\[ X \quad G - \text{CW-complex} \]

**Def:** A coefficient system is a functor

\[ O^G \longrightarrow Ab \]

e.g., \( F : \text{Top} \longrightarrow Ab \) is a functor, we let

\[ F(X)(G/H) = F(X^H) \]

"Bredon chains"

\[ C_n(X) = H_n\left(X^n, X^{n-1}, \mathbb{Z}\right) \]

\[ G/H \longrightarrow H_n\left((X^n)^H, (X^{n-1})^H, \mathbb{Z}\right) \]

The usual connecting homomorphism for \((X^n, X^{n-1}, X^{n-2})\) gives a natural transformation

\[ C_n(X) \longrightarrow C_{n-1}(X) \quad \text{with} \quad d^2 = 0 \]

So \( \underline{C}_\bullet(X) \) is a chain complex in coefficient systems.
If $M$ is another coefficient system, we define

$$C^*_G(X; M) := \bigoplus \Hom_{\mathcal{O}_G} (\mathcal{E}^n(X), M)$$

This is a chain complex of abelian groups with homology $H^*_G(X; M)$.

This is Bredon cohomology.

If $N: \mathcal{O}_G \to \text{Ab}$ is a covariant functor, we can define $C^*_N(X; N) = \bigoplus \Hom_{\mathcal{O}_G} (\mathcal{E}^n(X), N)$.

The homology of this chain complex is the Bredon homology $H^*_G(X; N)$.

Observation:

$H^*_G(-; M)$ and $H^*_G(-; N)$ are functorial for cellular $G$-maps and $G$-equivalences, and induce isomorphisms.

If $X$ is a $G$-space, we define

$H^*_G(X; M)$ and $H^*_G(X; N)$ using a cellular approximation $\Gamma X \to X$. 


Dimension axiom

\[ H^*_\mathbb{C}(G/H; M) \cong M(G/H) \]
\[ H^*_\mathbb{C}(G/H; N) \cong N(G/H) \]

Remark: \( C^*_\mathbb{C}(G/H) \) is concentrated in degree 0 (because we are working with finite groups)

\[ C^0(G/H, G/k) = H^0((G/H)^k, \mathbb{Z}) = \mathbb{Z}[G/k, G/H] \otimes G \]

\[ C^0(G/H) = \mathbb{Z}[-, G/H] \otimes G \]

Hence, \( H^0_G(G/H, M) = \text{Hom}_G(\mathbb{Z}[-, G/H], M) = M(G/H) \)

Similarly for homology.
Let \((X,A)\) be a \(G\)-CW-pair.

**Theorem:** There is a long exact sequence

\[
\begin{align*}
H^*_G(X; M) & \longrightarrow H^*_G(A; M) \\
\uparrow & \\
H^*_G(X; A; M) & \\
\end{align*}
\]

**Proposition:** \(C^*_G(X): \mathcal{O}_G^{\text{op}} \rightarrow \text{Ab} \) is a projective object of \(\text{Ab} \circ \mathcal{O}_G^{\text{op}}\).

**Proof:**

\[
\begin{align*}
C^*_G(X) & = H^*_G(X^n, X^{n-1}; \mathbb{Z}) \\
& = \widetilde{H}^*_G(X^n/X^{n-1}; \mathbb{Z})
\end{align*}
\]

but \(X^n/X^{n-1}\) is a wedge of \(G/H \wedge S^n\)'s.

Therefore we get

\[
\bigoplus \widetilde{H}^*_G(G/H \wedge S^n; \mathbb{Z})
\]

\[
\cong \bigoplus \widetilde{H}^*_G(G/H; \mathbb{Z})
\]

\[
\text{but } \widetilde{H}^*_G(G/H; \mathbb{Z}) = \mathbb{Z}[-, G/H] \circ \mathcal{O}_G
\]

\[
\begin{array}{c}
\begin{pmatrix}
g \in \text{MCG}(H) \\
g \in \text{N}(G/H)
\end{pmatrix} \quad \downarrow \\
\mathbb{Z}[-, G/H] \circ \mathcal{O}_G \quad \quad \downarrow \\
\mathbb{Z} \end{array}
\]

\(\square\)
Proof: There is a short exact sequence

\[ 0 \to C_0^*(A) \to C_0^*(X) \to C_0^*(X/A) \to 0 \]

Apply \( \text{Hom}_{G_0}(\_ , M) \). This preserves exactness as these are complexes of projectives.

We get

\[ 0 \leftarrow C_0^*(A; M) \leftarrow C_0^*(X; M) \leftarrow C_0^*(X/A; M) \leftarrow 0 \]

and then a long exact sequence in homology.

Example: Constant coefficient system.

\[ C_0^n(X; G) = C^n(X; G) \]

\[ H^* \]

\[ G = \mathbb{Z}/p \]

\[ N(G/H) = \{ 0 \} \]

Claim: \( H^*_G(X; N) = H^*(X; G) \)

\[ \text{OA: } G \to G \]

\[ \mathbb{Z}/p \]
\[ C^n(X) \xrightarrow{\partial} 0 \]
\[ C^n(x^G) \xrightarrow{+} \mathbb{Z} \]

\[ C^n(x; N) = C^n(x^G) \]

---

To recover ordinary cohomology, take \( M = \mathbb{R}^{[-1]} \) (free abelian group)

\[ H^*_G(x; M) = H^*(x; \mathbb{R}) \]

---

\( G = \mathbb{Z}(\rho) \quad x^G \rightarrow x \rightarrow x/x_G = :FX \)

If we take the coefficient system

\[ L(G/H) = \{ \mathbb{Z} \xrightarrow{\rho} G \} \]

then

\[ H^*_G(x; L) = H^*(FX/\rho) \]