Elmendorf's Theorem

Topology reading seminar
13/10 2010
10:15 - 11:15

References:
Alaska, Chapter V

Elmendorf: System of fixed point sets
TAMS 277 (1983) no.1
$G$ top. group.

$O_G$ orbit category

- full subcategory of $G$-spaces with objects $G/H$ for all closed subgroups $H \leq G$.

- topological category: $\text{Hom}_{O_G}(G/H, G/K) \equiv (G/K)^H$

An $O_G$-space is a continuous functor

$$T: O_G^{op} \to \text{Top}$$

$X$ $G_1$-space, $\mathbb{E}X$ is the $O_G$-space with

$$\mathbb{E}X(G/H) = X^H$$

**Theorem (Elmendorf)**

There is a functor

$$G_1$$

$O_G$-spaces $\xleftarrow{\sim} G_1$-spaces

and natural weak equivalences

$$\mathbb{E}X \xrightarrow{\sim} X$$

for $G_1$-spaces $X$ and $O_G$-spaces $T$.

**Note:** $T \to S$ is a weak equivalence in $O_G$-spaces if and only if $T(G/H) \xrightarrow{\sim} S(G/H)$ for all $G/H$.

$X \to Y$ is a weak equivalence in $G_1$-spaces iff

$$X^H \xrightarrow{\sim} Y^H$$

for all $H$, i.e., iff $\mathbb{E}X \xrightarrow{\sim} \mathbb{E}Y$. 
Applications

1) A family of subgroups of $G$ closed under conjugation and subgroups.

Then define an $O_G$-space $T$ by

$$T(G/H) = \bigcup_{H \in F} H \times H$$

(This is an $O_G$-space because there is a $G$-map $G/H \to G/K$ if and only if $g^{-1}Hg \subseteq K$ for some $g \in G$.)

Next, define $EF := 2^T$. Then

$$EF^H = \bigcup_{H \in F} T(G/H) \to T(G/H) = \bigcup_{H \in F} H \times H$$

and thus, we have constructed a universal $F$-isotropic space $EF$.

(Recall that $X$ is $F$-isotropic if $G_x \in EF$ for all $x \in X$.)

There is a unique homotopy class of maps $X \to EF$ for every $F$-isotropic $X$, of the homotopy type of a $G$-CW-complex.)
2) Eilenberg–Maclane spaces

The homotopy groups of a $G$-space $X$ is a functor $\Pi_n X : \mathcal{O}_G^\text{op} \to \text{Ab}$:

$$\Pi_n X = \pi_n(X^H) = \pi_n \circ \Delta^H X$$

Given a functor $\Pi : \mathcal{O}_G^\text{op} \to \text{Ab}$, we can construct an Eilenberg–Maclane $G$-space $K(\Pi, n)$ with $\Pi_n K(\Pi, n) = \Pi$ and $\Pi_m K(\Pi, n) = 0$ if $m \neq n$.

as follows:

Let $K(-, n) : \text{Ab} \to \text{Top}$ be a functorial Eilenberg–Maclane space construction (e.g., iterated bar construction). Then we get an $\mathcal{O}_G$-space

$$\mathcal{O}_G^\text{op} \xrightarrow{\Pi} \text{Ab} \xrightarrow{K(-, n)} \text{Top}$$

$\Psi(K(-, n) \circ \Pi)$ has the desired property.

$$\Pi_n \Psi(K(-, n) \circ \Pi) = \pi_n \circ \Delta^H \Psi(K(-, n) \circ \Pi)$$

$$\Pi \Psi K(-, n) \circ \Pi$$
The bar construction

\( C \) topological category, (small)

\( F: C^{op} \to \text{Top}, \ G: C \to \text{Top} \) continuous \( \text{functors} \).

\( \mathcal{B}_n(F, C, G) \) simplicial space with \( n+1 \) space

\((t \in F(x_0), \ x_0 \leftarrow x_1 \leftarrow \ldots \leftarrow x_n, \ c \in G(x_n) )\)

(topologized as a subspace of the product
\[ \prod_{x \in \text{ob} C} F(x) \times \prod_{x, y \in \text{ob} C} G(x) \times \prod_{x \in \text{ob} C} G(x) \]

\[ d_i = \begin{cases} 
(t \leftarrow x^i, \ x_0 \leftarrow \ldots \leftarrow x_n, \ c) & \text{if } i = 0 \\
(t, \ldots, x_i \leftarrow x_{i+1} \leftarrow \ldots, \ c) & \text{if } 0 < i < n \\
(t, x, c \leftarrow \ldots \leftarrow x_{n-1}, (f_n)_+ (c)) & \text{if } i = n
\end{cases}\]

\( s_i = (t, x_0 \leftarrow x_1 \leftarrow \ldots \leftarrow x_n, c) \)

**Special case:** \( G_! = \text{Hom}_C(A, -) \).

Then \( \mathcal{B}_n(F, C, \text{Hom}_C(A, -)) \) has an

"extra degeneracy" \( s_{n+1} = (t, x_0 \leftarrow \ldots \leftarrow x_n \leftarrow A, 1_A) \)

which can be used to contract

\[ \mathcal{B}_n(F, C, \text{Hom}_C(A, -)) \xrightarrow{\varepsilon} F(A) \]

\[ \varepsilon \varepsilon = \text{id}, \ \varepsilon \varepsilon = \text{id} \]
\[ E(t, x^1, \ldots, x^n, c) = (\phi_{t \circ \rho_c})^*(t) \in F(A) \]
\[ \eta(x) = (x, A, 1) \]

These maps are natural in \( F \) and \( A \), so we get a functor \( B : \text{c-spaces} \to \text{c-spaces} \)

\[ B(F) = \left[ B \cdot (F, c, \text{Hom}_c(A, -)) \right] \to FA \]

and a natural pointwise weak equivalence.
Proof of Elmendorf's theorem

\[ \mathcal{O}_G \text{-spaces} \xrightarrow{\Psi} \mathcal{B} \text{-spaces} \]

Then for any \( X \)

\[ \Psi(X) = \mathcal{B}(\Psi(X))(G/\mathfrak{g}e\mathfrak{g}) \overset{\sim}{\rightarrow} \Psi(X)(G/\mathfrak{g}e\mathfrak{g}) = X \]

Secondly, note that for any \( \mathcal{O}_G \)-space \( T \)

\[ \Psi(T)(G/H) = \Psi(T)^H \]

\[ = \left| \mathcal{B}_\mathcal{O}_G(T, e, \text{Hom}_{\mathcal{O}_G}(G_t, -)) \right|^H \]

\[ \overset{\sim}{\rightarrow} T(G/H) \]