Equiv Seminar, Lecture 5.

Self-maps of a representation sphere.  
Goal: Describe $[S^V, S^V]^0_G$ for $V$ "big enough".

$V = C G$. Let $S^V$ be given a $C$-CW-structure.

Note: $V = C \otimes W$, where $C = (CG)^G$.  $G \cong (CH)$.

$\dim (S^V)^G = 2$, $G \cong (CH)$.

$G \cong (CH)$.

$\dim (S^V)^G = \dim (S^V)^G - 2$.

$\dim (V^H) = \dim (S^V)^G - \dim (V^H)$.

Def: $\Phi: [S^V, S^V]^0_G \longrightarrow \prod_{(H) \in C(G)} [S^V]^H, (S^V)^H \otimes \mathbb{Z}$.

$C(G)$ is the class of all $G$-sets.

$\Phi([f]) = (\deg (f^H))_{(H) \in C(G)}$.

Theorem (Equivalency Hopf Theorem): If $S^V$ is sphere where $V = C G \otimes V'$.

1) $\Phi$ is injective.

2) $\text{Im}(\Phi) = \text{Im}(\psi)$ where $\psi: A(G) \longrightarrow \prod_{(H)} \mathbb{Z}$.

where $A(G)$ is the Burnside ring of finite classes of finite $C$-sets.

Addition: $++$;  
Multiplication $\times$.

Take the Grothendieck construction to get ring.

$\psi$ is induced by $X \mapsto (X^H, 1)_{(H) \in C(G)}$.  

\[\psi \circ X \mapsto (X^H, 1)_{(H) \in C(G)}\]
\( \text{Cov} : \pi_0^s(S^0) \cong \text{A}(G) \). 

Order \( (H_1), \ldots, (H_n) \) s.t. \( (H_i) \leq (H_j) \Leftrightarrow i \geq j \).

\[ \sum (H) \leq (K) \Leftrightarrow \exists H \leq K \text{ for some } j \leq n \].

If \( X \) is G-CW-complex, the cone \( (H_i), \ldots, (H_n) \) induces a G-CW-filtration

\[ X^0 = X_1 \leq \ldots \leq X_r \leq X \]

where \( X_i = \bigcup x \in X \mid (G_x) = (H_j) \) for some \( j \leq r \).

\[ = \bigcup_{j=1, \ldots, r} G/H_j \times D^n_x \]

Since if \( \exists \text{ mp } G/H \to G/K \) then \( (H) \leq (K) \).

Let \( H = H_i \), Notice \( X^H = X_i \),

and \( X^H_{i-1} = \bigcup_{x \in X} X^{(G_x) = (H_j)} \).

Let \( Y \) be another G-CW-complex, and suppose we have \( k : X_{i-1} \to Y \). Then we have a bijection of extensions:

\[
\begin{array}{rcl}
X^H_{i-1} \xrightarrow{k} Y & \xrightarrow{\text{cov}} & X^H_i \xrightarrow{\text{cov}} Y^H \\
\downarrow \text{G-equiv.} & \quad & \downarrow \text{G-equiv.} \\
X_{i-1} \xrightarrow{k} Y & \xrightarrow{\text{cov}} & X^H_{i-1} \xrightarrow{k} Y^H \\
\end{array}
\]

\[
\text{WH} = \text{WH}^H \quad \text{recov} \quad X^H \quad \text{WH-cov} \]

\[
\begin{array}{rcl}
X \xrightarrow{\text{cov}} & X^H & \xrightarrow{\text{cov}} Y^H \\
\end{array}
\]
Now, let $X = Y = S^V$. Suppose real $\Phi (\xi (f)) = (\deg (f^H))$.

**Thm 1)** $\Phi$ is surjective.

2) Let $[f, g] \in [S^V, S^V]_G$.

If $\deg (f^K) = \deg (g^K) \quad \forall K \geq H$ then $\deg (f^H) = \deg (g^H) \mod |W^H|$. 

Assume $[k] \in [(S^V)_{-i}, (S^V)_{i-1}]_G$.

Extensions of $[k]$ to $[X_i, Y_i]_G$ correspond to homotopy classes of $W^H$-extensions of $[k/\pi^H]$ to $x^H$.

Obstructions to existence lie in $H^i (X^H/\pi^H, x^H/\pi^H, \pi_{i-1})(Y^H)$.

Hence there groups $0$ for $i \leq n(H)$ and $0$ for $i > n(H)$ and $0$ for $i > n(H)$. Since $\dim x^H = n(H)$.

So $\exists$ extension.

Given a lift $[f]$ to $x^H$.

Homotopy classes of other lifts are in bijection with $H^i(W^H, x^H/\pi^H, \pi_{i-1})(Y^H)$ via "primary obstruction" $\deg (f, g)$.
Lemma. \( n = n(H) \) \( \Rightarrow \) \( H^n \left( X^H/WH, \bar{X}^H/WH; \pi_0(Y) \right) = 2 \).

Prove \( Z_H = \pi_0(Y) \) as \( WH \)-module.

\[
= \left( \text{Hom}_W \left( \mathcal{C}_* \left( X^H, \bar{X}^H \right), Z_H \right) \right) \quad \text{(Use cellular chain)}.
\]

\[
= \left( \text{Hom}_W \left( \mathcal{C}_* \left( X^H \right), Z_H \right) \right)
\]

Take \( \text{map to} \) \( \text{Hom}_W \left( \mathcal{C}_* \left( X^H \right), Z_H \right) \).

\[
t_\ell (\phi)(\sigma) = \sum_{w \in \mathcal{W}} \omega \cdot \phi(w \cdot \sigma)
\]

Induces map on \( \text{hom} \).

Only have free cells in deg \( n \). \( \mathcal{C}_n(x^H) \) free \( WH \)-mod.

So \( t_\ell \) is surjective.

Also for \( \phi \in \text{Hom}_W \left( \mathcal{C}_* \left( X^H \right), Z_H \right) \) we have \( t_\ell (\phi) = 1_{WH} \cdot \phi \).

i.e.,

\[
\begin{array}{ccc}
H^n \left( X^H/WH, \bar{Z}_H \right) & \xrightarrow{\tau^*} & H^n \left( X^H; Z \right) \\
& \xrightarrow{t_\ell} & H^n \left( X^H/WH, \bar{Z}_H \right)
\end{array}
\]

Therefore,

\[
H^n \left( X^H/WH, \bar{Z}_H \right)
\]

So \( H^n \left( X^H/WH, \bar{Z}_H \right) \).
Primary obstruction:

\[ d_c(f, g) \] is mapped by \( \gamma^* \) into an obstruction against a non-abelian set homotopy between \( f \) and \( g \).

Classical\ H^* \ d(f, g) = d(f) - d(g) .

So \( H^\ast(X^n; 2) \xrightarrow{\gamma} H^\ast(X^n/\text{WH}; 2_H) \).

So \( d(f, g) = \pm |\text{WH}| \cdot d_c(f, g) \).

So map. injective all have some hold on image. \( \Box \)