Moduli spaces of manifolds

Week 2

February 23, 2012

Problem 1

Recall the maps $\alpha = \alpha(n) : B_d(\mathbb{R}^n) \to \Omega^n \Psi_d(\mathbb{R}^n)$ defined in class. Consider $\mathbb{R}^n = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ and define the continuous map

$$\sigma(n) : \Psi_d(\mathbb{R}^n) \to \Omega \Psi_d(\mathbb{R}^{n+1})$$

$$W \mapsto \begin{cases} t \in \mathbb{R} \mapsto W + t \cdot e_{n+1} \\ \infty \mapsto \emptyset. \end{cases}$$

Show that the diagram

$$\begin{array}{ccc} B_d(\mathbb{R}^n) & \xrightarrow{\alpha(n)} & \Omega^n \Psi_d(\mathbb{R}^n) \\ \downarrow & & \downarrow \Omega^n \sigma(n) \\ B_d(\mathbb{R}^{n+1}) & \xrightarrow{\alpha(n+1)} & \Omega^{n+1} \Psi_d(\mathbb{R}^{n+1}) \end{array}$$

commutes.

Recall the inclusions $q(n) : Th(\gamma_{d,n}^+) \hookrightarrow \Psi_d(\mathbb{R}^n)$ defined in the first lecture. Construct maps $\bar{\sigma}(n) : Th(\gamma_{d,n}^+) \to \Omega Th(\gamma_{d,n+1}^+)$ so that the diagram

$$\begin{array}{ccc} Th(\gamma_{d,n}^+) & \xrightarrow{q(n)} & \Psi_d(\mathbb{R}^n) \\ \downarrow \bar{\sigma}(n) & & \downarrow \sigma(n) \\ \Omega Th(\gamma_{d,n+1}^+) & \xrightarrow{q(n+1)} & \Omega \Psi_d(\mathbb{R}^{n+1}) \end{array}$$

commutes.

The commutativity of these diagrams implies a map $q(\infty) : \Omega^\infty MTO(d) \to \Omega^\infty \Psi_d$, where

$$\Omega^\infty MTO(d) = \colim_{n \to \infty} \Omega^n Th(\gamma_{g,n}^+)$$

$$\Omega^\infty \Psi_d = \colim_{n \to \infty} \Omega^n \Psi_d(\mathbb{R}^n)$$

have the direct limit topology. Deduce that $q(\infty)$ is a weak equivalence.
Problem 2

In this exercise we will give a proof of *Ehresmann’s fibration theorem*: If $E$ and $X$ are smooth manifolds (without boundary) and $p : E \to X$ is a proper submersion, then for each $x \in X$ there exists an open neighbourhood $U = U_x \subseteq X$ and a diffeomorphism $f : U \times p^{-1}(x) \to p^{-1}(U)$ such that $p \circ f(u, e) = u$. (I.e. $p$ is a smooth fiber bundle.)

1. Show that it suffices to consider the case $X = \mathbb{R}^k$ and $x = 0$.

2. Prove the case $k = 1$ as follows. First prove that there exists a compactly supported vector field $V$ on $E$ such that $(D_ep)(V(e)) = \partial/\partial t \in T_{p(e)}\mathbb{R}$ for each $e \in p^{-1}(-1, 1)$. Then prove that the flow $F = F_V : \mathbb{R} \times E \to E$ of $V$ restricts to a diffeomorphism $f : (-1, 1) \times p^{-1}(0) \to p^{-1}(-1, 1)$ satisfying the required condition.

3. Prove the case $k > 1$ by induction. [Hint: the idea from (ii) can be reused to extend a trivialisation of $p^{-1}(\mathbb{R}^{k-1} \times \{0\})$ to $p^{-1}(\mathbb{R}^{k-1} \times (-1, 1))$.]