Naive $G$-spectrum

Spectrum with a $G$-action, i.e., $X = G X_n$ with $G \cdot 0 X_n$ and the structure maps $X_n \wedge S^m \to X_{n+m}$ are $G$-equivariant.

Recall: For $G$-spaces $X, Y$, we defined stable maps $G \{X, Y\}_G = \varprojlim \{S^V \wedge X, S^V \wedge Y\}_G$

We need another notion of equivariant spectra.

In the naive category, $[\Sigma^\infty X, \Sigma^\infty Y]_G \neq G \{X, Y\}_G$.

Lewis–May–Steinberger spectra

$U = \text{infinite dimensional inner product space}.$

Def: A prespectrum $X$ is a collection of based spaces $G \{X(V)\}_V \subset U$, (V f.d. subspace)

with structure maps $G \varrho_V W : X(V) \wedge S^{W-V} \to X(W)$ whenever $V \subset W$. $W-V$ is the orthogonal complement of $V$ in $W$.

$pS U$ = prespectrum indexed on $U$.
Def. A spectrum $X$ is a prespectrum in which the adjoints of the structure maps

$$\Phi_{v,w} : X(v) \to \Omega^{w-v} X(w)$$

are homeomorphisms.

$\mathcal{S}L$ - spectra indexed on $\mathcal{S}L$.

There is a left adjoint to the forgetful functor $\mathcal{S}L \to \mathcal{P}S\mathcal{L}$

$$\begin{array}{ccc}
\mathbb{L} & \dashv & \mathcal{S}L \\
\mathcal{P}S\mathcal{L} & \overset{\text{forgetful}}{\hookrightarrow} & \mathcal{S}L
\end{array}$$

In the case where $\Phi_{v,w}$ are all inclusions, then

$$\mathbb{L}(X)(v) = \colim_{v \in \mathcal{W}} \Omega^{w-v} X(w)$$

Then, if $X$ is a based topological space, we define

$$\Sigma^\infty X = \mathbb{L}(S^\infty X)$$

where $S^\infty(X)(v) = S^v \wedge X$
A \( G \)-universe is called a \( G \)-universe if it contains all its finite dimensional subrepresentations, infinitely often.

A \( G \)-universe is complete if it contains all irreducible representations (e.g., \( \bigoplus_{n \in \mathbb{N}} \mathbb{R}[G^n] \)).

We can define \( G_{PSU}, G_{SU} \).

Naive \( G \)-spectra = \( G \)-spectra where the action of \( G \) on \( U \) is trivial.

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Charge of universe functors

If \( \alpha : U \rightarrow U' \) then we can define

\[
\alpha^* : SU' \rightarrow SU
\]

\[
\alpha^*(x)(v) = x(\alpha(v))
\]

\[
\alpha_* : SU \rightarrow SU'
\]

\[
\alpha_*(x)(v) = x(\alpha^{-1}(v)) = (\alpha^{-1})^*(x)(v)
\]

If \( \alpha : U \rightarrow U' \) is an isometry, then

\[
\alpha^* : SU' \rightarrow SU \text{ is defined, but}
\]

\[
\hat{\alpha}^* : PU \rightarrow PU' \quad \hat{\alpha}^*(x)(v) = x(\alpha^{-1}(v)) v_{\alpha(v)}
\]
\( \alpha^* : \mathcal{F}U \to \mathcal{F}U' \)

\[ \alpha^*(X) := \mathcal{L}(\hat{\alpha}^*(X)) \]

**Proposition:** \( \alpha^* \) is left adjoint of \( \alpha^* \)

**Proof:**

\[
\mathcal{G}\mathcal{F}\mathcal{U}'(\alpha^*(X), Y) = \mathcal{G}\mathcal{F}\mathcal{U}'(\hat{\alpha}^*(X), Y)
\]

\[
= \text{colim} \ G_{-\text{Top}^*}(\hat{\alpha}^*(X)(v), Y(v)) \\
\quad \text{vc} U'
\]

\[
= \text{colim} \ G_{-\text{Top}^*}(X(\alpha^{-1}(v)) \times S(-), Y(v)) \\
\quad \text{vc} U'
\]

\[
= \text{colim} \ G_{-\text{Top}^*}(X(\alpha^{-1}(v)), Y(\alpha^{-1}(v))) \\
\quad \text{vc} U'
\]

\[
= \mathcal{G}\mathcal{F}\mathcal{U}'(X, \alpha^* Y).
\]

\( G \)-spectra are tensored over \( G_{-\text{Top}^*} \).

If \( E \in \mathcal{G}\mathcal{F}\mathcal{U}' \), \( X \in G_{-\text{Top}^*} \) we have

\[ E \wedge X, \ \mathcal{F}(X_{/E}) \]

\( \mathcal{F}(X_{/E}) \) can be defined by levelwise mapping spaces

\[ X \wedge \mathcal{E} \] on prespectra

\[(X \wedge \mathcal{E})(v) = X \wedge \mathcal{E}(v) \] Prespectrum instead of spectrum

\[ \text{Instead, define } X \wedge \mathcal{E} := \mathcal{L}(X \wedge \mathcal{E}) \]

Then \( \mathcal{G}\mathcal{F}\mathcal{U}'(X \wedge \mathcal{E}, \mathcal{E}') = \mathcal{G}\mathcal{F}\mathcal{U}'(E, \mathcal{F}(X_{/E}')) \)
A homotopy of maps $E \to E'$ of $G$-spectra is a map $E \times I_+ \to E'$ (I with trivial $G$-action). Denote

$$[E, E']_G = \text{homotopy classes of maps } E \to E'$$

Define: Homotopy groups of $G$-spectra

$$\pi^H_n(E) = [G_\mathbb{H}, I S^n, E]_G$$

A map $f : E \to E'$ is said to be a weak equivalence if $f_* : \pi^H_n(E) \to \pi^H_n(E')$ is an isomorphism for all $H \leq G$.

(Different maps give the same homotopy class)

**Theorem:** A map $f : E \to E'$ of $G$-spectra is a weak equivalence if and only if for all $V \leq U$, $f(V) : E(V) \to E'(V)$, is a weak equivalence.

$\leq$ If $n \geq 0$, $\pi^H_n(E) = \pi^H_n(E(0))$

$$\pi^{H-n}_n(E) = \pi^{H}_n(E(\mathbb{R}^n))$$

$\Rightarrow$

It suffices to prove this.

Prove this by induction on the subgroups of $G$:

If the theorem is true for all subgroups then it is true for the group.
\( t(v) : E(v) \to E'(v) \) is weak equivalence.

\( V = Z \oplus W, \ Z = V  \mathcal{G} \)

\( \Omega Z : E(v) = E(w) \)

\[ E(v) = \left( \Sigma^{-Z} E \right)(w) \]

\[ \Sigma^{-Z} \text{ is the functor: } (\text{on the universe shell } \mathcal{G} \text{ containing } Z) \]

\[ (\Sigma^{-Z} E)(Z \oplus A) = E(A) \]

\( \Sigma^{-Z} \) preserves weak equivalences.

So \( \forall t(v) \) is enough to show that \( \forall W \) such that \( W \mathcal{G} = \mathcal{G} \).

Let \( SW = \) unit sphere in \( W \)

\( DW = \) unit ball in \( W \)

\( SW^t = DW + SW^t \).

We get an associated fiber-sequence

\[ E(0) : F(SW, E^0) \to F(DW^+ , E^0) \to F(SW^t, E^0) \]

\[ \downarrow \quad \downarrow \quad \downarrow \]

\[ E(0) : F(SW^t, E(0)) \to F(DW^+ , E(0)) \to F(SW^t, E(0)) \]

By the 5-lemma, need to show that \( F(SW^t, E(w)) \to F(SW^t, E'(w)) \) is a weak eq. 
\[ [\Sigma^n S^w, E^w]_G \to [\Sigma^n S^w, E^w]_G \]

\( S^w \) has cells \( G/H \times \Sigma^n \) where \( H \triangleleft G \). \( \Box \)