\mathcal{P}(K)$ and $\mathcal{P}(J)$, but it involves a functor $\mathscr{C}_a^{\mathcal{P}(J)} \to \mathscr{C}_a^{\mathcal{P}(K)}$ different from p^*. This is discussed in A.5 below. In order to understand the interaction between p^* and cocartesian cubes we need to introduce a new functor. Let $p^{-1}(j) \subset K$ denote the fiber of an element $j \in J$, and consider the equivariant functor

$$\lambda\colon \Big(\prod_{j\in J} \mathcal{P}\big(p^{-1}(j)\big)\backslash p^{-1}(j)\Big) \times \mathcal{P}(J)\backslash J \to \mathcal{P}(K)\backslash K$$

that sends a pair $(\{U_j\}_{j\in J}, V)$ to $(\amalg_{j\in J} U_j) \cup p^{-1}(V)$. The product $\prod_{j\in J} \mathcal{P}\big(p^{-1}(j)\big)\backslash p^{-1}(j)$ is the limit of the G-diagram of categories $j \mapsto \mathcal{P}(p^{-1}(j)) \setminus p^{-1}(j)$ with the G-structure induced by the G-action on J.

The functor λ is a categorical analogue of a homeomorphism

$$\big(\prod_{j\in J} \Delta^{|p^{-1}(j)|-1}\big) \times \Delta^{|J|-1} \cong \Delta^{|K|-1}$$

Example A.2.

- If $p\colon K_+ \to 1_+$ is the pointed map that sends all the elements of K to 1, the product of the fibers is simply $\mathcal{P}(K)\backslash K$ and the functor

  $$\lambda\colon \mathcal{P}(K)\backslash K \times \mathcal{P}(1_+)\backslash 1_+ \to \mathcal{P}(K_+)\backslash K_+$$

  is analogous to a homeomorphism $\Delta^{\overline{K}} \times \Delta^1 \cong \Delta^K$ that splits off a copy of the trivial representation from the permutation representation of K. This is written on a more familiar form as $\overline{\mathbb{R}}[K] \times \mathbb{R} \cong \mathbb{R}[K]$. One could think of the product of the categories $\mathcal{P}\big(p^{-1}(j)\big)\backslash p^{-1}(j)$ as an orthogonal complement for the image of the embedding $p^{-1}(-)\colon \mathcal{P}(J)\backslash J \to \mathcal{P}(K)\backslash K$.
- Let I and J be finite G-sets, and consider the pointed projection $p\colon (I \amalg J)_+ \to J_+$ that sends J to J by the identity, and I to the basepoint $+$. The preimages over the elements of J consist of a single point, and the preimage over the basepoint is $p^{-1}(+) = I_+$. The functor λ above is the functor

  $$\lambda\colon \mathcal{P}(I_+)\backslash I_+ \times \mathcal{P}(J_+)\backslash J_+ \longrightarrow \mathcal{P}((I \amalg J)_+)\backslash (I \amalg J)_+$$

  that sends (U, V) to $U \cup V$. It is analogous to the standard homeomorphism of permutation representations $\mathbb{R}[I] \times \mathbb{R}[J] \cong \mathbb{R}[I \amalg J]$.

Proposition A.3: *For a surjective equivariant map $p\colon K \to J$, the functor λ above is right G-cofinal. Moreover, the functor $p^*\colon \mathscr{C}_a^{\mathcal{P}(J)} \to \mathscr{C}_a^{\mathcal{P}(K)}$ preserves homotopy cocartesian cubes.*

Proof. Let us first prove that λ is well defined, that is, it does not take the value K. Write for simplicity $\underline{U} = \{U_j\}_{j\in J}$ and $\amalg\underline{U} = \amalg_{j\in J} U_j$. Suppose that $\lambda(\underline{U}, V) = (\amalg\underline{U}) \cup p^{-1}(V) = K$. Take j in the complement of V in J. The fiber $p^{-1}(j) \subset K$ is disjoint from $p^{-1}(V)$, but it is covered by the collection $\underline{U}$. As each U_i is contained in $p^{-1}(i)$ we must have $U_j = p^{-1}(j)$, but this is absurd since U_j is a proper subset of $p^{-1}(j)$.

Now let W be an H-invariant proper subset of K. We show that the right fiber category W/λ is H-contractible by defining a zig-zag of natural transformations between the identity functor and the projection onto the H-invariant object $(\underline{\emptyset} = \{\emptyset\}_{j\in J}, p(W))$ of W/λ. This is a well defined object as $\lambda(\{\emptyset\}_{j\in J}, p(W)) = p^{-1}p(W)$ which contains W. The intermediate functor of the zig-zag is the equivariant functor $\tau\colon W/\lambda \to W/\lambda$ defined by

$$\tau(\underline{U}, V) = (\underline{U}, p(\amalg\underline{U}) \cup V)$$

The values of τ are indeed objects of W/λ since $\lambda(\tau(\underline{U}, V))$ clearly contains $(\amalg\underline{U}) \cup p^{-1}(V)$ which in turn contains W as $(\underline{U}, V)$ belongs to W/λ. There is a zig-zag of natural transformations

$$\mathrm{id} \longrightarrow \tau \longleftarrow (\underline{\emptyset}, p(W))$$

Both maps are obvious on the first component. The second component of the rightward pointing map is the inclusion $V \subset p(\amalg \underline{U}) \cup V$. The second component of the left pointing map is induced by the inclusion $W \subset \lambda(\underline{U}, V)$, that when projected down to J gives $p(W) \subset p(\amalg \underline{U}) \cup pp^{-1}(V) = p(\amalg \underline{U}) \cup V$. The zig-zag above realizes to a contracting H-invariant homotopy of the category W/λ showing that λ is right G-cofinal.

Now let $X \in \mathscr{C}_a^{\mathcal{P}(J)}$ be a cocartesian J-cube and $QX \xrightarrow{\simeq} X$ a point-wise cofibrant replacement. As in the proof of A.1, notice that $p^*QX \xrightarrow{\simeq} p^*X$ is a point-wise cofibrant replacement of p^*X. Let us compute the homotopy colimit of p^*QX over $\mathcal{P}(K)\backslash K$. By G-cofinality and 2.26 there are G-equivalences

$$\operatorname*{hocolim}_{\mathcal{P}(K)\backslash K} p^*QX \xleftarrow{\simeq} \operatorname*{hocolim}_{\left(\prod_{j\in J}\mathcal{P}\left(p^{-1}(j)\right)\backslash p^{-1}(j)\right)\times \mathcal{P}(J)\backslash J} \lambda^* p^*QX \xleftarrow{\simeq} \operatorname*{hocolim}_{\left(\prod_{j\in J}\mathcal{P}\left(p^{-1}(j)\right)\backslash p^{-1}(j)\right)} \operatorname*{hocolim}_{\mathcal{P}(J)\backslash J} \lambda^*p^*QX.$$

We claim that for every fixed collection $\underline{U}$ of subsets of the fibers, the canonical map

$$\phi_{\underline{U}} \colon \operatorname*{hocolim}_{\mathcal{P}(J)\backslash J} (\lambda^*p^*QX)_{(\underline{U},-)} \to X_J$$

is a $G_{\underline{U}}$-equivalence. From this claim it follows by homotopy invariance of the homotopy colimit that $\operatorname*{hocolim}_{\mathcal{P}(K)\backslash K} p^*QX$ is equivalent to the homotopy colimit over $\prod_{j\in J}\mathcal{P}\big(p^{-1}(j)\big)\backslash p^{-1}(j)$ of the constant G-diagram with value X_J. Since the indexing category is G-contractible (it has a G-invariant initial object) this is G-equivalent to $X_J = (p^*X)_K$, proving that p^*QX is homotopy cocartesian.

Let us show that $\phi_{\underline{U}}$ is a weak equivalence. Write $Z^{\underline{U}} = (\lambda^*p^*QX)_{(\underline{U},-)} = QX_{p(\amalg\underline{U})\cup(-)}$. This is a J-cube with the G-action on J restricted to the stabilizer group $G_{\underline{U}}$. Then $\phi_{\underline{U}}$ is an equivalence precisely when $Z^{\underline{U}}$ is homotopy cocartesian. If any of the sets U_j is non-empty, the maps $(Z^{\underline{U}})_V \to (Z^{\underline{U}})_{V\cup j}$ are identities for every subset $V \subset J$. We proved in 3.27 that in this case $Z^{\underline{U}}$ is homotopy cocartesian. For the family of empty sets $\underline{U} = \underline{\emptyset}$, the J-cube $Z^{\underline{\emptyset}}$ is the cube X, which is assumed to be homotopy cocartesian. □

Remark A.4. In general $p^* \colon \mathscr{C}_a^{\mathcal{P}(J)} \to \mathscr{C}_a^{\mathcal{P}(K)}$ does not detect homotopy cocartesian cubes. In the proof of A.3 we constructed an equivalence over X_J

$$\operatorname*{hocolim}_{\mathcal{P}(K)\backslash K} p^*QX \simeq \operatorname*{hocolim}_{\prod_{j\in J}\mathcal{P}\left(p^{-1}(j)\right)\backslash p^{-1}(j)} Y$$

where Y is the diagram that sends $\underline{\emptyset} = (\emptyset, \dots, \emptyset)$ to $\operatorname*{hocolim}_{\mathcal{P}(J)\backslash J} QX$ and all the other verticies to X_J. If p^*X is homotopy cocartesian the left hand side is also equivalent to X_J, but this is in general not enough to conclude that $Y_{\underline{\emptyset}}$ is equivalent to X_J. However, this is the case if $\mathscr{C}$ is the category of spectra, as homotopy cocartesian J-cubes are the same as homotopy cartesian J-cubes (cf. 3.32). Hence the functor $p^* \colon (\mathrm{Sp}^O)_a^{\mathcal{P}(J)} \to (\mathrm{Sp}^O)_a^{\mathcal{P}(K)}$ preserves and detects homotopy cocartesian cubes.

We end this section by discussing the duals of A.1 and A.3. For an equivariant surjective map of finite G-sets $p \colon K \to J$, let $\overline{p} \colon \mathcal{P}(K) \to \mathcal{P}(J)$ be the composite functor

$$\overline{p} \colon \mathcal{P}(K) \longrightarrow \mathcal{P}(K)^{op} \xrightarrow{p^{op}} \mathcal{P}(J)^{op} \longrightarrow \mathcal{P}(J)$$

that sends a subset U of K to $J\backslash p(K\backslash U)$. The dual of the functor λ is defined by a similar composition, and an easy calculation shows that it is the functor

$$\overline{\lambda} \colon \Big(\prod_{j\in J}\mathcal{P}\big(p^{-1}(j)\big)\backslash\emptyset\Big) \times \mathcal{P}(J)\backslash\emptyset \to \mathcal{P}(K)\backslash\emptyset$$

that sends $(\underline{U}, V)$ to $(\amalg\underline{U}) \cap p^{-1}(V)$. The dual proofs of A.1 and A.3 give the following.

Proposition A.5: *The restriction $\overline{p}\colon \mathcal{P}(K)\backslash K \to \mathcal{P}(J)\backslash J$ is right G-cofinal, and the functor $\overline{\lambda}$ is left G-cofinal. It follows that $\overline{p}^*\colon \mathscr{C}_a^{\mathcal{P}(J)} \to \mathscr{C}_a^{\mathcal{P}(K)}$ preserves and detects homotopy cocartesian cubes, and preserves homotopy cartesian cubes.*

We end by noticing that this picture does not have an analogue for injective equivariant maps $\iota\colon J \to K$. It is easy to see that restricting along ι does not preserve any cartesian nor cocartesian properties of cubes. The right thing to study seems to be the preimage functor $\iota^{-1}\colon \mathcal{P}(K) \longrightarrow \mathcal{P}(J)$, but this does not restrict to either $\mathcal{P}(K)\backslash\emptyset \longrightarrow \mathcal{P}(J)\backslash\emptyset$ nor $\mathcal{P}(K)\backslash K \longrightarrow \mathcal{P}(J)\backslash J$. However, if J and K are pointed and ι preserves the basepoint, there is a retraction $p\colon K \to J$ that collapses the complement of the image of ι onto the basepoint. In this case we can simply contemplate p^*.

A.2 Finite categories and cofibrant G-diagrams

We give a criterion for determining if a G-diagram is cofibrant in the model structure of 2.6, when the over-categories of the indexing category I have finite dimensional nerve. Such categories are sometimes called directed Reedy categories. The criterion is in terms of latching maps, and it is completely analogous to the classical theory (see e.g. [Hir03, §15]).

Let $\mathscr{C}$ be a cocomplete category. We denote by $(I/i)'$ the over-category I/i with the object $i = i$ removed. The latching diagram of a diagram $X\colon I \to \mathscr{C}$ is the diagram $L(X)\colon I \to \mathscr{C}$ given on objects by

$$L(X)_i = \operatorname{colim}((I/i)' \longrightarrow I \xrightarrow{X} \mathscr{C})$$

and on morphisms $f\colon i \to j$ by the map induced on colimits by $f_*\colon (I/i)' \to (I/j)'$. The inclusions $(I/i)' \hookrightarrow I/i$ induce a maps $L(X)_i \to \operatorname{colim}_{I/i} u_i^* X \cong X_i$ which give a natural transformation $L(X) \to X$.

For a G-diagram $X \in \mathscr{C}_a^I$, the latching diagram $L(X)$ inherits a G-structure. The structure maps are the composite maps

$$L(X)_i \xrightarrow{L(g_X)} \operatorname{colim}\Big((I/i)' \xrightarrow{g} (I/gi)' \longrightarrow I \xrightarrow{X} \mathscr{C}\Big) \longrightarrow L(X)_{gi}$$

induced by taking colimits of the compositions in the diagram

$$\begin{tikzcd}
(I/i)' \arrow[r] \arrow[d, "g"] & I \arrow[d, "g"] \arrow[dr, bend left, "X"] \arrow[d, phantom, shift left=4, "\swarrow g_X"] & \\
(I/gi)' \arrow[r] & I \arrow[r, "X"'] & \mathscr{C}
\end{tikzcd}$$

and the map canonical map $L(X) \to X$ is a map of G-diagrams.

Proposition A.6: *Let $\mathscr{C}$ be a G-model category (see 2.1), and I a category with G-action such that the simplicial set NI/i is finite dimensional for every object i in I. Let X be an object of $\mathscr{C}_a^I$ such that for every object i in I the map $L(X)_i \to X_i$ is a cofibration in $\mathscr{C}^{G_i}$. Then X is cofibrant in the model structure on $\mathscr{C}_a^I$ of 2.6.*

Proof. In order to show that X is cofibrant we need to define a lift for every diagram in $\mathscr{C}_a^I$

$$\begin{tikzcd}
 & Y \arrow[d, two heads, "\sim"] \\
X \arrow[r] \arrow[ur, dashed, "l"] & Z
\end{tikzcd}$$

where the vertical map is an acyclic fibration. We build this lift by induction on a filtration of I defined by the degree function $\deg\colon ObI \to \mathbb{N}$

$$\deg(i) = \dim NI/i$$

It is easy to see that the degree function is equivariant (where $\mathbb{N}$ has trivial action), and that if $\alpha\colon i \to j$ is a non-identity morphism then $\deg(i) < \deg(j)$. Let $I_{\leq n}$ be the full subcategory of I with objects of degree less than or equal to n. Since the degree function is equivariant, the G-action of I restricts to $I_{\leq n}$, and the G-structure on X restricts to a G-structure on the restricted diagram $X_{\leq n}\colon I_{\leq n} \to I \xrightarrow{X} \mathscr{C}$. We build the lift inductively on the diagrams $X_{\leq n}$.

For the base step, choose a section $s\colon ObI_{\leq 0}/G \to ObI_{\leq 0}$. For every orbit $\gamma \in ObI_{\leq 0}/G$ one can choose a $G_{s(\gamma)}$-equivariant lift

$$\begin{tikzcd} & Y_{s(\gamma)} \arrow[d, two heads, "\sim"] \\ X_{s(\gamma)} \arrow[r] \arrow[ur, dashed, "l_{s(\gamma)}"] & Z_{s(\gamma)} \end{tikzcd}$$

since the map $\emptyset = L(X)_{s(\gamma)} \to X_{s(\gamma)}$ is a cofibration in $\mathscr{C}^{G_{s(\gamma)}}$ by assumption (the map $Y_{s(\gamma)} \to Z_{s(\gamma)}$ is an acyclic fibration of $\mathscr{C}^{G_{s(\gamma)}}$ as equivalences and fibrations in $\mathscr{C}^I_a$ are point-wise). Given any object $i \in I_{\leq 0}$ outside the image of s, define $l_i\colon X_i \to Y_i$ as the composite

$$X_i \xrightarrow{g^{-1}} X_{s([i])} \xrightarrow{l_{s[i]}} Y_{s([i])} \xrightarrow{g} Y_i$$

for a choice of $g \in G$ with $gs[i] = i$. Since the category $I_{\leq 0}$ is discrete (a G-set) by the properties of the degree function, these lifts define a map of diagrams $l^0\colon X_{\leq 0} \to Y_{\leq 0}$ lifting $X_{\leq 0} \to Z_{\leq 0}$. Moreover l respects the G-structure since the lifts $l_{s(\gamma)}$ are $G_{s(\gamma)}$-equivariant.

Now suppose we defined a lift $l^{n-1}\colon X_{\leq n-1} \to Y_{\leq n-1}$. Let I_n be the full subcategory of I with objects of degree n. Choose a section $s^n\colon ObI_n/G \to ObI_n$, and for every $\gamma \in ObI_n/G$ a lift in $\mathscr{C}^{G_{s^n(\gamma)}}$

$$\begin{tikzcd} L(X)_{s^n(\gamma)} \arrow[r] \arrow[d, tail] & Y_{s^n(\gamma)} \arrow[d, two heads, "\sim"] \\ X_{s^n(\gamma)} \arrow[r] \arrow[ur, dashed, "l_{s^n(\gamma)}"'] & Z_{s^n(\gamma)} \end{tikzcd}$$

The top horizontal map is the canonical map given by the universal property of the colimits defining $L(X)$. Again, the lifts exist because $L(X)_{s^n(\gamma)} \to X_{s^n(\gamma)}$ is a cofibration. For a general object i of I_n define

$$l_i\colon X_i \xrightarrow{g^{-1}} X_{s([i])} \xrightarrow{l_{s[i]}} Y_{s([i])} \xrightarrow{g} Y_i$$

Commutativity of the diagram above insures that the resulting map $l^n\colon X_{\leq n} \to Y_{\leq n}$commutes with the structure maps of $X_{\leq n}$ and $Y_{\leq n}$. Moreover l^n respects the G-structure by $G_{s^n(\gamma)}$-equivariance of $l_{s(\gamma)}$. $\square$

A.3 Sequential homotopy colimits and finite G-homotopy limits

Definition A.7 ([Kel82]). A simplicial category $\mathscr{C}$ is locally finitely presentable if there is a set of objects Θ satisfying

1. For every $c \in \Theta$ the mapping space functor

$$Map_{\mathscr{C}}(c, -)\colon \mathscr{C} \longrightarrow sSet$$

preserves filtered colimits,

2. every object of $\mathscr{C}$ is isomorphic to a filtered colimit of objects in Θ.

When $\mathscr{C}$ is locally finitely presented the functor $\mathrm{map}_{\mathscr{C}}(K,-)$ commutes with filtered colimits if K is a finite simplicial set. This follows from the conditions above and an adjunction argument.

We consider the poset category $\mathbb{N}$ of natural numbers as a category with trivial G-action.

Proposition A.8: *Let $\mathscr{C}$ be a G-model category, and suppose that the underlying simplicial categories $\mathscr{C}^H$ are locally finitely presentable for all $H \leq G$. Let J be a finite G-set and $X\colon \mathbb{N} \times \mathcal{P}(J_+) \to \mathscr{C}$ a G-diagram with the property that for every $n \in \mathbb{N}$ the J_+-cube X_n is homotopy cartesian. Then the J_+-cube* $\mathrm{hocolim}_{\mathbb{N}}\, QX_n$ *is also homotopy cartesian.*

Proof. We must show that the top horizontal map in the commutative diagram

$$\begin{array}{ccc}
\underset{\mathbb{N}}{\mathrm{hocolim}}\, QX_{n,\emptyset} & \longrightarrow & \underset{S\in\mathcal{P}(J_+)\backslash\emptyset}{\mathrm{holim}} F\, \underset{\mathbb{N}}{\mathrm{hocolim}}\, QX_{n,S} \\
\simeq\downarrow & & \downarrow\simeq \\
\underset{\mathbb{N}}{\mathrm{colim}}\, X_{n,\emptyset} & \longrightarrow & \underset{S\in\mathcal{P}(J_+)\backslash\emptyset}{\mathrm{holim}} F\, \underset{\mathbb{N}}{\mathrm{colim}}\, X_{n,S}
\end{array}$$

is a weak equivalence in $\mathscr{C}^G$. The left hand vertical map is an equivalence since in the locally finitely presentable category $\mathscr{C}^G$ filtered colimits are homotopy invariant (see e.g. [Dug01, 7.3], or [BK72] for simplicial sets). Similarly, the right hand vertical map is the homotopy limit of an equivalence of pointwisefibrant G-diagrams, as each $\mathscr{C}^{G_S}$ is locally finitely presentable. The bottom map can be factored as

$$\begin{tikzcd}
\underset{\mathbb{N}}{\mathrm{colim}}\, X_{n,\emptyset} \arrow[r] \arrow[dr, "\simeq"'] & \underset{S\in\mathcal{P}(J_+)\backslash\emptyset}{\mathrm{holim}} F\, \underset{\mathbb{N}}{\mathrm{colim}}\, X_{n,S} \\
& \underset{\mathbb{N}}{\mathrm{colim}}\, \underset{S\in\mathcal{P}(J_+)\backslash\emptyset}{\mathrm{holim}} FX_{n,S} \arrow[u]
\end{tikzcd}$$

with the diagonal map an equivalence in $\mathscr{C}^G$ since X_n is homotopy cartesian and filtered colimits in $\mathscr{C}^G$ preserve equivalences. To show that the vertical map is an equivalence, we compute from the definition of homotopy limits. Denoting $K_S = N\mathcal{P}(S)\backslash\emptyset$ we have isomorphisms in $\mathscr{C}^G$

$$\underset{\mathbb{N}}{\mathrm{colim}}\, \underset{S\in\mathcal{P}(J_+)\backslash\emptyset}{\mathrm{holim}} FX_{n,S} = \underset{\mathbb{N}}{\mathrm{colim}}\lim\left(\prod_S \mathrm{map}_{\mathscr{C}}(K_S, FX_{n,S}) \rightrightarrows \prod_{S\to T}\mathrm{map}_{\mathscr{C}}(K_s, FX_{n,T})\right) \cong$$
$$\cong \lim\left(\prod_S \mathrm{map}_{\mathscr{C}}(K_S, \underset{\mathbb{N}}{\mathrm{colim}}\, FX_{n,S}) \rightrightarrows \prod_{S\to T}\mathrm{map}_{\mathscr{C}}(K_s, \underset{\mathbb{N}}{\mathrm{colim}}\, FX_{n,T})\right) = \underset{S\in\mathcal{P}(J_+)\backslash\emptyset}{\mathrm{holim}}\, \underset{\mathbb{N}}{\mathrm{colim}}\, FX_{n,S}$$

where the middle map is an isomorphism because sequential colimits commute with finite limits and with the functors $\mathrm{map}_{\mathscr{C}}(K_s,-)$, since each K_S is finite. Now let $\overline{FX} \overset{\sim}{\twoheadrightarrow} FX$ be a replacement of FX by a sequence of diagrams such that for each $S \subset J_+$ the sequence $\overline{FX}_S$ is a sequence of G_S-cofibrations. There is a commutative diagram,

$$\begin{tikzcd}
\underset{\mathbb{N}}{\mathrm{colim}}\, \underset{S\in\mathcal{P}(J_+)\backslash\emptyset}{\mathrm{holim}} FX_{n,S} \arrow[r, "\cong"] \arrow[d] & \underset{S\in\mathcal{P}(J_+)\backslash\emptyset}{\mathrm{holim}}\, \underset{\mathbb{N}}{\mathrm{colim}}\, FX_{n,S} \arrow[d] & \underset{S\in\mathcal{P}(J_+)\backslash\emptyset}{\mathrm{holim}}\, \underset{\mathbb{N}}{\mathrm{colim}}\, \overline{FX}_{n,S} \arrow[l, "\sim"'] \arrow[d, "\sim"] \\
\underset{S\in\mathcal{P}(J_+)\backslash\emptyset}{\mathrm{holim}} F\, \underset{\mathbb{N}}{\mathrm{colim}}\, X_{n,S} \arrow[r, "\sim"] & \underset{S\in\mathcal{P}(J_+)\backslash\emptyset}{\mathrm{holim}} F\, \underset{\mathbb{N}}{\mathrm{colim}}\, FX_{n,S} & \underset{S\in\mathcal{P}(J_+)\backslash\emptyset}{\mathrm{holim}} F\, \underset{\mathbb{N}}{\mathrm{colim}}\, \overline{FX}_{n,S} \arrow[l, "\sim"']
\end{tikzcd}$$

where the right hand vertical is a weak equivalence because $\operatorname{colim}_{\mathbb{N}} \overline{FX}_{n,S}$ is fibrant by an application of the small object argument in the cofibrantly generated model category $\mathscr{C}^G$ (see e.g. [Sch97, 1.3.2]). It follows that the left hand vertical map is a weak equivalence as desired.

$\square$

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