Equivariant $K$-theory ($G$ compact Lie group)

**Def.** A $G$-vector bundle is a $G$-map $E \rightarrow X$ such that $E$ is a vector bundle and the action $g : E_x \rightarrow E_{g \cdot x}$ is linear for all $x \in X$.

**Def.** For $X$ finite $G$-CW complex, we can define the $G$-equivariant $K$-theory of $X$:

$$K_G(X) = \text{Grothendieck construction on the semiring } \mathbb{R}^+ \text{ of classes of complex } G \text{-vector bundles over } X.$$

**Properties:**
- $K_G(X)$ is an $R[G]$-module.
- When $X$ is a $G$-free $G$-space:
  $$K_G(X) \cong K(X/G).$$
- When $X$ is a trivial $G$-space:
  $$K_G(X) \cong RG \otimes K(X).$$
  $$K_H(X) \cong K_G (G, X)$$ for $G \subset G$, $X$ $H$-CW, $G \subset $.
Theorem. (Thom isomorphism)
If $E \rightarrow X$ is a complex $G$-vector bundle, then there is a natural isomorphism

$$K_G(X) \rightarrow \tilde{K}_G(\text{Thom}(E))$$

$$u \ (\text{Thom class})$$

$$\tilde{K}_G(X) = \ker (K_G(X) \rightarrow K_G(\text{pt}))$$

Corollary. (Bott periodicity)
$$\tilde{K}_G(X) \cong \tilde{K}_G(\Sigma^V X)$$

for any representation $V$.

In particular, take $V = \mathbb{C} \rightarrow 2$-periodic cohomology theory $K_G^*(-)$ with

$$\tilde{K}_G^0(X) = \tilde{K}_G^0(X)$$

$$\tilde{K}_G^1(X) = \tilde{K}_G(\Sigma X).$$
Theorem 1 (Atiyah–Segal '69)

Let $X$ be a finite $G$-CW complex, then the projection map $\pi: E_G \times X \to X$ induces an isomorphism

$$K^*_G(I)^\wedge \xrightarrow{\pi^*} K^*_G(E_G \times X)$$

$I = \ker (R(G) \to \mathbb{Z})$

$$\cong K^*_G(E_G \times X)$$

By $K_*(X) = K_*(X \cup)$

for free $X$.

$$K^*_G(G_+ \wedge S^n) \cong K^*_G(S^n)$$

Complete with I.

Sketch of proof of Theorem

(following A–H–J–M '88)

Problems: $(-)^\wedge I$ is not exact

$$K^*_G(\text{colim } X_i) \neq \text{lim } K^*_G(X_i)$$
Def: \( K_0^*(X) := \bigcap K_0^*(X_\alpha) \) for \( X \) goes through all finite \( G \)-CW-complexes.

\[ K_0^*(X)^\wedge = \bigcap K_0^*(X_\alpha) / I^n \]

\[ K_0^*(-), K_0^*(*)^\wedge \] are both pro-\( R(G) \)-module valued cohomology theories.

**Theorem 2:** Suppose \( X, Y \) are \( G \)-CW-complexes, \( f: X \to Y \) a \( G \)-map and non-equivariant homotopy equivalence. Then \( f \) induces an isomorphism

\[ f^*: K_0^*(Y)^\wedge \to K_0^*(X)^\wedge \]

**Sketch of proof of Theorem 2:**

Enough to show that

\[ K_0^*(Z) = 0 \text{ if } Z = pt \text{ non-equivariantly.} \]

(prode \( Z = G_\# \))
Use induction on subgroups of $G_r$.
For $G_r = \ast$, follows from $\mathbb{Z} \cong \ast$ pt.
Assume the claim holds for all proper $H < G_r$ (closed subgroup)
$$K^*_{/H}(\mathbb{Z})^1 = 0.$$  

Let $U$ be a complete $G_r$-universe,  
and let $U' = U - U G_r$. (Cartesian complement of $U G_r$ in $U$).

Define
$$T = \colim_{V < U'} S^V$$
finite dimensional $G_r$-subrepresentations.

There is a cofiber sequence
$$S^0 \rightarrow T \rightarrow T/S^0 \rightarrow$$
$$\rightarrow T G_r$$
This gives rise to a cofiber sequence
\[ \mathbb{Z} \rightarrow \mathbb{Z} \times T \rightarrow \mathbb{Z} \times T / S_0 \]

so it is enough to show that

I) \[ \overset{\sim}{K}^*_G(\mathbb{Z} \times T)_{+} \rightarrow 0 \]

II) \[ \overset{\sim}{K}^*_G(\mathbb{Z} \times T / S_0)_{+} \rightarrow 0 \]

For I), we need

Lemma 1: \[ T \cong H^{	ext{pt}} \quad \text{for } H \neq G. \]

(fairly easy)

Lemma 2: \[ \overset{\sim}{K}^*_G(T)_{+} \rightarrow 0 \]

This is a key step in the proof.

(uses the equivariant Bott periodicity theorem)

To prove I), decompose \( \mathbb{Z} \) into cells \( \rightarrow \)

\[ \overset{\sim}{K}^*_G((G/H)_+ \times T)_{+} \rightarrow 0 \]

For \( H = G \), use Lemma 2

For \( H \neq G \), use \[ \overset{\sim}{K}^*_G(G/(H_+ \times T)) \cong \overset{\sim}{K}^*_G((G_+ \times T)_{+} \rightarrow \overset{\sim}{K}^*_H(T)_{+} \rightarrow 0 \]

(cheegergromov)
Proof at II)

Decompose $T/S^0$ into cells $\Rightarrow$

$\Rightarrow K^*_G(\mathbb{Z} \wedge (G/H)_+)^\wedge_1 = 0$

for $H \leq G$. Then we have that

$K^*_G(\mathbb{Z} \wedge (G/H)_+)^\wedge_1 :\mathcal{K}^*_H(\mathbb{Z})^{\wedge_1} = 0$

induction assumption

Sketch of proof that Theorem 2 $\Rightarrow$ Theorem 1.

Lemma: If acts freely on $X$, then

$K^*_G(X)^\wedge_1 = K^*_G(X)$

Main ingredient

$K^*_G(X) \cong K^*(X/G)$

Pick a model for $EG$ such that

$EG = \bigcup_{k=1}^{\infty} E G_k$, $E G_k \subset E G$

finite $G$-CW complexes.
We have the diagram

\[ \begin{array}{ccc}
K^*_G(X) \uparrow & \xrightarrow{\pi^*} & K^*_G(EG \times X) \uparrow \\
\uparrow & \quad & \uparrow \\
\mathbb{E}_{K^*_G(U \times X / \text{In}^* \mathfrak{S}_n)} & \cong & K^*_G(EG \times X) \\
\end{array} \]

Take \( \text{lim} \) on both sides \( \Rightarrow \)

\[ K^*_G(X) \uparrow \xrightarrow{\text{lim}} \text{lim}_{k} K^*_G(EG_k \times X) \]

\( \text{lim} \) - problem remains for \( \mathbb{E}(K^*_G(EG_k \times X)) \), because it vanishes for the system \( K \)

'some other pro-system \( \mathbb{E}(K^*_G(X) \uparrow \text{In}^* \mathfrak{S}_n) \)

as this satisfies Mittag-Leffler.