A GROUPOID APPROACH TO CUNTZ-KRIEGER ALGEBRAS

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Abstract

An irreducible $N \times N$-matrix $A$ over $\{0, 1\}$ gives rise to a plethora of interesting mathematical objects. One can construct the one-sided topological Markov shift $X_A$ on which the shift $\sigma_A$ acts. Alternatively, one can construct a $C^*$-algebra, the Cuntz-Krieger algebra $\mathcal{O}_A$, or a groupoid $G_A$. In 2010, K. Matsumoto introduced the notion of continuous orbit equivalence between topological Markov shifts and showed that this can be characterized in terms of $C^*$-isomorphisms between the Cuntz-Kriegers algebras which preserve a certain abelian $C^*$-subalgebra. In a recent paper, published in 2014, K. Matsumoto and H. Matui provided the final piece of a characterization of continuous orbit equivalence between Markov shifts. This includes the notions of $C^*$-algebras, groups and groupoids. In this thesis, we give a proof of this classification result. Along the way, we will see how to realize $\mathcal{O}_A$ as a groupoid $C^*$-algebra.

Resumé

Introduction

In their paper [CK80], J. Cuntz and W. Krieger introduced the Cuntz-Krieger algebra $O_A$ determined by some $N \times N$ matrix $A$ over $\{0,1\}$ as the $C^*$-algebra generated by partial isometries $S_1, \ldots, S_N$ acting on a Hilbert space such that

$$1 = \sum_{j=1}^{N} S_j S_j^*, \quad S_i^* S_i = \sum_{j=1}^{N} A(i,j) S_j S_j^*. $$

Under suitable conditions on $A$, the $C^*$-algebra $O_A$ is independent of the specific Hilbert space. The diagonal in $O_A$ is a distinguished abelian $C^*$-subalgebra, $D_A$. The matrix $A$ also determines a compact space $X_A = \{(x_n)_{n} \in \{1, \ldots, N\}^\mathbb{N} \mid A(x_n, x_{n+1}) = 1\}$ on which the shift operation $\sigma_A : X_A \to X_A$ acts by $\sigma_A((x_n)_n) = (x_{n+1})_n$. In our case, $X_A$ is always a Cantor space. The pair $(X_A, \sigma_A)$ is called the one-sided topological Markov shift. The Cuntz-Krieger algebras and the topological Markov shifts are intimately connected.

In [Mat10], K. Matsumoto defined shift spaces $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ to be continuous orbit equivalent if there exists a homeomorphism $h : X_A \to X_B$ which respects the orbits in a continuous way. He proved that $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are continuously orbit equivalent if and only if there is a $C^*$-isomorphism between $O_A$ and $O_B$ which maps the diagonal, $D_A$, onto $D_B$. The shift space $(X_A, \sigma_A)$ also gives rise to an étale LCH groupoid, $G_A$, which is essentially principal. It turns out that there is a groupoid isomorphism $G_A \cong G_B$ if and only if there is a diagonal preserving $C^*$-isomorphism between the corresponding Cuntz-Krieger algebras (see, e.g., [Ren08]). In [CM00], H. Matui studied the homology (with constant coefficients) of étale LCH groupoids and showed that the homology groups of $G_A$ coincides with the $K$-theory of $O_A$. Finally, the canonical groupoid $C^*$-algebra of $G_A$ is exactly $O_A$. We can therefore think of $O_A$ as arising from a groupoid.

In the paper, [MM14], the authors K. Matsumoto and H. Matui presented the final piece in the classification of one-sided irreducible topological Markov shifts up to continuous orbit equivalence. This is the main paper of this thesis and our main objective is to prove this classification theorem (Theorem 5.3.10).

**Theorem.** Let $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ be two irreducible one-sided topological Markov shifts. The following are equivalent:

1. $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are continuously orbit equivalent,
The étale LCH groupoids $G_A$ and $G_B$ are isomorphic,

(3) There is a $C^*$-isomorphism $\Psi: (\mathcal{O}_A, \mathcal{D}_A) \rightarrow (\mathcal{O}_B, \mathcal{D}_A)$,

(4) The Cuntz-Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are isomorphic and $\det(I - A) = \det(I - B)$,

(5) There is an isomorphism $(\text{BF}(A^t), u_A) \cong (\text{BF}(B^t), u_B)$ and $\det(I - A) = \det(I - B)$.

Here, $\text{BF}(A^t) = \mathbb{Z}^N/(I - A^t)\mathbb{Z}$ is the Bowen-Franks group of the matrix $A^t$ and $u_A$ is the image in $\text{BF}(A^t)$ of $(1, \ldots, 1) \in \mathbb{Z}^N$. The main contribution of [MM14] is the implication $(1) \Rightarrow (4)$.

The reader is assumed to be well-acquainted with the theory of operator algebras, more specifically that of $C^*$-algebras and the $K$-theory of operator algebras. The reader is also expected to have a basic knowledge of category theory.

The thesis is structured in the following way:

Chapter 1: We set the stage and establish the basics of dynamical systems. The two-sided and one-sided topological Markov shifts are defined in terms of a square matrix $A$ and we introduce the notion of continuous orbit equivalence between such shift spaces.

Chapter 2: The Cuntz-Krieger algebra $\mathcal{O}_A$ is defined as a universal $C^*$-algebra generated by a number of partial isometries which are subject to the so-called CK-relations determined by the matrix $A$. A more concrete description of this algebra is given and we define a certain abelian $C^*$-subalgebra called the diagonal. We show that two topological Markov shifts are continuously orbit equivalent if and only if there is a diagonal preserving $C^*$-isomorphism between the two corresponding Cuntz-Krieger algebras. We also compute the $K$-theory of $\mathcal{O}_A$.

Chapter 3: The notion of topological groupoids is introduced and we show how to construct a $C^*$-algebra from an étale locally compact Hausdorff groupoid. It turns out that this construction is a complete invariant of étale locally compact Hausdorff groupoids which are also essentially principal. It is possible to associate a groupoid $G_A$ to a topological Markov shift and the induced $C^*$-algebra is exactly the Cuntz-Krieger algebra. This paves the way for a groupoid interpretation of $\mathcal{O}_A$.

Chapter 4: We dive further into the study of topological groupoids by establishing a homology and cohomology theory with constant coefficients. With some work we show that the homology of the groupoid $G_A$ coincides with the $K$-theory of $\mathcal{O}_A$.

Chapter 5: We finally provide the reader with a proof of the five-term classification theorem which shows how continuous orbit equivalence between topological Markov shifts can be stated and understood in terms of $C^*$-algebras, groupoids or groups. This requires a brief discussion of flow equivalence and we import some theorems on this topic.

A small note on notation: Most of the notation is carefully explained within the thesis but some choices are implicit in the text. We let $\mathbb{Z}$ denote the integers and let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the natural numbers while $\mathbb{Z}_+ = \{0, 1, 2 \ldots\}$. Furthermore, $\mathbb{R}$ and $\mathbb{C}$ will denote the field of real and complex numbers, respectively. Matrices will be written with latin capital letters and $A^t$ denotes the transpose of the matrix $A$. Abstract $C^*$-algebras are mostly written with a calligraphic font, e.g., $\mathcal{A}, \mathcal{B}, \ldots$ while abstract (topological) groups are noted with capital greek letters, e.g., $\Gamma, \Lambda, \ldots$ The characteristic map of a set $S$ is denoted $\chi_S$. Additional notation and symbols are listed at the end of the thesis.
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Topological dynamical systems

1.1 The setting

We start by introducing the relevant concepts and establishing some notation. Pick your favorite positive integer \( N > 1 \). Any \( N \times N \) matrix \( A = [A(i,j)]_{i,j=1}^N \) over \( \{0,1\} \) defines an alphabet \( \mathcal{A}(A) = \mathcal{A} = \{1, \ldots, N\} \) as well as compact Hausdorff spaces

\[
X_A = \{ x = (x_n)_{n \in \mathbb{N}} \in \mathcal{A}^\mathbb{N} \mid A(x_n, x_{n+1}) = 1 \}, \\
\bar{X}_A = \{ x = (x_n)_{n \in \mathbb{Z}} \in \mathcal{A}^\mathbb{Z} \mid A(x_n, x_{n+1}) = 1 \},
\]

with respect to the product topology.

The shift operations \( \sigma_A: X_A \rightarrow X_A \) and \( \bar{\sigma}_A: \bar{X}_A \rightarrow \bar{X}_A \) given by

\[
\sigma_A((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}, \quad \bar{\sigma}_A((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}},
\]

respectively, are continuous surjections. Actually, \( \bar{\sigma}_A \) is a homeomorphism. The topological dynamical system \((X_A, \sigma_A)\) (resp. \((\bar{X}_A, \bar{\sigma}_A)\)) is called the one-sided (resp. two-sided) topological Markov shift. When \( A(i,j) = 1 \) for every \( i, j \in \mathcal{A} \), we say that \( A \) is the full shift. The canonical continuous surjection \( \rho: \bar{X}_A \rightarrow X_A \) given by \( \rho((x_n)_{n \in \mathbb{Z}}) = (x_n)_{n \in \mathbb{N}} \) is called the cut. The interested reader is invited to see \([LM95]\) for a nice introduction to the subject of symbolic dynamics.

Two one-sided topological Markov shifts \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are said to be conjugate if there exists a homeomorphism \( h: X_A \rightarrow X_B \) which intertwines the shift operation; that is, if the diagram

\[
\begin{array}{ccc}
X_A & \xrightarrow{\sigma_A} & X_A \\
\downarrow{h} & & \downarrow{h} \\
X_B & \xrightarrow{\sigma_B} & X_B
\end{array}
\]

commutes.

We shall refer to finite strings of letters in \( \mathcal{A} \) as words. A word \( \alpha = \alpha_1 \cdots \alpha_k \ (\alpha_i \in \mathcal{A}) \) is admissible if it appears somewhere in some \( x \in X_A \). Its length is \( |\alpha| = k \). We let \( \mathcal{A}_k \) be the set of admissible words of length \( k \geq 1 \); the collection of all admissible words is \( \mathcal{A}_{\leq N} = \bigcup_{k=0}^\infty \mathcal{A}_k \).

\[^1\]We allow the empty word \( \emptyset \), the unique element in \( \mathcal{A}_0 \).
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and

\[ \mathfrak{A}_{\leq n} = \bigcup_{k=0}^{n} \mathfrak{A}_k, \quad \mathfrak{A}_{\geq n} = \bigcup_{k=n}^{\infty} \mathfrak{A}_k. \]

Words are composed by concatenation. If \( x \in X_A \) and \( k < l \) are positive integers, then we may consider \( x_{[k,l]} = x_k \cdots x_{l-1} x_l \in \mathfrak{A}_{l-k+1} \) as well as the half-infinite sequence \( x_{[k,\infty]} = x_k x_{k+1} \cdots \). Furthermore, if \( j \in \mathfrak{A} \) and \( k \in \mathbb{N} \cup \{\infty\} \), then we write \( j^k \) for the word \( j \cdots j \in \mathfrak{A} \) (\( k \) factors).

The cylinder set of \( \alpha \in \mathfrak{A}_k \) is

\[ N_\alpha = \{(x_n)_n \in X_A \mid x_{[1,|\alpha|]} = \alpha\}. \quad (1.1) \]

If \( (X_A, \sigma_A) \) is a one-sided topological Markov shift, then the orbit of \( x = (x_n)_n \in X_A \) is

\[ \text{orb}_{\sigma_A}(x) = \text{orb}_A(x) = \bigcup_{k=0}^{\infty} \bigcup_{l=0}^{\infty} \sigma_A^{-k}(\sigma_A^l(x)) \subseteq X_A. \quad (1.2) \]

That is, \( y = (y_n)_n \in X_A \) is in the orbit of \( x \) if and only if

\[ y = \alpha_1 \cdots \alpha_k x_{l+1} x_{l+2} \cdots \]

for some \( k, l \in \mathbb{Z}_+ \) and \( \alpha_1 \cdots \alpha_k \in \mathfrak{A}_k \).

1.2 Continuous orbit equivalence

**Definition 1.2.1.** Let \( \text{Homeo}(X_A) \) denote the group of homeomorphisms on \( X_A \). The full group \( [\sigma_A] \subseteq \text{Homeo}(X_A) \) is the subgroup consisting of homeomorphisms \( \tau \) on \( X_A \) such that \( \tau(x) \in \text{orb}_A(x) \), for all \( x \in X_A \).

The topological full group\(^{2}\) \( [\sigma_A]_c \subseteq [\sigma_A] \) is the subgroup of homeomorphism \( \tau \in [\sigma_A] \) to which we may associate continuous orbit cocycles \( k_\tau, l_\tau : X_A \rightarrow \mathbb{Z}_+ \) satisfying

\[ \sigma_A^{k_\tau}(x)(\tau(x)) = \sigma_A^{l_\tau}(x), \quad (1.3) \]

for \( x \in X_A \).

In order to ensure that these dynamical systems are »well-behaved« we shall impose some conditions on the matrix \( A \). Define a subset \( \mathfrak{A}' \subseteq \mathfrak{A} \) by declaring \( i \in \mathfrak{A}' \) if there are distinct admissible words \( \alpha = \alpha_1 \cdots \alpha_k \) and \( \beta = \beta_1 \cdots \beta_l \) such that \( i = \alpha_1 = \alpha_k = \beta_1 = \beta_l \) while \( i \neq \alpha_s, \beta_t \) when \( 1 < s < k \) and \( 1 < t < l \).

**Definition 1.2.2.** The matrix \( A \) satisfies condition (I)\(^{3}\) if given \( i \in \mathfrak{A} \) there exists an admissible word \( \alpha = \alpha_1 \cdots \alpha_k \) such that \( i = \alpha_1 \) and \( \alpha_k \in \mathfrak{A}' \).

This is a somewhat technical restriction on \( A \). Fortunately, condition (I) is equivalent to the assumption that \( X_A \) is a Cantor space, by which we mean a perfect non-empty, compact metrizable and totally disconnected space. It is well-known that all such spaces are homeomorphic (see e.g. [Kec95]). The cylinder sets \( (1.1) \) comprise a basis of open and compact sets,

\[^{2}\text{In some papers, e.g., [MM14] the topological full group is denoted } \Gamma_A.\]

\[^{3}\text{As named by Cuntz and Krieger in [CK80].}\]
whenever $X_A$ is a Cantor space and $N_\beta \subseteq N_\alpha$ exactly when $\beta$ is an extension of $\alpha$ (in the sense that $\beta = \alpha \alpha'$, for some $\alpha' \in \mathbb{A}_{<\mathbb{N}}$). In many cases, we will also require $A$ to be irreducible.

It turns out that the notion of conjugate shift spaces is too restrictive for our analysis. We shall now introduce two weaker notions of equivalence.

**Definition 1.2.3.** Two one-sided topological Markov shift $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are **topologically orbit equivalent** if there exists a homeomorphism $h: X_A \rightarrow X_B$ such that $h(\text{orb}_A(x)) = \text{orb}_B(h(x))$, for all $x \in X_A$.

When two shift spaces are topologically orbit equivalent, there are functions (relative to $h$) $k_1, l_1: X_A \rightarrow \mathbb{Z}_+$ and $k_2, l_2: X_B \rightarrow \mathbb{Z}_+$ satisfying the formulae

$$
\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)),
$$

$$\sigma_A^{k_2(y)}(h^{-1}(\sigma_B(y))) = \sigma_A^{l_2(y)}(h^{-1}(y)),
$$

for $x \in X_A$ and $y \in X_B$.

**Definition 1.2.4.** Two topologically orbit equivalent shift spaces $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are **continuously orbit equivalent** if the functions satisfying (1.4) and (1.5) can be chosen to be continuous.

In this thesis, the notion of continuous orbit equivalent shift spaces will be of central importance. We shall start by characterizing continuous orbit equivalence in terms of the topological full group. Let us first explore some properties of the latter object.

**Lemma 1.2.5.** Assume that $A$ is irreducible and satisfies condition (I). For every $\alpha \in \mathbb{A}_n$, there are $\tau \in [\sigma_A]_c$ and continuous $k, l: X_A \rightarrow \mathbb{Z}_+$ such that

$$\sigma_A^{k(x)}(\tau(x)) = \sigma_A^{l(x)}(x)$$

for $x \in X_A$. Furthermore, $k(x) = 0$ and $l(x) = 1$, whenever $x \in N_\alpha$.

In particular, the one-sided shift operation is a local homeomorphism.

**Proof.** For notational purposes, we prove the lemma for $n = 2$; the argument extends to any non-negative integer $n \geq 2$. Suppose first that $\alpha = aa \in \mathbb{A}_2$ and pick $b \in \mathbb{A} \setminus \{a\}$ such that $A(b, a) = 1$. This is possible because $A$ is irreducible. Now form the subset $\{b_1, \ldots, b_m\} \subseteq \mathbb{A} \setminus \{a\}$ of elements satisfying $A(a, b_i) = 1$, $i = 1, \ldots, m$. This is non-empty because $A$ satisfies condition (I). We may now define a map $\tau: X_A \rightarrow X_A$ by

$$
\tau(x) = \begin{cases} 
\sigma_A(x) & x \in N_{aa}, \\
ba_b x_{[3, \infty)} & x \in N_{ab}, \\
baax_{[3, \infty)} & x \in N_{ba}, \\
x & \text{otherwise}.
\end{cases}
$$

This is a homeomorphism. We define continuous $k, l: X_A \rightarrow \mathbb{Z}_+$ by

$$
k(x) = \begin{cases} 
0 & x \in N_{aa}, \\
1 & x \in N_{ab}, \\
2 & x \in N_{ba}, \\
0 & \text{otherwise,}
\end{cases} \quad l(x) = \begin{cases} 
1 & x \in N_{aa}, \\
0 & x \in N_{ab}, \\
1 & x \in N_{ba}, \\
0 & \text{otherwise.}
\end{cases}
$$
These maps satisfy \(1.6\).

On the other hand, suppose \(\alpha = ab \in \mathfrak{A}_2\) with \(a \neq b\). In this case, we define a homeomorphism \(\tau: X_A \rightarrow X_A\) by putting

\[
\tau(x) = \begin{cases} 
\sigma_A(x) & x \in N_{ab}, \\
ax & x \in N_b, \\
x & \text{otherwise}
\end{cases}
\]

This is a homeomorphism in \([\sigma_A]_c\).

Remark 1.2.6. In the following, we shall always assume that \(A\) satisfies condition (I).

Lemma 1.2.7. For every \(x \in X_A\) and \(j \in \mathfrak{A}\) with \(jx \in X_A\), there exists \(\tau \in [\sigma_A]_c\) with \(\tau(x) = jx\).

Proof. If \(x = j^\infty\) then simply choose the identity. If this is not the case, choose \(k \in \mathbb{N}\) and \(i \in \mathfrak{A} \setminus \{j\}\) such that \(x_k = i\) and \(x_n = j\) for every \(n < k\) \((n\) may be zero in which case \(i = x_1\)). If we put \(\alpha = j^{k-1}i \in \mathfrak{A}_k\), then \(x \in N_{\alpha}\) and we define \(\tau: X_A \rightarrow X_A\) by

\[
\tau(y) = \begin{cases} 
jy & y \in N_{\alpha}, \\
\sigma_A(y) & y \in N_{ja}, \\
y & \text{else.}
\end{cases}
\]

This is a homeomorphism in \([\sigma_A]_c\).

Lemma 1.2.8. For each \(x \in X_A\), let \([\sigma_A]_c(x) = \{\tau(x) \mid \tau \in [\sigma_A]_c\}\). Then \([\sigma_A]_c(x) = \text{orb}_A(x)\).

Proof. For every \(\tau \in [\sigma_A]_c\), \(\tau(x) \in \text{orb}_A(x)\) so one inclusion is clear. Conversely, pick \(x \in X_A\) and \(j \in \mathfrak{A}\) with \(jx \in X_A\). By Lemmas 1.2.7 and 1.2.5 we may pick \(\tau_1, \tau_2 \in [\sigma_A]_c\) such that

\[
\tau_1(x) = jx, \quad \tau_2(x) = \sigma_A(x).
\]

In particular, \(jx, \sigma_A(x) \in [\sigma_A]_c(x)\). As \([\sigma_A]_c\) is closed under composition, an iteration of this process shows that for every \(k, l \in \mathbb{Z}_+\) and \(\alpha \in \mathfrak{A}_k\) we have \(\alpha x_{[l+1,\infty)} \in [\sigma_A]_c(x)\) (provided that \(\alpha x_{[l+1,\infty)} \in X_A\), of course). Hence \(\text{orb}_A(x) \subseteq [\sigma_A]_c(x)\).

A homeomorphism \(h: X_A \rightarrow X_B\) is said to intertwine the topological full group if \(h \circ [\sigma_A]_c \circ h^{-1} = [\sigma_B]_c\). We will now show that the existence of such an intertwining homeomorphism is equivalent to the shift spaces being continuously orbit equivalent.

Proposition 1.2.9. Suppose \(h: X_A \rightarrow X_B\) is a homeomorphism intertwining the topological full group. Then \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are continuously orbit equivalent.
Lemma 1.2.10. purely computational and omitted. homorphism the two shift spaces are topologically orbit equivalent. This follows from the computation

\[ h(\text{orb}_A(x)) = h([\sigma_A]_c(x)) = [\sigma_B]_c(h(x)) = \text{orb}_B(h(x)) \] (1.7)

for every \( x \in X_A \). Fix a word \( \alpha \in \mathcal{A}_2 \) and choose \( \tau \in [\sigma_A]_c \) such that \( \tau(x) = \sigma_A(x) \), for every \( x \in N_\alpha \) (cf. Lemma 1.2.5). Then \( \tau_h := h \circ \tau \circ h^{-1} \in [\sigma_B]_c \), so there are continuous \( k_\alpha, l_\alpha : X_B \rightarrow \mathbb{Z}_+ \) satisfying

\[ \sigma_B^{k_\alpha}(y)(\tau_h(y)) = \sigma_B^{l_\alpha}(y)(y), \]

for \( y \in X_B \). However, a swift computation shows that \( h(\sigma_A(x)) = \tau_h(h(x)) \) whenever \( x \in N_\alpha \), so

\[ \sigma_B^{k_\alpha(h(x))}(h(\sigma_A(x))) = \sigma_B^{l_\alpha(h(x))}(h(x)), \] (1.8)

for \( x \in N_\alpha \). This process may be applied to each \( \alpha \in \mathcal{A}_2 \) and since the cylinder sets \( N_\alpha \) are clopen and partition \( X_A \), the images \( h(N_\alpha) \) are clopen and partition \( X_B \). We may now define continuous maps \( k_1, l_1 : X_A \rightarrow \mathbb{Z}_+ \) by

\[ k_1(x) = k_\alpha \circ h(x), \quad l_1(x) = l_\alpha \circ h(x), \]

for \( x \in N_\alpha \). Similarly, one can find continuous \( k_2, l_2 : X_B \rightarrow \mathbb{Z}_+ \). By (1.8), we see that these maps will witness the continuous orbit equivalence of \( (X_A, \sigma_A) \) and \( (X_B, \sigma_B) \). \( \square \)

In order to prove the converse to the above proposition, we need a lemma. The proof is purely computational and omitted.

**Lemma 1.2.10.** Let \( (X_A, \sigma_A) \) and \( (X_B, \sigma_B) \) be topologically orbit equivalent. Choose a homeomorphism \( h : X_A \rightarrow X_B \) and continuous \( k, l : X_A \rightarrow \mathbb{Z}_+ \) satisfying (1.4). If

\[ k^n(x) = \sum_{i=0}^{n-1} k(\sigma_A^i(x)), \quad l^n(x) = \sum_{i=0}^{n-1} l(\sigma_A^i(x)) \]

then

\[ \sigma_B^{k^n}(h(\sigma_A^n(x))) = \sigma_B^{l^n}(h(x)), \]

for \( x \in X_A \) and \( n \in \mathbb{N} \).

**Proposition 1.2.11.** Suppose \( (X_A, \sigma_A) \) and \( (X_B, \sigma_B) \) are continuously orbit equivalent. Then there exists a homeomorphism \( h : X_A \rightarrow X_A \) intertwining the topological full group.

**Proof.** By hypothesis, there exists a homeomorphism \( h : X_A \rightarrow X_B \) together with continuous \( k, l : X_A \rightarrow \mathbb{Z}_+ \) satisfying (1.4). For any \( \tau \in [\sigma_A]_c \), there are also continuous orbit cocycles \( k_\tau, l_\tau : X_A \rightarrow \mathbb{Z}_+ \) satisfying (1.3). We wish to show that \( \tau_h = h \circ \tau \circ h^{-1} \) is in \([\sigma_B]_c\).

Let \( y \in X_B \) and consider \( x = h^{-1}(y) \). Put \( m = k_\tau(x) \) and \( n = l_\tau(x) \). By the previous lemma, we have

\[ \sigma_B^{k^m(\tau(x))}h(\sigma_A^n(x)) = \sigma_B^{l^m(\tau(x))}(h(\sigma_A^n(\tau(x)))) = \sigma_B^{l^m(\tau(x))}(h(\tau(x))). \]
Applying $\sigma^k(x)$ on both sides and using the previous lemma again, we obtain

$$\sigma^k(x + l)(h(x)) = \sigma^k(x + k(h(x))) = \sigma^l(h(x)).$$

Hence

$$\sigma^{k\tau_h}(y) = \sigma^l(y),$$

where $k\tau_h(y) = k\tau(x) + l\tau(x)$ and $l\tau_h(y) = k\tau(x) + l\tau(x)$ and $x = h^{-1}(y)$. Therefore, $h \circ [\sigma_A]_c \circ h^{-1} \subseteq [\sigma_B]_c$. A similar argument using (1.5) will show the other inclusion. 

We state this result as a theorem for later reference.

**Theorem 1.2.12.** The one-sided topological Markov shifts $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are continuously orbit equivalent if and only if there is a homeomorphism $h: X_A \rightarrow X_B$ such that $h \circ [\sigma_A]_c \circ h^{-1} = [\sigma_B]_c$. 

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2.1 Cuntz-Krieger algebras

Let $A$ be an irreducible $N \times N$ matrix over \{0, 1\} satisfying condition (I). In Section 1.1 we used the data of $A$ to construct a topological dynamical system. In this section, we use it to construct a $C^*$-algebra.

The matrix $A$ uniquely determines a (finite) directed graph $(\mathcal{V}_A, \mathcal{E}_A) = (\mathcal{V}, \mathcal{E})$ in the following way: Let $\mathcal{V} = \mathbb{A}(A)$ be the (finite) set of vertices; that is, each vertex is uniquely labeled by an integer $1, \ldots, N$. There exists an edge from $i$ to $j$ if and only if $A(i, j) = 1$. In this way the set $\mathcal{E} = \mathbb{A}_2$ of admissible words of length 2 corresponds to the collection of edges in the graph. Given an edge $e = e_1e_2 \in \mathcal{E}$, we may identify its range and source as $r(e) = e_2$ and $s(e) = e_1$, respectively, and we put

$$\mathcal{E}_i = \{e \in \mathcal{E} | s(e) = i\}, \quad \mathcal{E}^i = \{e \in \mathcal{E} | r(e) = i\}$$

and $\mathcal{E}^j_i = \mathcal{E}_i \cap \mathcal{E}^j$ for $i, j \in \mathcal{V}$. In this context, we shall refer to the matrix $A$ as the adjacency matrix of the graph. The graph is essential (there are no sinks nor sources) when $A$ is irreducible.

**Definition 2.1.1.** Let $(\mathcal{V}, \mathcal{E})$ be the directed graph determined by the matrix $A$. The Cuntz-Krieger algebra $\mathcal{O}_A$ is the universal $C^*$-algebra generated by orthogonal projections $\{P_i\}_{i \in \mathcal{V}}$ and orthogonal partial isometries $\{T_e\}_{e \in \mathcal{E}}$ subject to the conditions

$$P_{r(e)} = T_e^*T_e, \quad P_i = \sum_{e \in \mathcal{E}_i} T_eT_e^*,$$

(2.1)

for every $i \in \mathcal{V}$. We shall refer to the generators $\{P_i\}_i$ and $\{T_e\}_e$ as a CK family. We use the notation $1_A$ for the unit in $\mathcal{O}_A$.

**Remark 2.1.2.** In this sense, $\mathcal{O}_A$ is an example of a graph $C^*$-algebra. Note that the explicit mention of the projections $\{P_i\}_i$ is superfluous. However, in practical computations and to make clear the connection to the underlying graph, we shall keep track of the projections as well as the partial isometries.

**Example 2.1.3.** The reader will note that when $A(i, j) = 1$ for every $i, j \in \mathcal{V}$, then $\mathcal{O}_A$ is the Cuntz algebra $\mathcal{O}_N$. 

— 2 —

Cuntz-Krieger algebras
It is readily verified that $T_e^*T_f \neq 0$ only when $e = f$ and that $T_e = T_e P_{r(e)}$. Furthermore, $T_e T_f \neq 0$ forces $r(e) = s(f)$ and $T_f^* \neq 0$ only when $r(e) = r(f)$. We may consider an admissible word $\alpha = \alpha_1 \cdots \alpha_k$ with $k \geq 2$ to be a path of edges $(\alpha_1, \alpha_2)(\alpha_2, \alpha_3) \cdots (\alpha_{k-1}, \alpha_k)$ in $\mathcal{E}$ and write $T_\alpha := T_{(\alpha_1, \alpha_2)}T_{(\alpha_2, \alpha_3)} \cdots T_{(\alpha_{k-1}, \alpha_k)}$. The range and source maps easily extend so that $r(\alpha) = \alpha_k$ and $s(\alpha) = \alpha_1$, respectively. Given paths $\alpha, \beta \in \mathbb{A}_{\geq 2}$ the composition $T_\alpha T_\beta$ is non-zero only when $r(\alpha) = s(\beta)$ and $T_\alpha T_\beta^* \neq 0$ forces $r(\alpha) = r(\beta)$. Note also that

$$T_\alpha^* T_\beta = \begin{cases} T_\beta^* & \beta = \alpha \alpha', \\ T_\alpha^* & \alpha = \beta \alpha', \\ P_{r(\alpha)} & \alpha = \beta, \\ 0 & \text{otherwise}. \end{cases} \quad (2.2)$$

We now give a more concrete description of $O_A$.

**Lemma 2.1.4.** We have

$$O_A = \text{span}\{T_\alpha T_\beta^* \mid \alpha, \beta \in \mathbb{A}_{\leq N}\} = \text{span}\{T_\alpha T_\beta^* \mid \alpha, \beta \in \mathbb{A}_{\leq N}, \ r(\alpha) = r(\beta)\}. \quad (2.3)$$

**Proof.** The second equality follows from the fact that $T_\alpha T_\beta^*$ vanishes when $r(\alpha) \neq r(\beta)$. Concerning the first equality, the right hand side is a $C^*$-subalgebra of $O_A$ so it suffices to find the generators there. For every edge $e \in \mathcal{E}$, we have

$$T_e = T_e P_{r(e)} = \sum_{f \in \mathcal{E}_{r(e)}} (T_e T_f) T_f^*$$

which is contained in the right hand side of (2.3). \qed

Below, we import a uniqueness-result without proof, see e.g., [CK80] or [Szy02].

**Theorem 2.1.5.** If $B$ is any $C^*$-algebra and $\varphi : O_A \rightarrow B$ is a $\ast$-homomorphism such that $\varphi(P_i) \neq 0$ for every $i \in \mathcal{V}$, then $\varphi$ is injective.

By an ideal in a $C^*$-algebra, we will always mean a norm-closed and two-sided ideal.

**Proposition 2.1.6.** Suppose $A$ is irreducible. Then $O_A$ is simple.

**Proof.** Pick a non-zero ideal $I \triangleleft O_A$ and let $\varphi : O_A \rightarrow B$ be a $\ast$-homomorphism such that $\ker \varphi = I$. By the above theorem, there is a vertex $i \in \mathcal{V}$ such that $P_i \in I$. Now, let $j \in \mathcal{V}$ and pick a path $\alpha \in \mathbb{A}_{\geq 2}$ from $i$ to $j$. That is, $s(\alpha) = i$ and $r(\alpha) = j$. This is possible because the graph $(\mathcal{V}, \mathcal{E})$ is irreducible. Then

$$P_j = P_{r(\alpha)} = T_\alpha^* T_\alpha = T_\alpha^* (T_\alpha T_\alpha^*) T_\alpha = T_\alpha^* (P_i T_\alpha T_\alpha^*) T_\alpha \in I.$$  

As $j \in \mathcal{V}$ was arbitrary it follows that $1 = \sum_{j \in \mathcal{V}} P_j \in I$ and so $I = O_A$. \qed

In order to gain a better understanding the structure of the Cuntz-Krieger algebras, we shall examine the Gauge action. In the following, we shall adopt the abbreviation $e_n(t) = e^{\sqrt{-1}nt}$, for $n \in \mathbb{Z}$ and $t \in [0, 2\pi]$ and let $T = \{e_1(t) \mid t \in [0, 2\pi]\}$ be the circle group.
Definition 2.1.7. Define an action \( \rho: \mathbb{T} \curvearrowright \mathcal{O}_A \) by automorphisms by putting \( \rho(e_1(t)) = \rho_t \), where

\[
\rho_t(T_e) = e_1(t)T_e, \quad \rho_t(P_i) = P_i,
\]

for \( t \in [0,2\pi] \), \( e \in \mathcal{E} \) and \( i \in \mathcal{V} \). This is the gauge action. Let \( \mathcal{F}_A \subseteq \mathcal{O}_A \) denote the fixed point algebra under this action.

The existence of such an action is clear since the operators \( \{e_1(t)T_e\}_e \) satisfy the CK relations and \( \rho_0 = \text{id} \) while \( \rho_t^{-1} = \rho_{2\pi - t} \) for \( t \in (0,2\pi) \).

Lemma 2.1.8. The map \( E: \mathcal{O}_A \rightarrow \mathcal{F}_A \) given by

\[
E(a) = \int_0^{2\pi} \rho_t(a) \, dt,
\]

for \( a \in \mathcal{O}_A \), defines a faithful conditional expectation whose image is exactly the fixed point algebra, \( \mathcal{F}_A \). Here, \( dt \) is normalized Haar measure and the integral is a Bochner integral\(^1\).

Proof. By construction (of the integral) \( E \) is unital, positive and contractive. Given \( t_0 \in [0,2\pi] \), we have

\[
\rho_t(E(a)) = \rho_0 \left( \int_0^{2\pi} \rho_t(a) \, dt \right) = \int_0^{2\pi} \rho_{t+t_0}(a) \, dt = E(a),
\]

so \( \text{im} \, E \subseteq \mathcal{F}_A \). The sum is modulo \( 2\pi \). Conversely, if \( a \in \mathcal{F}_A \), then \( E(a) = a \). Hence \( E \) is a conditional expectation onto \( \mathcal{F}_A \). Furthermore, if \( a \in \mathcal{O}_A \) is strictly positive, then \( \rho_t(a) \) is strictly positive, so the integral \( E(a) \) is strictly positive. That is, \( E \) is faithful.

We can interpret the fixed point algebra as the inductive limit of certain finite dimensional matrix algebras. Start by defining

\[
\mathcal{F}_0 = \text{span}\{P_i \mid i \in \mathcal{V}\}, \quad \mathcal{F}_n = \text{span}\{T_\alpha T_\beta^* \mid \alpha, \beta \in \mathfrak{A}_{n+1}, \ r(\alpha) = r(\beta)\},
\]

for \( n \in \mathbb{N} \). Then \( \mathcal{F}_n = \bigoplus_{i=1}^N \mathcal{F}^i_n \), where

\[
\mathcal{F}^i_n = \text{span}\{T_\alpha T_\beta^* \mid \alpha, \beta \in \mathfrak{A}_{n+1}, \ r(\alpha) = i = r(\beta)\}.
\]

Furthermore, given \( \alpha, \beta, \alpha', \beta' \in \mathfrak{A}_{n+1} \) with \( r(\alpha) = r(\beta) = r(\alpha') = r(\beta') \), we have

\[
T_\alpha T_\beta^* T_{\alpha'} T_{\beta'}^* = \begin{cases} T_\alpha T_{\beta'}^* & \beta = \alpha', \\ 0 & \text{otherwise}, \end{cases}
\]

and so \( \{T_\alpha T_\beta^* \mid \alpha, \beta \in \mathfrak{A}_{n+1}, \ r(\alpha) = i = r(\beta)\} \) constitutes a set of matrix units. For each fixed \( i \in \mathcal{V} \), \( \mathcal{F}^i_n \) is isomorphic to \( M_{k(n,i)}(\mathbb{C}) \) as \( C^* \)-algebras. Here, \( k(n,i) = |\{\alpha \in \mathfrak{A}_{n+1} \mid r(\alpha) = i\}| \).

We may therefore identify \( \mathcal{F}_n \) with the (not necessarily simple) finite dimensional \( C^* \)-algebra \( \bigoplus_{i=1}^N M_{k(n,i)}(\mathbb{C}) \). Now if \( \alpha, \beta \in \mathfrak{A}_{n+1} \) with \( r(\alpha) = i = r(\beta) \), then

\[
T_\alpha T_\beta^* = T_\alpha P_i T_{\beta}^* = \sum_{f \in \mathcal{E}_i} T_{\alpha f} T_{\beta f}^* \in \mathcal{F}_{n+1},
\]

hence \( \mathcal{F}_n \subseteq \mathcal{F}_{n+1} \). The sum is non-zero, since \( A \) is irreducible.

\(^1\)The reader is invited to consult Appendix B.6 in [DEIL] for an introduction to vector-valued integrals.
Lemma 2.1.9. The fixed point algebra \( F_A \) is the inductive limit of \( F_1 \subseteq F_2 \subseteq \cdots \). That is, \( F_A = \bigcup_{n \in \mathbb{N}} F_n \). In particular, \( F_A \) is an AF-algebra.

Proof. Take \( T_\alpha T_\beta^* \in F_n \) and observe that \( \rho_n(T_\alpha T_\beta^*) = T_\alpha T_\beta^* \), so \( F_n \subseteq F_A \) for every \( n \in \mathbb{N} \). Since \( F_n \subseteq F_{n+1} \) and \( F_A \) is norm-closed, this shows one inclusion. Conversely, note that \( E(x) \in \bigcup_{n \in \mathbb{N}} F_n \) when \( x \in \text{span}\{T_\alpha T_\beta^* \mid \alpha, \beta \in \mathcal{A}_n\} \). As this is dense in \( O_A \) and \( E \) is continuous, this proves the other inclusion. \( \square \)

Definition 2.1.10. Let \( \mathcal{D}_A \subseteq F_A \subseteq \mathcal{O}_A \) be the abelian \( C^* \)-subalgebra generated by elements of the form \( T_\alpha T_\alpha^* \), for \( \alpha \in \mathcal{A}_\geq 2 \).

Recall that a \( C^* \)-subalgebra \( \mathcal{D} \) of a \( C^* \)-algebra \( \mathcal{B} \) is a maximal abelian subalgebra (masa) if it is abelian and not properly contained in any other abelian subalgebra. Equivalently, \( \mathcal{D} \) equals the collection of elements \( b \in \mathcal{B} \) for which \( bd = db \) for every \( d \in \mathcal{D} \).

Lemma 2.1.11. The \( C^* \)-algebra \( \mathcal{D}_A \) is a masa in \( \mathcal{O}_A \).

Proof. Let \( \alpha, \beta \in \mathcal{A}_\geq 2 \) be paths and suppose \( T_\beta T_\alpha^* \) is non-zero. Assuming that \( T_\beta T_\alpha^* \) commutes with \( \mathcal{D}_A \) it follows that

\[
T_\beta T_\alpha^* = (T_\beta T_\alpha^*) T_\beta T_\alpha^* = T_\beta (T_\alpha^* T_\beta) T_\beta^*.
\]

If \( \beta = \alpha \beta' \) for some \( \beta' \in \mathcal{A}_\geq 2 \), then \( T_\beta T_\alpha^* = T_\beta T_\beta^* T_\beta = 0 \) since \( r(\beta) \neq s(\beta') \). This follows from \( \ref{2.2} \). Similarly, \( T_\beta T_\alpha^* \) vanishes if \( \alpha \) extends \( \beta \). Therefore, \( \alpha = \beta \) and \( T_\beta T_\alpha^* \in \mathcal{D}_A \). That is, \( \mathcal{D}_A \) is maximal abelian. \( \square \)

2.2 Continuous orbit equivalence and Cuntz-Krieger algebras

In this section, we wish to relate continuous orbit equivalence of topological Markov shifts to their corresponding Cuntz-Krieger algebras. More precisely, we aim to prove the following theorem.

Theorem 2.2.1. Two one-sided topological Markov shifts \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are continuously orbit equivalent if and only if there is a \( C^* \)-isomorphism \( \Psi : (\mathcal{O}_A, \mathcal{D}_A) \to (\mathcal{O}_B, \mathcal{D}_B) \).

Remark 2.2.2. A \( C^* \)-isomorphism \( \Psi : \mathcal{O}_A \to \mathcal{O}_B \) satisfying \( \Psi(\mathcal{D}_A) = \mathcal{D}_B \) is said to be diagonal preserving.

In order to do this, we follow the exposition of K. Matsumoto in \cite{Mat10}. Let us first record an alternative definition of the Cuntz-Krieger algebra determined by the \( N \times N \)-matrix \( A \).

The fact that the two definitions agree is explained in Appendix \[\ref{A}\].

Definition 2.2.3. The Cuntz-Krieger algebra \( \mathcal{O}_A \) is the universal \( C^* \)-algebra generated by \( N \) partial isometries \( S_1, \ldots, S_N \) subject to the relations:

\[
\sum_{j=1}^{N} S_j S_j^* = 1, \quad S_i^* S_i = \sum_{j=1}^{N} A(i, j) S_j S_j^* \quad \text{(C-K)}
\]

for each \( i = 1, \ldots, N \).

\[\text{This is actually the original definition which Cuntz and Krieger gave in } \cite{CK80}.\]
With this definition in mind, we may realize $\mathcal{O}_A$ in the following way: Let $\mathcal{H}_A$ be the (non-separable) Hilbert space with orthonormal basis $\{e_x\}_{x \in X_A}$ indexed by the sequences in $X_A$. The operators $S_i : \mathcal{H}_A \to \mathcal{H}_A$ given by

$$S_i e_x = \begin{cases} e_{ix} & \text{if } ix \in X_A, \\ 0 & \text{otherwise} \end{cases}$$

satisfy the relations (C-K). This determines a faithful representation of $\mathcal{O}_A$ on $\mathcal{H}_A$. Note also that

$$S_i^* e_x = \begin{cases} e_{\sigma_A(x)} & x \in N_i, \\ 0 & \text{otherwise}. \end{cases}$$

**Lemma 2.2.4.** The C*-algebra $\mathcal{D}_A$ is canonically isomorphic to $C(X_A)$ via the map defined by $\varphi : S_{\alpha} S_{\alpha}^* \mapsto \chi_{N_{\alpha}}$, for every $\alpha \in \mathcal{A}_{<\infty}$.

**Proof.** The map $\varphi$ respects the product. Indeed, let $\alpha, \beta \in \mathcal{A}_{<\infty}$ and suppose $\alpha$ extends $\beta$. Then

$$\varphi : (S_{\alpha} S_{\alpha}^*)(S_{\beta} S_{\beta}^*) = S_{\alpha} S_{\alpha}^* \mapsto \chi_{N_{\alpha}} = \chi_{N_{\alpha}} \chi_{N_{\beta}}$$

and $\varphi : (S_{\alpha} S_{\alpha}^*)(S_{\beta} S_{\beta}^*) = S_{\beta} S_{\beta}^* \mapsto \chi_{N_{\beta}} = \chi_{N_{\alpha}} \chi_{N_{\beta}}$, when $\beta$ extends $\alpha$. The map $\varphi$ clearly respects the involution and so it extends by continuity to a C*-isomorphism. \hfill \Box

One implication of Theorem 2.2.1 can be verified with the tools we have established hitherto. In the proof below, we let $S_i^A$ and $S_i^B$ denote the generating partial isometries of $\mathcal{O}_A$ and $\mathcal{O}_B$. We view them as operators on the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ which have bases $\{e_x^A\}_{x \in X_A}$ and $\{e_x^B\}_{y \in X_B}$, respectively.

**Proposition 2.2.5.** Suppose $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are continuously orbit equivalent. Then there exists a C*-isomorphism $\Psi : (\mathcal{O}_A, \mathcal{D}_A) \to (\mathcal{O}_B, \mathcal{D}_B)$.

**Proof.** Let $h : X_A \to X_B$ be a homeomorphism witnessing the orbit equivalence of the two shift spaces and choose continuous functions $k_1, l_1 : X_A \to \mathbb{Z}_+$ satisfying

$$\sigma_{k_1(x)}^B(h(\sigma_A(x))) = \sigma_{l_1(x)}^A(h(x)), \quad (2.6)$$

for $x \in X_A$. This is [1.4]. Define an operator $u_h : \mathcal{H}_A \to \mathcal{H}_B$ by putting $u_h e_x^A = e_{h(x)}^B$, for every $x \in X_A$. Since $h$ is a bijection, $u_h$ is a unitary. We will show that

1. $\text{Ad}(u_h)(\mathcal{O}_A) = \mathcal{O}_B$,
2. $\text{Ad}(u_h)(f) = f \circ h^{-1},$

for every $f \in \mathcal{D}_A$. For (1) it suffices to show that $\text{Ad}(u_h)(S_i^A) \in \mathcal{O}_B$ and this is what we will do. As a preliminary observation, note that

$$u_h S_i^A u_h^* e_y^B = \begin{cases} e_{h^{-1}(y)}^B & ih^{-1}(y) \in X_A, \\ 0 & \text{otherwise}. \end{cases}$$
Set $X^{(i)}_B = \{ y \in X_B \mid ih^{-1}(y) \in X_A \}$. The range of the source projection of $\text{Ad}(u_h)(S_i^A)$ is spanned by the basis vectors indexed over $X^{(i)}_B$. Since $h(\sigma_A(N^A_i)) = X^{(i)}_B$, this space is compact and open. Let $P^{(i)} = \chi_{X^{(i)}_B}$ be the corresponding projection in $D_B$.

Fix $i \in \mathfrak{A}(B)$. To avoid notational clutter, we put $z = ih^{-1}(y) \in X_A$, when $y \in X^{(i)}_B$. Let us now agree to put

$$\tilde{k} = \max\{k_1(z) \mid y \in X^{(i)}_B\}, \quad \tilde{t} = \max\{l_1(z) \mid y \in X^{(i)}_B\}.$$  

These are maxima of continuous images of compact sets. Now, since $h(\sigma_A(z)) = y$, formula (2.6) shows that $\sigma^{k_1(z)}_B(y) = \sigma^{l_1(z)}_B(h(z))$. To each $y \in X^{(i)}_B$, we may associate a unique word $\mu(z) \in \mathfrak{A}^{\tilde{k}}_{\tilde{t}(z)}$ such that

$$h(z) = \mu(z)y_{[k_1(z)+1,\infty)}.$$  

This defines a continuous map $\mu : X^{(i)}_B \to \mathfrak{A}^{\tilde{k}}_{\tilde{t}(z)}$ when the codomain is equipped with the discrete topology.

For every word $\alpha \in \mathfrak{A}_{\tilde{t}(z)}$ and $0 \leq n \leq \tilde{k}$, we shall consider the clopen subsets of $X^{(i)}_B$

$$E^{(i)}_{\alpha} = \{ y \in X^{(i)}_B \mid \alpha = \mu(z)_{[1,l_1(z)]} \}, \quad F^{(i)}_n = \{ y \in X^{(i)}_B \mid k_1(z) = n \},$$

together with the associated projections $Q^{(i)}_{\alpha} = \chi_{E^{(i)}_{\alpha}}$ and $P^{(i)}_n = \chi_{F^{(i)}_n}$ in $D_B$. The fact that

$$X^{(i)}_B = \coprod_{\alpha \in \mathfrak{A}_{\tilde{t}(z)}} E^{(i)}_{\alpha} = \coprod_{n=0}^{\tilde{k}} F^{(i)}_n$$

implies that

$$P^{(i)} = \sum_{\alpha \in \mathfrak{A}_{\tilde{t}(z)}} Q^{(i)}_{\alpha} = \sum_{n=0}^{\tilde{k}} P^{(i)}_n.$$  

Let us finally fix $y \in X^{(i)}_B$. There are unique $\alpha \in \mathfrak{A}^{\tilde{k}}_{\tilde{t}(z)}$ and $0 \leq n \leq \tilde{k}$ such that $y \in E^{(i)}_{\alpha} \cap F^{(i)}_n$ and it follows that $u_h S_i^A u_h^* e_y = e_y^{B} e^{B}_{h(ih^{-1}(y))} = S_i^B e_y^{B} e^{B}_{\sigma_B^B(y)}$. Therefore,

$$u_h S_i^A u_h^* e_y = \sum_{\alpha \in \mathfrak{A}^{\tilde{k}}_{\tilde{t}(z)}} \sum_{n=0}^{\tilde{k}} \left( S^B_{\alpha} \sum_{\beta \in \mathfrak{A}_n} (S^B_{\beta})^* \right) Q^{(i)}_{\alpha} P^{(i)}_n e_y,$$

so

$$u_h S_i^A u_h^* = \sum_{\alpha \in \mathfrak{A}^{\tilde{k}}_{\tilde{t}(z)}} \sum_{n=0}^{\tilde{k}} \left( S^B_{\alpha} \sum_{\beta \in \mathfrak{A}_n} (S^B_{\beta})^* \right) Q^{(i)}_{\alpha} P^{(i)}_n P^{(i)}$$

is in $\mathcal{O}_B$. A similar argument invoking continuous functions $k_2, l_2 : X_B \to \mathbb{Z}_+$ satisfying (1.5), yields the inclusion $\text{Ad}(u_h^*)(\mathcal{O}_B) \subseteq \mathcal{O}_A$. This proves (1).
It suffices to show (2) for generating elements \( f = S_\alpha S_\alpha^* \in \mathcal{D}_A \). Such an element corresponds to the characteristic function \( \chi_{N_\alpha} = \chi_\alpha \). Observe that

\[
u_h S_\alpha S_\alpha^* \nu_h^* e_B = \begin{cases} e_y & h^{-1}(y) \in N_\alpha, \\ 0 & \text{otherwise}, \end{cases}
\]

for each \( y \in X_B \). So \( \text{Ad}(\nu_h)(S_\alpha S_\alpha^*) \) is the orthogonal projection onto the subspace of \( \mathcal{H}_B \) spanned by \( e_y^B \) with \( y \in h(N_\alpha) \). This corresponds to \( \chi_{h(N_\alpha)} = \chi_{N_\alpha} \circ h^{-1} \). This finishes the proof.

The other implication requires some more work. We shall make use of the collection of normalizing unitaries

\[
\mathcal{U}(\mathcal{O}_A, \mathcal{D}_A) = \{ v \in \mathcal{U}(\mathcal{O}_A) \mid v \mathcal{D}_A v^* = \mathcal{D}_A \},
\]

where \( \mathcal{U}(\mathcal{O}_A) \) is the group of unitaries in \( \mathcal{O}_A \). Given \( v \in \mathcal{U}(\mathcal{O}_A, \mathcal{D}_A) \), the adjoint representation \( \text{Ad}(v) \) defines an automorphism on \( (\mathcal{O}_A, \mathcal{D}_A) \). In particular, \( \text{Ad}(v) : \mathcal{D}_A \to \mathcal{D}_A \) produces a unique homeomorphism \( \tau_v : X_A \to X_A \) subject to the condition that

\[
\text{Ad}(v)(f) = f \circ \tau_v^{-1},
\]

for \( f \in \mathcal{D}_A \). We will show that \( \tau_v \) actually defines an element in the topological full group. This is the first step in proving the other implication of Theorem 2.2.1. First we need some technical lemmas. Here,

\[
N_s(\mathcal{O}_A, \mathcal{D}_A) = \{ v \in \mathcal{O}_A \mid v \text{ is a partial isometry, } v \mathcal{D}_A v^* \subseteq \mathcal{D}_A, \ v^* \mathcal{D}_A v \subseteq \mathcal{D}_A \}
\]

is the normalizing partial isometries of the subalgebra \( \mathcal{D}_A \) in \( \mathcal{O}_A \). The subscript »s« indicates that \( N_s(\mathcal{O}_A, \mathcal{D}_A) \) has a natural structure of a semigroup. \(^4\)

**Lemma 2.2.6.** Fix \( v \in \mathcal{U}(\mathcal{O}_A, \mathcal{D}_A) \). There exist orthogonal partial isometries \( \{v_m\}_{m=-K}^K \), for some \( K \in \mathbb{N} \), subject to following conditions:

1. \( v = \sum_{m=-K}^K v_m \),
2. \( v_m \in N_s(\mathcal{O}_A, \mathcal{D}_A) \) for every \( m = -K, \ldots, K \),
3. \( v_m v_m^* \) and \( v_m^* v_m \) are projections in \( \mathcal{D}_A \) for every \( m = -K, \ldots, K \),
4. \( v_0 \in \mathcal{F}_A \).

**Proof.** Define a function \( g : [0, 2\pi] \to \mathcal{O}_A \) by \( g(t) = v^* \rho_t(v) \) and observe that

\[
v^* \rho_t(v) f = v^* \rho_t(v f v^*) \rho_t(v) = f v^* \rho_t(v),
\]

\(^3\)Our notation differs slightly from the one of Mat10.

\(^4\)Actually, \( N_s(\mathcal{O}_A, \mathcal{D}_A) \) has the structure of an inverse semigroup (see Section 3.3) but we will not be needing this. Compare this to the definition of the normalizer (Definition 3.3.9) in which the elements are not assumed to be partial isometries.
since \(vf^*v^* \in \mathcal{D}_A\) is fixed under the gauge action when \(f \in \mathcal{D}_A\). So \(g(t)\) commutes with \(\mathcal{D}_A\). As \(\mathcal{D}_A\) is a masa, \(g\) takes values in \(\mathcal{D}_A\). In particular, \(g(t)\) is fixed under the gauge action for any \(t \in [0, 2\pi]\). It follows that \(g(t)^* = g(2\pi - t)\) and \(g(t)g(s) = g(t + s)\) for any \(t, s \in [0, 2\pi]\), since

\[
g(t)^* = \rho_t(v^*v) = \rho_t(v^*\rho_{2\pi - t}(v)) = v^*\rho_{2\pi - t}(v),
\]

\[
g(t)g(s) = v^*\rho_t(v^*\rho_s(v)) = v^*\rho_{t+s}(v).
\]

The sum is modulo \(2\pi\). In particular, \(g(t)\) is a unitary in \(\mathcal{D}_A\). The \(m\)'th Fourier coefficient

\[
\hat{g}(m) = \langle g, e_m \rangle = \int_0^{2\pi} g(t)e_{-m}(t) \, dt
\]

(this is again a Bochner integral) satisfies the relations

\[
\hat{g}(m)^* = \int_0^{2\pi} g(-t)e_{m}(t) \, dt = \int_0^{2\pi} g(-t)e_{-m}(-t) \, dt = \langle g, e_m \rangle = \hat{g}(m),
\]

\[
\hat{g}(m)^2 = \hat{g}\ast\hat{g}(m) = \langle g, e_m \rangle = \hat{g}(m),
\]

since \(g \ast g(t) = \int_0^{2\pi} g(t - s)g(s) \, ds = \int_0^{2\pi} g(t) \, ds = g(t)\). Hence \(\hat{g}(m)\) is a projection in \(\mathcal{D}_A\) for every \(m \in \mathbb{Z}\). Viewed as a continuous map on \(X_A\) it thus takes values in \(\{0, 1\}\). Put \(v_m = \hat{v}\hat{g}(m)\) and observe that \(v^*_mv_m = \hat{v}\hat{g}(m)v^*\) and \(v_mv^*_m = \hat{g}(m)\) are projections in \(\mathcal{D}_A\). In particular, \(v_m\) is a partial isometry for every \(m \in \mathbb{Z}\). Note that

\[
v_0 = \hat{v}\hat{g}(0) = \int_0^{2\pi} \rho_t(v) \, dt = E(v) \in \mathcal{F}_A.
\]

and so (4) holds.

Fix \(x \in X_A\) and define \(g_x : [0, 2\pi] \to \mathbb{C}\) by \(g_x(t) = g(t)(x)\). Then

\[
|g_x(t)|^2 = \langle g(t)e_x, g(t)e_x \rangle = \langle e_x, e_x \rangle = 1
\]

so by Parseval’s identity

\[
1 = \int_0^{2\pi} |g_x(t)|^2 \, dt = \sum_{m \in \mathbb{Z}} |\langle g_x, e_m \rangle|^2 = \sum_{m \in \mathbb{Z}} \int_0^{2\pi} g_x(t)e_{-m}(t) \, dt|^2 = \sum_{m \in \mathbb{Z}} |\hat{g}(m)(x)|^2.
\]

That is, for each \(x \in X_A\) there is a unique integer \(m\) such that \(\hat{g}(m)(x) = 1\) (since \(\hat{g}(m)\) takes values in \(\{0, 1\}\)). It follows that \((\hat{g}(m))_m\) is a family of mutually orthogonal projections. Therefore, the partial isometries \((v_m)_m\) are also mutually orthogonal.

The support \(E_m = \text{supp}(\hat{g}(m))\) is clopen for every \(m \in \mathbb{Z}\) and these sets partition \(X_A\). As \(X_A\) is compact, there exists a \(K \in \mathbb{N}\) such that \(X_A = \bigsqcup_{m=-K}^{K} E_m\). Consequently, \(v_m = 0\) when \(|m| > K\). In addition

\[
\sum_{m=-K}^{K} v_m = v \sum_{m=-K}^{K} \hat{g}(m) = v.
\]

This proves (1). Note also that

\[
v_mv^*_m = \hat{v}\hat{g}(m)\mathcal{D}_A\hat{g}(m)v^*, \quad v^*_mv_m = \hat{g}(m)v^*\mathcal{D}_A\hat{v}\hat{g}(m)
\]

are both contained in \(\mathcal{D}_A\) since \(v\) normalizes \(\mathcal{D}_A\). Hence \(v_m \in \mathcal{N}_s(\mathcal{O}_A, \mathcal{D}_A)\). \qed
Lemma 2.2.7. Let $v \in U(O_A, D_A)$ and $(v_m^K)_{m=1}^K$ be as above and fix $m = 1, \ldots, K$. For $\alpha \in A_m$ there exist partial isometries $v_\alpha, v_{-\alpha} \in F_A$ subject to the following conditions:

1. $v_m = \sum_{\alpha \in A_m} S_{\alpha} v_\alpha$ and $v_m = \sum_{\alpha \in A_m} v_{-\alpha} S_{\alpha}^*$.
2. $v_\alpha^* v_\alpha, S_{\alpha} v_\alpha v_\alpha^* S_{\alpha}^*, S_{\alpha} v_{-\alpha} S_{\alpha}^* S_{\alpha}$ and $v_{-\alpha} v_{-\alpha}^*$ are projections in $D_A$ satisfying
   \begin{align}
   &v_m v_m = \sum_{\alpha \in A_m} v_{\alpha}^* v_{\alpha}, & v_m v_m = \sum_{\alpha \in A_m} S_{\alpha} v_\alpha v_{\alpha}^* S_{\alpha}^* \\
   &v_m v_m = \sum_{\alpha \in A_m} S_{\alpha} v_{-\alpha} S_{\alpha}^*, & v_m v_m = \sum_{\alpha \in A_m} v_{-\alpha} v_{-\alpha}^* \
   \end{align}

3. $v_\alpha v_\beta^* = v_{-\alpha} v_{-\beta} = 0$ for distinct $\alpha, \beta \in A_m$.
4. $v_\alpha, v_{-\alpha} \in N_A(O_A, D_A)$.

Here, all the sums are indexed over $\alpha \in A_m$.

Proof. Define $v_\alpha = E(S_{\alpha}^* v) \in F_A$ and $v_{-\alpha} = E(v S_\alpha) \in F_A$ and observe that

\begin{align}
S_{\alpha}^* v_m &= \int_0^{2\pi} S_{\alpha}^* v e_m(t) dt = \int_0^{2\pi} v (S_{\alpha}^* v)e_m(t) dt = v_\alpha, \\
v_m S_{\alpha} &= \int_0^{2\pi} v(t) e_m(t) S_{\alpha} dt = v_{-\alpha}.
\end{align}

Therefore,

\begin{align}
\sum_{\alpha \in A_m} S_{\alpha} v_\alpha &= \sum_{\alpha \in A_m} S_{\alpha} S_{\alpha}^* v_m = v_m, \\
\sum_{\alpha \in A_m} v_{-\alpha} S_{\alpha}^* &= \sum_{\alpha \in A_m} v_m S_{\alpha}^* S_{\alpha} = \sum_{\alpha \in A_m} v_m S_{\alpha} S_{\alpha}^* = v_m,
\end{align}

and this proves (1). Note that

\begin{align}
v_{\alpha}^* v_\alpha &= v_m^* S_{\alpha}^* S_{\alpha} v_m = \widehat{g}(m) S_{\alpha}^* S_{\alpha} v(\widehat{g})(m), \\
S_{\alpha} v_\alpha v_{\alpha}^* S_{\alpha} &= S_{\alpha} S_{\alpha} v_m^* S_{\alpha} v_{\alpha}^* S_{\alpha} = S_{\alpha} S_{\alpha} v(\widehat{g})(m) v_{\alpha}^* S_{\alpha} v_{\alpha}^*
\end{align}

are projections in $D_A$. Similar computations apply for $S_{\alpha} v_{-\alpha} v_{-\alpha}^* S_{\alpha}$ and $v_{-\alpha} v_{-\alpha}^*$. Hence

\begin{align}
\sum_{\alpha \in A_m} v_{\alpha}^* S_{\alpha} &= \sum_{\alpha \in A_m} v_m S_{\alpha} S_{\alpha}^* v_m = v_m^* v_m, \\
\sum_{\alpha \in A_m} v_{-\alpha} S_{\alpha}^* &= \sum_{\alpha \in A_m} v_m S_{\alpha} S_{\alpha}^* v_m = v_m v_m.
\end{align}

This proves (2.9); (2.10) follows from similar computations. So (2) holds. We have

\begin{align}
v_\alpha v_\beta^* &= S_{\alpha}^* (v_m v_m) S_{\beta} = \sum_{\xi \in A_m} S_{\xi}^* \xi v_\xi v_\xi^* S_{\xi} S_{\beta} \\
\end{align}

which vanishes when $\alpha \neq \beta$. This is (3). Finally,

\begin{align}
v_\alpha &= S_{\alpha}^* v_m = S_{\alpha}^* \widehat{g}(m), & v_{-\alpha} &= v_m S_{\alpha} = \widehat{g}(m) S_{\alpha}
\end{align}
from which we see that
\[ v_{\alpha}D_{A}v_{\alpha}^{*} = S_{\alpha}^{*}v_{\hat{g}}(m)D_{A}v_{\hat{g}}(m)v_{\alpha}^{*}, \quad v_{\alpha}^{*}D_{A}v_{\alpha} = \hat{g}(m)v_{\alpha}^{*}S_{\alpha}D_{A}S_{\alpha}v_{\hat{g}}(m) \]
are both contained in \( D_{A} \) since both \( v \) and \( S_{\alpha} \) normalize \( D_{A} \). Similarly, one see that \( v_{-\alpha} \)
normalizes \( D_{A} \). Thus (4) holds and this finishes the proof.

If \( u \in N_{s}(O_{A}, D_{A}) \) is a partial isometry, then the support of the range and source projections of \( u \) are clopen subsets \( X_{A} \). Furthermore, the adjoint representation defines an isomorphism \( Ad(u) : D_{A}u^{*}u \rightarrow D_{A}u^{*}u^{*} \). We shall identify \( D_{A}u^{*}u \) and \( D_{A}u^{*}u^{*} \) with \( C(\text{supp}(u^{*}u)) \) and \( C(\text{supp}(uu^{*})) \), respectively. Hence we obtain a (unique) homeomorphism \( h_{u} : \text{supp}(u^{*}u) \rightarrow \text{supp}(uu^{*}) \) satisfying
\[ Ad(u)(f) = f \circ h_{u}^{-1}, \quad (2.11) \]
for every \( f \in D_{A}u^{*}u^{*}u \).

Now, given \( x \in \text{supp}(u^{*}u) \) let \( y = h_{u}(x) \in \text{supp}(uu^{*}) \) and consider the partial isometry
\[ uT_{n} = S_{y[1,n]}^{*}uS_{x[1,n]}, \quad (2.12) \]
for \( n \in \mathbb{N} \). To simplify the notation, put \( \chi = \chi_{N_{y[1,n]}} \). Then \( uS_{x[1,n]}^{*}u^{*} \) in \( D_{A} \) corresponds to \( Ad(u)(\chi) = \chi \circ h_{u}^{-1} \) in \( C(X_{A}) \) and
\[ Ad(u)(\chi)e_{y} = \chi \circ h_{u}^{-1}(y) = \chi(x) = 1. \]
Consequently,
\[ \langle S_{y[1,n]}^{*}uS_{x[1,n]}^{*}S_{y[1,n]}^{*}u^{*}(S_{y[1,n]}^{*}e_{y}(y)), e_{\sigma_{\alpha}^{A}(y)} \rangle = \langle e_{\sigma_{\alpha}^{A}(y)} \rangle = 1. \quad (2.13) \]
Hence the partial isometry \( uT_{n} \) is non-zero. In particular, \( ||uT_{n}|| = 1 \), for every \( n \in \mathbb{N} \).

**Lemma 2.2.8.** Let \( u \in N(O_{A}, D_{A}) \) be a partial isometry. Assume furthermore that \( u \in F_{A} \). Then \( \text{supp}(u^{*}u) \) and \( \text{supp}(uu^{*}) \) uniformly agree eventually. That is, there is \( k_{u} \in \mathbb{N} \) such that
\[ \sigma_{A}^{k_{u}}(h_{u}(x)) = \sigma_{A}^{k}(x), \]
for every \( x \in \text{supp}(u^{*}u) \).

**Proof.** Let \( x \in \text{supp}(u^{*}u) \) and put \( y = h_{u}(x) \). Pick also \( k \in \mathbb{N} \) and \( u' \in F_{A}^{k} \) such that \( ||u - u'|| < 1/2 \). In order to reach a contradiction, we shall assume the existence of \( N > k \) such that \( x_{N} \neq y_{N} \). We may write
\[ u' = \sum_{\alpha, \beta \in \mathbb{A}_{N-1}} c_{\alpha, \beta} S_{\alpha}S_{\beta}^{*} \in F_{A}^{N-1}, \]
for some coefficients \( c_{\alpha, \beta} \in \mathbb{C} \). Then (using the notation of (2.12))
\[ u'T_{N-1} = \sum_{\alpha, \beta \in \mathbb{A}_{N-1}} c_{\alpha, \beta} S_{\alpha}^{*}S_{y[1,N-1]}^{*}S_{x[1,N-1]}^{*}S_{\beta}S_{x[1,N-1]}, \]
so \( u'T_{N} = S_{y[1,N]}^{*}(u'T_{N-1})S_{x[1,N]} = 0 \) since \( x_{N} \neq y_{N} \). Hence
\[ ||uT_{n}|| = ||(u - u')T_{n}|| = ||u - u'|| < 1/2 \]
provided that \( n > N \). This contradicts the above observation (2.13). \[\Box\]
Proposition 2.2.9. Let \( v \in \mathcal{U}(\mathcal{O}_A, \mathcal{D}_A) \). The homeomorphism \( \tau_v \) introduced in (2.8) is an element of \( [\sigma_A]_c \).

Proof. Start by writing \( v = \sum_{m=-K}^{K} v_m \) for some orthogonal partial isometries \( v_{-K}, \ldots, v_K \) chosen in accordance with Lemma 2.2.7. The sets
\[
X_A^{(m)} = \operatorname{supp}(v_m^* v_m), \quad Y_A^{(m)} = \operatorname{supp}(v_m v_m^*)
\]
are clopen and partition \( X_A \). For each \( 1 \leq m \leq K \), we also pick partial isometries \( v_\alpha, v_{-\alpha} \in \mathcal{F}_A \) according to Lemma 2.2.7. Then
\[
X_A^{(m,\alpha)} = \operatorname{supp}(v_\alpha^* v_\alpha) \quad X_A^{(-m,\alpha)} = \operatorname{supp}(v_{-\alpha}^* v_{-\alpha})
\]
\[
Y_A^{(m,\alpha)} = \operatorname{supp}(S_\alpha v_\alpha v_\alpha^* s_\alpha^*) \quad Y_A^{(-m,\alpha)} = \operatorname{supp}(v_{-\alpha} v_{-\alpha}^*)
\]
are also clopen in \( X_A \). Furthermore, \( X_A^{(m)} = \bigsqcup_{\alpha \in \mathbb{N}_m} X_A^{(m,\alpha)} \) and \( Y_A^{(m)} = \bigsqcup_{\alpha \in \mathbb{N}_m} Y_A^{(m,\alpha)} \).

We shall now set out to find orbit cocycles for \( \tau_v \). Observe that for \( f \in \mathcal{D}_A \) we have
\[
\operatorname{Ad}(v)(f) = \sum_{m=1}^{K} v_m f v_m^* + v_0 f v_0^* + \sum_{m=1}^{K} v_{-m} f v_{-m}^*
\]
and the case \( m = 0 \) is easily taken care of. Since \( v_0 \in \mathcal{F}_A \) we simply apply Lemma 2.2.8 to pick up \( k_0 \in \mathbb{N} \) such that \( \sigma_A^{k_0}(\tau_0(x)) = \sigma_A^{k_0}(x) \), for \( x \in X_A^{(0)} \).

For each \( v_\alpha \in \mathcal{F}_A \), there is a homeomorphism \( h_{v_\alpha} \) such that \( \operatorname{supp}(v_\alpha^* v_\alpha) \) and \( \operatorname{supp}(v_\alpha v_\alpha^*) \) uniformly agree eventually, cf. Lemma 2.2.8. However, since
\[
v_m f v_m^* = \sum_{\alpha \in \mathbb{N}_m} S_\alpha v_\alpha f v_\alpha^* S_\alpha^*, \quad v_{-m} f v_{-m} = \sum_{\alpha \in \mathbb{N}_m} v_{-\alpha}^* S_\alpha f S_\alpha v_{-\alpha}^*,
\]
we are interested in \( \operatorname{Ad}(S_\alpha v_\alpha) \). If we put \( \tau_{(m,\alpha)} := h_{S_\alpha} \circ h_{v_\alpha} \), then we see that given \( x \in X_A^{(m,\alpha)} \) the sequences \( \sigma_A^n(x) \) and \( \tau_{(m,\alpha)}(x) \in Y_A^{(m,\alpha)} \) agree eventually. That is, there exists \( k_{(m,\alpha)} \in \mathbb{N} \) such that
\[
\sigma_A^{k_{(m,\alpha)}}(\tau_{(m,\alpha)}(x)) = \sigma_A^{k_{(m,\alpha)}+m}(x).
\]
Since
\[
\tau_v(x) = \begin{cases} 
\tau_{(m,\alpha)}(x) & x \in X_A^{(m,\alpha)} \\
\tau_0(x) & x \in X_A^{(0)} \\
\tau_{(-m,\alpha)} & x \in X_A^{(-m,\alpha)} 
\end{cases}
\]
this finishes the proof. \( \square \)

Proposition 2.2.10. To each \( \tau \in [\sigma_A]_c \), there is a unitary \( u_\tau \in \mathcal{U}(\mathcal{O}_A, \mathcal{D}_A) \) satisfying
\[
\operatorname{Ad}(u_\tau)(f) = f \circ \tau^{-1}, \quad (2.14)
\]
for every \( f \in \mathcal{D}_A \). The association \( \tau \mapsto u_\tau \) is a group homomorphism.
Proof. Define \( u_\tau : \mathcal{O}_A \to \mathcal{O}_A \) by \( u_\tau e_x = e_{\tau(x)} \), for every \( x \in X_A \). Since \( \tau \) is a bijection, \( u_\tau \) is a unitary. It is clear that this defines a group homomorphism. We will show that \( u_\tau \in \mathcal{O}_A \).

Let \( k, l : X_A \to \mathbb{Z}_+ \) be orbit cocycles for \( \tau \) satisfying \( \sigma_A^{k(x)}(\tau(x)) = \sigma_A^{l(x)}(x) \) for \( x \in X_A \) and put
\[
\tilde{k} = \max\{k(X_A)\}, \quad \tilde{l} = \max\{l(X_A)\}.
\]
Furthermore, we let \( \mu(x) \in \mathfrak{A}_{\leq \tilde{k}} \) denote the unique admissible word satisfying
\[
\tau(x) = \mu(x)[l(x)+1, \infty).
\]
This defines a continuous map \( \mu : X_A \to \mathfrak{A}_{\leq \tilde{k}} \) when \( \mathfrak{A}_{\leq \tilde{k}} \) is equipped with the discrete topology. For each admissible word \( \alpha \in \mathfrak{A}_{\leq \tilde{k}} \) and \( 0 \leq n \leq \tilde{l} \), we may now consider the clopen subsets of \( X_A \) given by
\[
E_\alpha = \{ x \in X_A \mid \alpha = \mu(x)[1, k(x)] \}, \quad F_n = \{ x \in X_A \mid l(x) = n \}
\]
together with their associated projections \( Q_\alpha = \chi_{E_\alpha} \) and \( P_n = \chi_{F_n} \) in \( \mathcal{D}_A \). Since subsets of this form partition \( X_A \), we have
\[
1 = \sum_{\alpha \in \mathfrak{A}_{\leq \tilde{k}}} Q_\alpha = \sum_{n=0}^{\tilde{l}} P_n.
\]
Now, given \( x \in X_A \), there is a unique \( \alpha \in \mathfrak{A}_{\leq \tilde{k}} \) and \( 0 \leq n \leq \tilde{l} \) such that \( x \in E_\alpha \cap F_n \) in which case \( u_\tau e_x = e_{\tau(x)} = S_\alpha e_{\sigma_A^n(x)} \). Hence
\[
u_\tau e_x = e_{\tau(x)} = \sum_{\alpha \in \mathfrak{A}_{\leq \tilde{k}}} \sum_{n=0}^{\tilde{l}} \left( S_\alpha \sum_{\beta \in \mathfrak{A}_n} S_{S_\beta}^* \right) Q_\alpha P_n e_x.
\]
In particular, \( u_\tau \in \mathcal{O}_A \).

The relation (2.14) follows from the observation that
\[
u_\tau S_\alpha S_{S_\alpha}^* u_\tau e_x = \begin{cases} e_x & \tau^{-1}(x) \in N_\alpha, \\ 0 & \text{else} \end{cases}
\]
so \( \text{Ad}(u_\tau)(S_\alpha S_{S_\alpha}^*) \) is the orthogonal projection onto the subspace of \( \mathcal{O}_A \) spanned by \( e_x \) for \( x \in \tau(N_\alpha) \). This corresponds to the projection \( \chi_{\tau(N_\alpha)} = \chi_{N_\alpha} \circ \tau^{-1} \) in \( \mathcal{D}_A \). As every \( g \in \mathcal{D}_A \) is of the form \( f \circ \tau^{-1} \), for some \( f \in \mathcal{D}_A \), this shows that \( u_\tau \in \mathcal{U}(\mathcal{O}_A, \mathcal{D}_A) \).

Theorem 2.2.11. The sequence
\[
0 \to \mathcal{U}(\mathcal{D}_A) \to \mathcal{U}(\mathcal{O}_A, \mathcal{D}_A) \xrightarrow{\tau_*} [\sigma_A]_* \to 0
\]
is a split short exact sequence of groups.

Proof. Propositions 2.2.9 and 2.2.10 shows that \( \tau_* : \mathcal{U}(\mathcal{O}_A, \mathcal{D}_A) \to [\sigma_A]_* \) given by \( v \mapsto \tau_v \) is a well-defined and surjective homomorphism. The map \( \mathcal{U}(\mathcal{D}_A) \to \mathcal{U}(\mathcal{O}_A, \mathcal{D}_A) \) is just inclusion. If \( v \in \mathcal{U}(\mathcal{D}_A) \), then \( \text{Ad}(v) = \text{id} \) on \( \mathcal{D}_A \) since \( \mathcal{D}_A \) is commutative. It follows that \( \tau_v = \text{id} \). Conversely, if \( \tau_v = \text{id} \) then \( \text{Ad}(v) = \text{id} \) on \( \mathcal{D}_A \), that is, \( v \) commutes with every element of \( \mathcal{D}_A \). Since \( \mathcal{D}_A \) is a masa in \( \mathcal{O}_A \), we conclude that \( v \in \mathcal{D}_A \). Hence the above sequence is exact. It is clear that the group homomorphism \( \tau \mapsto u_\tau \) of Proposition 2.2.10 constitutes a section of the sequence. \( \square \)
In particular, we have an explicit description \([\sigma_A]_c = \{\tau_v \mid v \in \mathcal{U}(\mathcal{O}_A, \mathcal{D}_A)\}\) of the topological full group.

**Proposition 2.2.12.** Suppose there exists a \(C^\ast\)-isomorphism \(\Psi: (\mathcal{O}_A, \mathcal{D}_A) \to (\mathcal{O}_B, \mathcal{D}_B)\). Then there is a homeomorphism \(h: X_A \to X_B\) which intertwines the topological full group.

**Proof.** Since \(\Psi\) is an isomorphism between \(\mathcal{D}_A\) and \(\mathcal{D}_B\), we may choose a unique homeomorphism \(h: X_A \to X_B\) satisfying \(\Psi(f) = f \circ h^{-1}\), for \(f \in \mathcal{D}_A\). Furthermore, \(\Psi\) restricts to isomorphisms of groups

\[
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{U}(\mathcal{D}_A) \longrightarrow \mathcal{U}(\mathcal{O}_A, \mathcal{D}_A) \\
\Psi & \quad & \Psi \\
0 & \longrightarrow & \mathcal{U}(\mathcal{D}_B) \longrightarrow \mathcal{U}(\mathcal{O}_B, \mathcal{D}_B)
\end{array}
\]

and by the 5-lemma, the indicated dashed line is an isomorphism of groups. In fact, commutativity of the diagram shows that \(\tilde{\Psi}(\tau_v) = \tau_{\Psi(v)}\), for every \(v \in \mathcal{U}(\mathcal{O}_A, \mathcal{D}_A)\). Now, fix \(v \in \mathcal{U}(\mathcal{O}_A, \mathcal{D}_A)\) and observe that \(\text{Ad}((\Psi(v))(g)) = \Psi(v)g\Psi(v)^* = \Psi(v\Psi^{-1}(g)v^*)\), for all \(g \in \mathcal{D}_B\). Pictorially,

\[
\begin{array}{ccc}
\mathcal{D}_B & \overset{\Psi^{-1}}{\longrightarrow} & \mathcal{D}_A \\
\text{Ad}(\Psi(v)) & \downarrow & \text{Ad}(v) \\
\mathcal{D}_B & \overset{\Psi}{\longrightarrow} & \mathcal{D}_A
\end{array}
\]

Traversing the diagram clockwise, we obtain

\[
g = f \circ h^{-1} \overset{\Psi^{-1}}{\longrightarrow} f \overset{\text{Ad}(v)}{\longrightarrow} f \circ \tau_v^{-1} \overset{\Psi}{\longrightarrow} f \circ \tau_v^{-1} \circ h^{-1} = g \circ h \circ \tau_v^{-1} \circ h^{-1},
\]

since any \(g \in \mathcal{D}_B\) is of the form \(f \circ h^{-1}\), for some \(f \in \mathcal{D}_A\). By commutativity, the above expression equals

\[
\text{Ad}(\Psi(v))(g) = g \circ \tau_v^{-1} = g \circ \tilde{\Psi}(\tau_v)^{-1}.
\]

That is, \(\tilde{\Psi}(\tau_v)^{-1} = h \circ \tau_v \circ h^{-1}\). As \(\tilde{\Psi}\) is an isomorphism, this finishes the proof. \qed

Recalling Theorem 1.2.12 we have now proved Theorem 2.2.1.

### 2.3 \(K\)-theory

We shall now compute the \(K\)-theory of the Cuntz-Krieger algebra. See [Rø95] for an introduction to \(K\)-theory for \(C^\ast\)-algebras. We mainly follow the strategy of Cuntz in [Cun81a] so the task at hand is two-fold. We first show that \(\mathcal{O}_A\) is a corner in a certain crossed product and then use the Pimsner-Voiculescu sequence to actually compute the \(K\)-groups. Unlike in the Cuntz algebra case, \(K_1(\mathcal{O}_A)\) need not be trivial.

Consider once again the graph \((\mathcal{V}, \mathcal{E})\) determined by the adjacency matrix \(A\) together with the Cuntz-Krieger algebra \(\mathcal{O}_A\). We start på showing that \(\mathcal{O}_A\) is a crossed product by an endomorphism.
Consider the operator

\[ S = \sum_{e \in \mathcal{E}} |\mathcal{E}^{r(e)}|^{-\frac{1}{2}} T_e. \]

This is an isometry, as

\[ S^*S = \sum_{e,f \in \mathcal{E}} |\mathcal{E}^{r(e)}|^{-\frac{1}{2}} |\mathcal{E}^{r(f)}|^{-\frac{1}{2}} T_e^*T_f = \sum_{e \in \mathcal{E}} |\mathcal{E}^{r(e)}|^{-1} T_e^*T_e = \sum_{i=1}^{N} P_i = 1. \]

Observe also that

\[ SS^* = \sum_{e,f \in \mathcal{E}} |\mathcal{E}^{r(e)}|^{-1/2} |\mathcal{E}^{r(f)}|^{-1/2} T_e T_f^* = \sum_{i=1}^{N} \left( |\mathcal{E}^{i}|^{-1} \sum_{e,f \in \mathcal{E}^{i}} T_e T_f^* \right) \]

where each summand in the outer sum is a rank-one projection in \( \mathcal{F}_1 \). Let \( \beta_n : \mathcal{F}_n \rightarrow \mathcal{F}_{n+1} \) be the map \( \beta_n(a) = SaS^* \), for \( a \in \mathcal{F}_n \). This is well-defined since

\[ ST_\alpha T_\alpha^* S^* = \sum_{e,f \in \mathcal{E}} |\mathcal{E}^{r(e)}|^{-\frac{1}{2}} |\mathcal{E}^{r(f)}|^{-\frac{1}{2}} T_e^*T_f, \]

and so \( SF_n S^* \subseteq \mathcal{F}_{n+1} \) for any \( n \in \mathbb{N} \). Consequently, we have a well-defined (non-unital) endomorphism \( \beta : \mathcal{F}_A \rightarrow \mathcal{F}_A \) given by \( \beta(a) = SaS^* \). This is injective though not surjective. Pictorially,

\[ \begin{array}{ccccccc}
\mathcal{F}_1 & \xrightarrow{\beta_1} & \mathcal{F}_2 & \xrightarrow{\beta_2} & \mathcal{F}_3 & \xrightarrow{\beta_3} & \cdots \xrightarrow{\beta_n} & \mathcal{F}_A \\
\mathcal{F}_1 & \rightarrow & \mathcal{F}_2 & \rightarrow & \mathcal{F}_3 & \rightarrow & \cdots & \rightarrow & \mathcal{F}_A,
\end{array} \]

in which the horizontal maps are inclusions. We say that \( S \) implements the action \( \beta \).

**Theorem 2.3.1.** There exists an isomorphism

\[ \mathcal{O}_A \cong \mathcal{F}_A \rtimes_{\beta} \mathbb{N}, \]

where \( \mathcal{F}_A \rtimes_{\beta} \mathbb{N} := C^* (\mathcal{F}_A, T \mid T^*_TT = 1, TaT^* = \beta(a), a \in \mathcal{F}_A) \).

**Proof.** For each \( e \in \mathcal{E} \) we consider the operator \( R_e = |\mathcal{E}^{r(e)}|^{\frac{1}{2}} (T_e T_e^*) (T_e T_e^*)^* \). We want to define a *-homomorphism \( \varphi : \mathcal{O}_A \rightarrow C^* (\mathcal{F}_A, T) \) by

\[ \varphi : P_i \mapsto P_i, \quad T_e \mapsto R_e, \]

for \( i \in \mathcal{V} \) and \( e \in \mathcal{E} \). As a preliminary observation, note that \( \beta(T_e^* T_e) = \sum_{f,g \in \mathcal{E}^{r(e)}} |\mathcal{E}^{r(e)}|^{-1} T_f^* T_g^* \). Therefore,

\[ R_e^* R_e = |\mathcal{E}^{r(e)}| T^* (TT_e^* T_e T_e^*) (T_e T_e^*)^* (TT_e^* T_e T_e^*) T_e^* (TT_e^* T_e T_e^*) T = |\mathcal{E}^{r(e)}|^{-1} T_e \left( \sum_{f,g \in \mathcal{E}^{r(e)}} T_f^* T_g^* T_e \right) \left( \sum_{f',g' \in \mathcal{E}^{r(e)}} T_{f'}^* T_{g'}^* T_e \right) T_e^* \beta(T_e^* T_e) = T_e^* T_e \]
and
\[ \sum_{e \in E_i} R_e R_e^* = \sum_{e \in E_i} T_e T_e^* (TT_e^* T_e T_e^* T_e T_e^*) T_e T_e^* \]
\[ = \sum_{e \in E_i} |E^{(e)}|^{-1} \sum_{f,g \in E^{(e)}} T_e T_e^* T_f T_f^* T_e T_e^* \]
\[ = \sum_{e \in E_i} T_e T_e^* = P_i \]

for every \( i \in \mathcal{V} \). By the universal property of \( \mathcal{O}_A \), the map \( \varphi \) is well-defined. Similarly, since \( S \) implements the action \( \beta \) on \( \mathcal{F}_A \), there is a \( * \)-homomorphism \( \psi : C^*(\mathcal{F}_A, T) \to \mathcal{O}_A \) given by

\[ \psi : a \mapsto a, \quad T \mapsto S, \]

for every \( a \in \mathcal{F}_A \). The two maps are inverses of each other. Indeed,

\[ \psi(R_e) = \sum_{f \in E} (T_e T_e^*) T_f (T_e T_e^*) = T_e. \]

On the other hand,

\[ \varphi(T_e T_e^*) = R_e R_e^* = |E^{(e)}| T_e T_e^* (TT_e^* T_e T_e^*) (TT_e^* T_e T_e^*) T_e T_e^* \]
\[ = |E^{(e)}|^{-1} \sum_{h \in E^{(e)}} T_e T_e^* T_h T_h^* = T_e T_e^* \]

and

\[ \varphi(S) = \sum_{e \in E} (T_e T_e^*) (TT_e^* T_e T_e^*) T = \left( \sum_{e \in E} \sum_{g \in E^{(e)}} |E^{(e)}|^{-1} T_e T_g^* \right) T = \beta(1)T = T. \]

This establishes the isomorphism \( \mathcal{O}_A \cong \mathcal{F}_A \times_{\beta} \mathbb{N} \).

We have the following preliminary \( K \)-theoretic observations.

**Lemma 2.3.2.** We have

\[ K_0(\mathcal{F}_A) = \lim \left( \mathbb{Z}^N \xrightarrow{A^t} \mathbb{Z}^N \xrightarrow{A^t} \cdots \right) =: \lim \mathbb{Z}^N, \quad K_1(\mathcal{F}_A) = 0. \]

**Proof.** Let \( \iota_n : \mathcal{F}_n \to \mathcal{F}_{n+1} \) denote the inclusion and recall from Lemma 2.1.9 that \( \mathcal{F}_A \) is the inductive limit of the inductive system \( \mathcal{F}_1 \xrightarrow{\iota_1} \mathcal{F}_2 \xrightarrow{\iota_2} \cdots \). Each \( \mathcal{F}_n \) is a finite dimensional \( C^* \)-algebra and \( K_0(\mathcal{F}_n) = \mathbb{Z}^N \). Furthermore, \( K_0(\iota_n) = A^t \) for every \( n \in \mathbb{N} \). Indeed, fix \( n \in \mathbb{N} \) and put \( \iota = \iota_n \). Given \( i, j \in \mathcal{V} \), let \( \iota_{ij} \) denote the composition

\[ M_{k(n,i)}(\mathbb{C}) \xrightarrow{\iota} \bigoplus_{m=1}^N M_{k(n+1,m)}(\mathbb{C}) \to M_{k(n+1,j)}(\mathbb{C}), \]

where the second map is the projection onto the \( j \)’th summand. Now, if \( \alpha \in \mathfrak{A}_{n+1} \) with \( r(\alpha) = i \), then \( T_\alpha T_\alpha^* \) is a rank one projection in \( \mathcal{F}_n \) and

\[ \iota(T_\alpha T_\alpha^*) = T_\alpha T_\alpha^* = \sum_{e \in E} T_{ae} T_{ae}^* = \sum_{j=1}^N \left( \sum_{e \in E^j} T_{ae} T_{ae}^* \right) \]
where the summand in the parenthesis is contained in \( \mathcal{F}_{n+1}^j \). Hence
\[
\text{rank}(\iota_{j,i}(T_{\alpha}T_{\alpha}^*)) = \text{rank}(\sum_{e \in \mathcal{E}_i} T_{\alpha e}T_{\alpha e}^*) = |\mathcal{E}_i| = A(i,j)
\]
and so \( K_0(\iota) = A^t \). Continuity of \( K \)-theory (see e.g., [RLL00]) implies that
\[
K_0(\mathcal{F}_A) = K_0(\mathcal{F}_1 \to \mathcal{F}_2 \to \cdots) = \lim_{\to} (\mathbb{Z}^N \to \mathbb{Z}^N \to \cdots) =: \lim_{\to} \mathbb{Z}^N.
\]
We realize the inductive limit, \( \lim_{\to} \mathbb{Z}^N \), as the equivalence classes of sequences \((x_n)_n\) in \( \mathbb{Z}^N \) where \( x_{n+1} = \beta(x_n) \) for every \( n > n_0 \) and some \( n_0 \in \mathbb{N} \); two sequences are identified if they differ at finitely many points. The fact that \( K_1(\mathcal{F}_A) \) is trivial follows from the fact that \( \mathcal{F}_A \) is an AF algebra.

Via computations similar to the proof above, one can show that \( K_0(\beta_n) = I_N \) independent of \( n \in \mathbb{N} \). Let us now move one to show that \( \mathcal{O}_A \) is a corner in a crossed product by \( \mathbb{Z} \) via an appropriate extension \( \bar{\beta} \) of \( \beta \).

Consider the stationary inductive system
\[
\mathcal{F}_A \xrightarrow{\beta} \mathcal{F}_A \xrightarrow{\beta} \mathcal{F}_A \xrightarrow{\beta} \cdots \xrightarrow{\beta} \bar{\mathcal{F}}_A.
\]
and let \( \beta_\infty : \mathcal{F}_A \to \bar{\mathcal{F}}_A \) be the canonical map from the \( n \)’th term in the inductive system. This is injective, since \( \beta \) is injective and \( \bar{\mathcal{F}}_A \) has no unit, since \( \beta \) is non-unital. The inductive limit, \( \bar{\mathcal{F}}_A \), is realized as the equivalence classes of sequences \((a_n)_n\) in \( \mathbb{Z}^N \) where \( a_{n+1} = \beta(a_n) \) for every \( n > n_0 \) and some \( n_0 \in \mathbb{N} \); two sequences are identified if they differ at finitely many coordinates. The morphism of inductive systems depicted in the diagram
\[
\begin{array}{ccc}
\mathcal{F}_A & \xrightarrow{\beta} & \mathcal{F}_A \\
\downarrow{id} & & \downarrow{id} \\
\mathcal{F}_A & \xrightarrow{\beta} & \mathcal{F}_A \\
\downarrow{id} & & \downarrow{id} \\
\mathcal{F}_A & \xrightarrow{\beta} & \mathcal{F}_A \\
\downarrow{id} & & \downarrow{id} \\
\mathcal{F}_A & \xrightarrow{\beta} & \mathcal{F}_A \\
\downarrow{\beta} & & \downarrow{\bar{\beta}} \\
\bar{\mathcal{F}}_A & & \bar{\mathcal{F}}_A
\end{array}
\]
induces a well-defined injective map, \( \bar{\beta} \), in the limit. This takes the class of the sequence \((\ldots, a, \beta(a), \ldots)\) with \( a \in \mathcal{F}_A \) in the \( n \)’th coordinate to the class of \((\ldots, a, \beta(a), \ldots)\). That is, \( \bar{\beta} \) is the shift to the left. Equivalently, \( \bar{\beta} \) applies \( \beta \) to each coordinate in the sequence. This is an automorphism: If \( \bar{a} = (\ldots, a, \beta(a), \ldots) \), then the class of \((\ldots, a, \beta(a), \ldots)\) is mapped to the class of \( \bar{a} \) via \( \bar{\beta} \).

**Lemma 2.3.3.** Let \( \varphi : \mathcal{B} \to \mathcal{B} \) be an endomorphism of a \( C^* \)-algebra \( \mathcal{B} \) and let \( \mathcal{B}_\varphi \) be the inductive limit of the stationary inductive system \( \mathcal{B} \xrightarrow{\varphi} \mathcal{B} \xrightarrow{\varphi} \cdots \). If \( K_*(\varphi) \) is an isomorphism, then the inclusion \( \mathcal{B} \to \mathcal{B}_\varphi \) induces an isomorphism \( K_*(\mathcal{B}) \cong K_*(\mathcal{B}_\varphi) \).

**Proof.** There is an induced inductive system
\[
\begin{array}{cccccc}
K_*(B) & \xrightarrow{K_*(\varphi)} & K_*(B) & \xrightarrow{K_*(\varphi)} & K_*(B) & \xrightarrow{\cdots} & K_*(B) \\
\end{array}
\]

in the category of abelian groups. The homomorphism \( j: K_*(B) \rightarrow K_*(B_{\varphi}) \) given by mapping \( b \in K_*(B) \) to the class of \( (b, K_*(\varphi)(b), \ldots) \) is induced by the inclusion \( B \rightarrow B_{\varphi} \). This is injective since \( K_*(\varphi) \) is injective. It is also surjective: Let \( b = (\ldots, b, K_*(\varphi)(b), \ldots) \), for some \( b \in K_*(B) \), be a sequence in \( K_*(B) \) and put \( b' = K_*(\varphi)^{-n}(b) \). Then

\[
j(b') = (b', \ldots, K_*(\varphi)^{n}(b'), \ldots) = (b', \ldots, b, K_*(\varphi)(b), \ldots)
\]

which is identified with \( \bar{b} \) in \( K_*(B_{\varphi}) \).

\[\square\]

**Lemma 2.3.4.** The endomorphism \( \beta: \mathcal{F}_A \rightarrow \mathcal{F}_A \) induces an isomorphism on \( K \)-theory. Consequently, there is an isomorphism \( K_*(\mathcal{F}_A) \cong K_*(\bar{\mathcal{F}}_A) \).

**Proof.** It is clear that \( K_1(\beta) = 0 \) since \( \mathcal{F}_A \) is an AF algebra and \( K_1 \) is a functor. Via the above lemma, it thus suffices to show that \( K_0(\beta): \lim_{A^l} \mathbb{Z}^N \rightarrow \lim_{A^l} \mathbb{Z}^N \) is an isomorphism.

We remarked earlier that \( K_0(\beta_n) = I_N: \mathbb{Z}^N \rightarrow \mathbb{Z}^N \). The diagram (2.16) induces a commutative diagram of abelian groups

\[
\begin{array}{cccccc}
\mathbb{Z}^N & \xrightarrow{I_N} & \mathbb{Z}^N & \xrightarrow{I_N} & \mathbb{Z}^N & \xrightarrow{\cdots} & \lim_{A^l} \mathbb{Z}^N \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}^N & \xrightarrow{I_N} & \mathbb{Z}^N & \xrightarrow{I_N} & \mathbb{Z}^N & \xrightarrow{\cdots} & \lim_{A^l} \mathbb{Z}^N \\
\end{array}
\]

in which the horizontal maps are given by \( A^l \). This shows that \( K_0(\beta) \) is the shift to the right. This is an isomorphism; an explicit inverse is given as the shift to the left, \( \sigma^r \).

\[\square\]

We record the following general fact about C*-algebras. Recall that a projection \( p \) in a C*-algebra \( B \) is full if it is not contained in any proper ideal of \( B \).

**Theorem 2.3.5.** Let \( p \) be a full projection in a unital C*-algebra \( B \). Then \( B \otimes \mathbb{K} \cong pBP \otimes \mathbb{K} \). Here, \( \mathbb{K} \) is the compact operators on some separable infinite dimensional Hilbert space.

The above result is due to Brown, cf. [Bro77].

**Theorem 2.3.6.** Let \( \mathcal{F}_A \) and \( \bar{\beta} \) be as above. Then \( \mathcal{O}_A \otimes \mathbb{K} \cong (\mathcal{F}_A \rtimes_{\beta} \mathbb{Z}) \otimes \mathbb{K} \).

**Proof.** We wish to apply Brown’s theorem, so the first objective is to show that \( \beta_1^\infty(1_A) \) is a full projection in \( \mathcal{F}_A \). Since \( \mathcal{F}_A \) is the inductive limit of a stationary inductive system and \( \beta \) is injective, it suffices to show that \( \beta(1_A) \) is full in \( \mathcal{F}_A \). Observe from (2.15) that \( \beta(1_A) = \sum_{i=1}^N (|\mathcal{E}_i|^{-1} \sum_{e \in \mathcal{F}_A} T_i T_j^* \sigma^r) \) in which each term is non-zero since \( A \) is irreducible. Let \( I \prec \mathcal{F}_A \) be an ideal and suppose \( \beta(1_A) \in I \). Since \( \mathcal{F}_A = \bigcup_{n=1}^\infty \mathcal{F}_n \), the ideal \( I \) is of the form

\[
I = \bigcup_{n=1}^\infty (F_n \cap I) = \bigcup_{n=1}^\infty \left( \bigoplus_{i=1}^N (\mathcal{F}_n \cap I) \right).
\]
Now, by the above observation, $\mathcal{F}_n^i \cap I \neq \emptyset$ for every $i \in \mathfrak{A}$ and $n \in \mathbb{N}$. Since $\mathcal{F}_n^i$ is simple (it is a matrix algebra), this implies that $\mathcal{F}_n^i \cap I = \mathcal{F}_n^i$. Hence $I = \bigcup_{n=1}^{\infty} \mathcal{F}_n = \mathcal{F}_A$ and so $\beta(1_A)$ is a full projection in $\mathcal{F}_A$. Hence $p := \beta_1^\infty(1_A)$ is a full projection in $\mathcal{F}_A$.

Let $U$ be the unitary implementing the action $\beta$; that is, $U a U^* = \beta(a)$, for every $a \in \mathcal{F}_A$. Then $\mathcal{F}_A \rtimes_\beta \mathbb{Z} \cong C^*(\mathcal{F}_A, U)$. Note that $\beta_1^\infty(\mathcal{F}_A)$ is a copy of $\mathcal{F}_A$ inside $\mathcal{F}_A$ in which $p = \beta_1^\infty(1_A)$ plays the rôle of the identity. Moreover,

\[ p(U^* p U) p = p \beta^{-1}_1(p) p = p \]

and

\[ p(U p \beta_1^\infty(x) p U^*) p = p \beta_1^\infty(\beta_1^\infty(x)) p = p \beta_1^\infty(\beta(x)) p = \beta_1^\infty(x), \]

for every $x \in \mathcal{F}_A$. Hence there is a well-defined *-homomorphism $C^*(\mathcal{F}_A, T) \to C^*(\mathcal{F}_A, U)$ defined by

\[ x \mapsto \beta_1^\infty(x), \quad T \mapsto p U p, \]

for $x \in \mathcal{F}_A$. Since $p \mathcal{F}_A \mathcal{P} = \beta_1^\infty(\mathcal{F}_A)$, this shows that $p C^*(\mathcal{F}_A, U) p$ is a copy of $C^*(\mathcal{F}_A, T) \cong \mathcal{O}_A$ inside $C^*(\mathcal{F}_A, U)$. The conclusion follows from Brown’s theorem.

We are now ready to compute the $K$-groups of the Cuntz-Krieger algebra. We let $\hat{\sigma} = K_0(\beta)$ denote the shift to the left on $\lim_{\mathcal{A}^i} \mathbb{Z}^N$. The Pimsner-Voiculescu exact sequence (see e.g., Theorem 2.4 in [PV80]) associated to the crossed product $\mathcal{F}_A \rtimes_\beta \mathbb{Z}$ takes the form

\[
\begin{array}{ccccccc}
K_0(\mathcal{F}_A) & \xrightarrow{id - K_0(\beta)} & K_0(\mathcal{F}_A) & \xrightarrow{} & K_0(\mathcal{F}_A \rtimes_\beta \mathbb{Z}) \\
\downarrow & & \downarrow & & \\
K_1(\mathcal{F}_A \rtimes_\beta \mathbb{Z}) & \leftarrow & K_1(\mathcal{F}_A) & \xrightarrow{id - K_1(\beta)} & K_1(\mathcal{F}_A).
\end{array}
\]

By the work we have done hitherto, this reduces to the exact sequence

\[ 0 \to K_1(\mathcal{O}_A) \to \lim_{\mathcal{A}^i} \mathbb{Z}^N \xrightarrow{id - \hat{\sigma}} \lim_{\mathcal{A}^i} \mathbb{Z}^N \to K_0(\mathcal{O}_A) \to 0 \]

from which it follows that

\[ K_0(\mathcal{O}_A) \cong \lim_{\mathcal{A}^i} \mathbb{Z}^N / (id - \hat{\sigma}) \lim_{\mathcal{A}^i} \mathbb{Z}^N, \quad K_1(\mathcal{O}_A) \cong \ker(id - \hat{\sigma}) \subseteq \lim_{\mathcal{A}^i} \mathbb{Z}^N. \quad (2.18) \]

**Lemma 2.3.7.** Define a map $j: \mathbb{Z}^N \to \lim_{\mathcal{A}^i} \mathbb{Z}^N$ given by $j(x) = (x, A^i x, (A^i)^2 x, \ldots)$ for every $x \in \mathbb{Z}^N$. Then every element in $\lim_{\mathcal{A}^i} \mathbb{Z}^N$ is equivalent to an element in $j(\mathbb{Z}^N)$ modulo $(id - \hat{\sigma}) \lim_{\mathcal{A}^i} \mathbb{Z}^N$. Hence

\[ K_0(\mathcal{O}_A) \cong \frac{j(\mathbb{Z}^N)}{(id - \hat{\sigma}) \lim_{\mathcal{A}^i} \mathbb{Z}^N}. \quad (2.19) \]
Proof. Note first that \((\mathrm{id} - \hat{\sigma}) \lim_{A^t} \mathbb{Z}^N \subseteq (\mathrm{id} - \sigma) \lim_{A^t} \mathbb{Z}^N\) for every \(k \in \mathbb{N}\). Indeed, for \(k = 2\) we have
\[
(x, y - (A')^2x, A'x - (A')^3x, \ldots) = (x - A'x, A'x - (A')^2x, \ldots)
\] and this argument generalizes to any \(k \in \mathbb{N}\). Now suppose an element in \(\lim_{A^t} \mathbb{Z}^N\) is represented by the sequence \(\mathbf{x} = (\cdots, x_1, A'x, \ldots)\), for some \(x \in \mathbb{Z}^N\) in the \((k + 1)\)st coordinate. Then
\[
\hat{x} - j(x) = (\cdots, x - (A')^k x, (A')^k x - (A')^{k+1} x, \ldots) \in (\mathrm{id} - \hat{\sigma}) \lim_{A^t} \mathbb{Z}^N
\] and this shows the first assertion of the lemma. The second now follows directly from (2.18).

\[\text{Theorem 2.3.8.} \quad \text{Consider the Cuntz-Krieger algebra } \mathcal{O}_A \text{ determined by the } N \times N \text{-matrix } A. \text{ Then}
\]
\[K_0(\mathcal{O}_A) \cong \frac{\mathbb{Z}^N}{(I_N - A') \mathbb{Z}^N}, \quad K_1(\mathcal{O}_A) \cong \ker(I_N - A') \subseteq \mathbb{Z}^N.
\]

If \(e_i\) denotes the canonical basis of \(\mathbb{Z}^N\), then the above isomorphism takes the class of the range projection of \(S_i\) to the class of \(e_i\) in the quotient group. In particular, it takes the class of \(1_A\) to the class of \((1, \ldots, 1)\).

Proof. Let \(j\) and \(\hat{\sigma}\) be as above and consider the diagram
\[
\begin{array}{ccc}
\mathbb{Z}^N & \xrightarrow{j} & \mathbb{Z}^N \\
\| & \downarrow \hat{\sigma} & \downarrow j \\
\lim_{A^t} \mathbb{Z}^N & \xrightarrow{(\mathrm{id} - \hat{\sigma})} & \lim_{A^t} \mathbb{Z}^N.
\end{array}
\]

Given \(x \in \mathbb{Z}^N\) the sequence \(j(x - A'x) = (x - A'x, A'x - (A')^2x, \ldots)\) coincides with the element \((\mathrm{id} - \hat{\sigma})(x, A'x, \ldots) = (x - A'x, A'x - (A')^2x, \ldots)\) and so the diagram commutes. Now, suppose \(j(x) \in (\mathrm{id} - \hat{\sigma}) \lim_{A^t} \mathbb{Z}^N\) for some \(x \in \mathbb{Z}^N\). That is,
\[
j(x) = (x_1, (A')^2 x_2, \ldots) = (\cdots, y - A^i y, A^i y - (A^i)^2 y, \ldots),
\]
so \((A^i)^k x = y - A^i y = (I_N - A^i) y\), for some \(y \in \mathbb{Z}^N\) and \(k \in \mathbb{N}\). It follows that \(x \in (I_N - A^i) \mathbb{Z}^N\) since
\[
x = (I_N - A^i)(y + (A^i)^{k-1} x + \cdots + A^i x + x).
\]
Hence if \(j(x) \in (\mathrm{id} - \hat{\sigma}) \lim_{A^t} \mathbb{Z}^N\) then \(x\) was already in \((I_N - A^i) \mathbb{Z}^N\). With (2.19) in mind, we now see that
\[
K_0(\mathcal{O}_A) \cong \frac{\mathbb{Z}^N}{(I_N - A') \mathbb{Z}^N}.
\]
Finally, suppose \((\text{id} - \sigma)\tilde{x} = 0\), for some \(\tilde{x} \in \lim_{\rightarrow} \mathbb{Z}^N\), then \(\tilde{x}\) is represented by a constant sequence \((x, x, \ldots)\) with \(x \in \ker(I_N - A^t) \subseteq \mathbb{Z}^N\). From (2.18) we conclude that \(K_1(O_A) \cong \ker(I_N - A^t)\). \qed
Étale groupoids

3.1 What is a groupoid?

There is a concise category-theoretic answer:

A groupoid is a small category with inverses.

Even though this is a valid and relevant definition, we shall spend most of this section diving in to a more algebraic description of these objects.

Definition 3.1.1. A groupoid is a pair \((G,X)\) where \(G\) is the set of morphisms and \(X \subseteq G\) is a distinguished subset of objects (called the unit space) together with structure maps \(r,s \colon G \to X\) assigning the range and source of each morphism. The collection of composable morphisms is the set \(G^{(2)} = \{ (\gamma, \gamma') \in G \times G \mid s(\gamma) = r(\gamma') \} \subseteq G \times G\). Composition \(G^{(2)} \to G\) is associative and denoted by \((\gamma, \gamma') \mapsto \gamma \gamma' \in G\). Furthermore, inversion is a bijection \(-1 : G \to G\). This data is subject to the following conditions:

1. \(s(\gamma \gamma') = s(\gamma')\) and \(r(\gamma \gamma') = r(\gamma)\),
2. \(s(x) = r(x) = x\),
3. \(\gamma s(\gamma) = \gamma = r(\gamma) \gamma\),
4. \(\gamma^{-1} = r(\gamma)\) and \(\gamma^{-1} \gamma = s(\gamma)\),

for every \(\gamma, \gamma' \in G\) and \(x \in X\). Due to (4), the structure maps can be recovered from composition and inversion. A pair \((G', X') \subseteq (G, X)\) is a subgroupoid if \((G', X')\) is itself a groupoid such that composition and inversion agrees with \((G, X)\).

Remark 3.1.2. In the literature, the unit space is often denoted \(G^{(0)}\) and we shall also make use of this terminology when convenient. In most cases, however, we shall honour the unit space by explicitly mentioning it.

For each unit \(x \in X\), we define the fibers

\[
G^x = r^{-1}(\{x\}) = \{ \gamma \in G \mid r(\gamma) = x \},
\]

\[
G_x = s^{-1}(\{x\}) = \{ \gamma \in G \mid s(\gamma) = x \}.
\]
3. Étale groupoids

Their intersection naturally has a group structure and we denote this by $\Omega(X, x)$, the loop group or isotropy group. We shall refer to the union

$$\Omega X = \coprod_{x \in X} \Omega(X, x) \subseteq G$$

as the loop bundle or isotropy bundle.

**Example 3.1.3.** Let $(G, X)$ be a groupoid. Given $x \in X$, we can consider the loop group $\Omega(X, x)$ to be a subgroupoid of $(G, X)$ whose unit space is the singleton $\{x\}$. In fact, any group $\Gamma$ with identity element $e$ can be interpreted as the groupoid $(\Gamma, \{e\})$.

A topological groupoid is a groupoid endowed with a topology such that composition and inversion is continuous. This entails continuity of the structure maps as well. The unit space is given the subspace topology while $G^{(2)}$ inherits the topology from the product topology on $G \times G$.

**Definition 3.1.4.** A groupoid $(G, X)$ is principal if $X = \Omega X$. If $(G, X)$ is topological and $X = \text{Int}(\Omega X)$, then $(G, X)$ is said to be essentially principal.

**Example 3.1.5.** Let $R \subseteq X \times X$ be an equivalence relation on a set $X$. The pair $(R, X)$ is naturally a groupoid in which any loop $(x, x) \in R$ is identified with the unit $x \in X$. Hence any equivalence relation can be interpreted as a principal groupoid. Conversely, if $(G, X)$ is a principal groupoid, then we may consider $G$ to be an equivalence relation on the set $X$ by declaring $x \sim y$ if and only if $(x, y) \in G^{(2)}$.

**Example 3.1.6.** Another (somewhat degenerate) example of a groupoid is given by the pair $(X, X)$ where $X$ is some topological space. We shall refer to such groupoids as trivial. If one wants, one can consider trivial groupoids to be given by the equivalence relation on $X$ which is equality.

We are interested in more topological structure. In particular, we shall focus on groupoids which are both locally compact and Hausdorff, we abbreviate this by LCH. Unfortunately, not all locally compact groupoids are Hausdorff, see e.g. [Pat99], so we must be careful with the definition.

**Definition 3.1.7.** A topological groupoid $(G, X)$ is locally compact if

1. The unit space $X$ is LCH in $G$,
2. There exists a countable basis for the topology on $G$ consisting of relatively compact sets,
3. The fibers $G^x$ and $G_x$ are LCH in $G$.

It is implicit in the definition of locally compact groupoids that they are second countable as topological spaces. In particular, the fibers $G^x$ and $G_x$ have countable bases. Some authors include the existence of a left Haar system in the definition of locally compact groupoids, see e.g. [Pat99]. Though we will not do the same, we shall consider the notion of Haar systems on LCH groupoids in Appendix B.

**Definition 3.1.8.** A topological groupoid is said to be étale if the structure maps $r$ and $s$ are local homeomorphisms.
Example 3.1.9. Let \( R \) be an equivalence relation on \( X \) and suppose \( R \) is endowed with a locally compact and Hausdorff topology such that the product operation is continuous and the inversion is a homeomorphism. Then \( (R, X) \) is an LCH principal groupoid (cf. Example 3.1.5). In particular, the equivalence relation is étale when the range map \( r: R \to X \) is a local homeomorphism. Furthermore, if \( \Delta \subseteq X \times X \) is the diagonal and \( X \setminus \Delta \) is compact, then \( (R, X) \) is said to be a compact étale equivalence relation (CEER).

It is known (see Lemma 3.4 in [GPS04]) that any CEER \( R \) on a totally disconnected space \( X \) is the finite disjoint union of CEERs \( (R_i, X_i) \), where \( X_i \) is (homeomorphic to) the product \( Y_i \times \{1, \ldots, m_i\} \) (for some \( m_i \in \mathbb{N} \)) and \( R_i \) is the trivial equivalence relation on \( \{1, \ldots, m_i\} \) (all points are identified) and the cotrivial equivalence relation (the equality relation) on \( Y_i \). In addition, each \( X_i \) is \( R \)-invariant in the sense that if \( x \in X_i \), then the whole equivalence class \([x]_R\) is in \( X_i \). We shall make use of this fact in Proposition 4.2.7 in our discussion of Kakutani equivalence.

A subset \( U \subseteq G \) is called an \textbf{r-section} (resp. \textbf{s-section}) when \( r \) (resp. \( s \)) restricts to a homeomorphism on \( U \). A subset \( S \subseteq G \) which is both an \( r \)-section and \( s \)-section is said to be a \textbf{bisection}. We let \( \mathcal{S} = \mathcal{S}_G \) be the collection of all open bisections of \( G \). If \( \gamma \) is a morphism in an étale groupoid \( (G, X) \), then there are open neighborhoods \( U \) and \( V \) of \( \gamma \) on which \( r \) and \( s \) restrict to homeomorphisms. Their intersection is an open bisection and so the elements of \( \mathcal{S} \) cover \( G \). Observe that if \( S, T \in \mathcal{S} \) then

\[
ST = \{ \gamma \gamma' \mid (\gamma, \gamma') \in (S \times T) \cap G^{(2)} \}
\]

(3.1)
is again an open bisection and this defines an associative product on \( \mathcal{S} \). Furthermore,

\[
T = S^{-1} = \{ \gamma^{-1} \mid \gamma \in S \}
\]
is an open bisection and \( STS = S, TST = T \) and \( (ST)^{-1} = T^{-1}S^{-1} \). Any \( S \in \mathcal{S} \) defines a homeomorphism \( \zeta_S: s(S) \to r(S) \) given by

\[
\zeta_S(x) = r(\gamma_x),
\]

(3.2)
for all \( x \in s(S) \). Here, \( \gamma_x \) is the unique morphism in \( S \) with \( s(\gamma_x) = x \). We shall refer to this map as the \textbf{canonical action} of \( \mathcal{S} \) on \( X \). We shall elaborate more on this in Section 3.3.

Definition 3.1.10. A \textbf{groupoid homomorphism} between topological groupoids is a \textit{continuous} map \( \varphi: (G, X) \to (H, Y) \) satisfying

\[
(\varphi(\gamma_1), \varphi(\gamma_2)) \in H^{(2)} \quad \text{and} \quad \varphi(\gamma_1) \varphi(\gamma_2) = \varphi(\gamma_1 \gamma_2),
\]

whenever \( (\gamma_1, \gamma_2) \in G^{(2)} \). A groupoid homomorphism is \textbf{étale} when it is a local homeomorphism. We let \( \text{Hom}(G, H) \) denote the collection of all groupoid homomorphisms \( G \to H \).

Any groupoid homomorphism respects the unit space in the sense that \( \varphi(X) \subseteq Y \). Therefore, we shall sometimes suppress the unit space in the notation. Two groupoids \( G \) and \( H \) are \textbf{isomorphic} if there are homomorphisms \( G \to H \) and \( H \to G \) which compose (both ways) to the identity in which case we write \( G \cong H \).
3. Étale groupoids

Example 3.1.11. Suppose \((G, X)\) is a groupoid and \(\Gamma\) is a countable discrete group (written multiplicatively). A groupoid homomorphism \(\rho: G \to \Gamma\) gives rise to the skew product \(G \times_{\rho} \Gamma\). This has a groupoid structure in the following way: As a set \(G \times_{\rho} \Gamma\) is the cartesian product \(G \times \Gamma\) with unit space \(X \times \Gamma\). Two pairs \((\gamma, g), (\gamma', g') \in G \times_{\rho} \Gamma\) are composable if and only if \((\gamma, \gamma') \in G^{(2)}\) and \(gp(\gamma) = g'\) in which case

\[(\gamma, g)(\gamma', g\rho) = (\gamma', g)\]

Inversion is given as \((\gamma, g)^{-1} = (\gamma^{-1}, g\rho(\gamma))\). It follows that

\[r(\gamma, g) = (r(\gamma), g), \quad s(\gamma, g) = (s(\gamma), g\rho(\gamma))\]

for every \((\gamma, g) \in G \times_{\rho} \mathbb{Z}\). Note that \(G \times_{\rho} \mathbb{Z}\) is étale whenever \(G\) is.

We end this section with some immediate observations regarding étale groupoids.

Lemma 3.1.12. If \((G, X)\) is a Hausdorff topological groupoid, then \(X\) is closed in \(G\) and \(G^{(2)}\) is closed in \(G \times G\). When \((G, X)\) is étale, \(X\) is also open.

Proof. Consider the continuous map \(\bar{r}: G \to G \times G\) given by \(\bar{r}(\gamma) = (\gamma, \gamma^{-1}) = (\gamma, r(\gamma))\). As \(G\) is Hausdorff, the diagonal \(\Delta = \{(\gamma, \gamma) \mid \gamma \in G\}\) is closed in \(G \times G\). It follows that \(X = \bar{r}^{-1}(\Delta)\) is closed in \(G\). Similarly, \(\Delta_X = \{(x, x) \mid x \in X\}\) is closed in \(X \times X\) and \(r \times s : G \times G \to X \times X\) is continuous. Hence \(G^{(2)} = (r \times s)^{-1}(\Delta_X)\) is closed in \(G\).

Next, let \(x \in X\). Assuming that \((G, X)\) is étale, there exists an open s-section \(U \subseteq G\) of \(x\). In particular, \(X \cap U\) is open in \(X\). It follows that \(V = U \cap s^{-1}(X \cap U) \subseteq G\) is an open neighborhood of \(x\). We show that \(V \subseteq X\). If \(\gamma \in V\), then \(s(\gamma) \in U\) and \(s(\gamma) = s(s(\gamma))\). As \(s\) is injective on \(U\), we conclude that \(\gamma = s(\gamma)\), so \(V \subseteq X\). Hence \(X\) is open in \(G\).

Lemma 3.1.13. Let \((G, X)\) be an étale LCH groupoid. Then the fibers \(G^x\) and \(G_x\) are discrete\(^1\) for every \(x \in X\).

Proof. Let \(\gamma \in G^x\) and take an open r-section \(U\) containing \(\gamma\). Then \(U \cap G^x = \{\gamma\}\) since \(r\) is injective on \(U\) and this is open in \(G^x\). As \(G^x\) has a countable basis, it follows that \(G^x\) is countable and thus discrete. Similarly, we find that \(G_x\) is discrete.

Definition 3.1.14. Let \((G, X)\) be a topological groupoid and let \(Y \subseteq X\) be an open subset. The reduction of \(G\) to \(Y\), written \(G|Y\), is the topological groupoid \(r^{-1}(Y) \cap s^{-1}(Y)\).

We shall study the reduction later. Note that \(G|Y\) is étale LCH whenever \((G, X)\) is.

3.2 The groupoid \(C^*\)-algebra

In order to construct \(C^*\)-algebras from (étale) groupoids, we first need to discuss the convolution algebra. In this section, we mainly follow the exposition in [BO08]. Let \((G, X)\) be a fixed étale LCH groupoid.

\(^1\)This is the reason why some authors refer to étale groupoids as \(r\)-discrete.
Definition 3.2.1. The convolution algebra $C_c(G)$ is the collection of complex-valued continuous maps $G \to \mathbb{C}$ with compact support. Given $f, g \in C_c(G)$, the convolution product

$$ (f \ast g)(\gamma) = \sum_{\beta \in G_x(\gamma)} f(\gamma \beta^{-1})g(\beta) \quad (3.3) $$

together with the *-involution $f^*(\gamma) = \overline{f(\gamma^{-1})}$ gives $C_c(G)$ the structure of a *-algebra.

The terms in the sum \[(3.3)\] vanish outside of a compact set and since the fibers $G_x$ are discrete, this means that there are at most finitely many non-zero terms. Hence the product is well-defined and $f \ast g \in C_c(G)$. A simple computation shows that the product is associative: If $f, g, h \in C_c(G)$, then

$$ f \ast (g \ast h)(\gamma) = \sum_{\beta, \alpha \in G_x(\gamma)} f(\gamma \beta^{-1})g(\beta \alpha^{-1})h(\alpha), $$

and

$$ (f \ast g) \ast h(\gamma) = \sum_{\alpha \in G_x(\gamma)} \sum_{\beta \in G_x(\gamma)} f(\alpha \beta^{-1})g(\alpha)h(\beta) = \sum_{\delta, \beta \in G_x(\gamma)} f(\gamma \delta^{-1})g(\delta \beta^{-1})h(\beta), $$

for every $\gamma \in G$. In the last equality, we used the substitution $\delta = \alpha \beta$. Note that

$$ f^* \ast g^*(\gamma) = \sum_{\alpha \in G_x(\gamma)} \overline{f(\alpha)}g(\alpha^*), \quad (3.4) $$

so that

$$ (g \ast f)^*(\gamma) = \sum_{\alpha \in G_x(\gamma^{-1})} \overline{f(\alpha)}g(\gamma^{-1} \alpha^{-1}) = \sum_{\alpha \in G_x(\gamma)} \overline{f(\alpha)}g^*(\alpha^*) = f^* \ast g^*(\gamma) \quad (3.5) $$

for every $\gamma \in G$. Hence $(g \ast f)^* = f^* \ast g^*$.

We may view $C_c(G)$ as a $C_0(X)$-module via the right action

$$ f.\xi(\gamma) = f(\gamma)\xi(s(\gamma)), $$

for $f \in C_c(G), \xi \in C_0(X)$ and $\gamma \in G$. Furthermore, there is a $C_0(X)$-valued inner product $\langle \cdot, \cdot \rangle: C_c(G) \times C_c(G) \to C_0(X)$ given by

$$ \langle f, g \rangle(x) = f^* \ast g(x) = \sum_{\beta \in G_x} \overline{f(\beta)}g(\beta), $$

where $f, g \in C_c(G)$ and $x \in X$. Note that $\langle f, g \rangle$ is just the restriction of the convolution $f^* \ast g$ to the unit space $X$. It is clear that $\langle \cdot, \cdot \rangle$ is linear in the second coordinate and conjugate linear in the first. Furthermore, $\langle f, f \rangle \geq 0$ for every $f \in C_c(G)$ with equality if and only if $f = 0$. The above computation \[(3.5)\] shows that $\langle f, g \rangle = (g, f)^*$. Finally, if $\xi \in C_0(X)$, then

$$ \langle f, g.\xi \rangle(x) = \sum_{\alpha \in G_x} \overline{f(\alpha)}g(\alpha)\xi(x) = \langle f, g \rangle(x)\xi(x) $$

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for every \( x \in X \). This shows that \( C_c(G) \) is, indeed, a pre-Hilbert module over \( C_0(X) \), (see e.g., [Lan95]).

The inner product induces the norm on \( C_c(G) \)
\[
||f||^2 = \langle f, f \rangle = \sup_{x \in X} \sum_{\beta \in G_x} |f(\beta)|^2.
\]

Let \( L^2G \) be the Hilbert \( C_0(X) \)-module completion of \( C_c(G) \) with respect to this norm. Define the left regular representation \( \lambda: C_c(G) \to \mathbb{B}(L^2G) \) by
\[
\lambda(f)\xi = f \star \xi,
\]
for \( f, \xi \in C_c(G) \). If \( x \in X \), then
\[
\sum_{\gamma \in G_x} |f \star \xi(\gamma)|^2 \leq \sum_{\gamma, \beta \in G_x} |f(\gamma^{-1})\beta| |\xi(\beta)|^2 \leq \left( \sum_{\gamma \in G_x} |f(\gamma)|^2 \right) \left( \sum_{\beta \in G_x} |\xi(\beta)|^2 \right)
\]
and therefore \( ||f \star \xi|| \leq ||f|| \cdot ||\xi|| \). In particular, \( ||\lambda(f)|| \leq ||f|| \). Furthermore, we see that
\[
\langle \lambda(f)\xi, \eta \rangle = (f \star \xi)^* \eta = \xi^* (f^* \star \eta) = \langle \xi, \lambda(f^*)\eta \rangle
\]
and so \( \lambda(f^*) = \lambda(f)^* \).

**Definition 3.2.2.** The reduced groupoid \( C^*-algebra \) \( C_r^*(G) \) of the étale LCH groupoid \( (G, X) \) is the norm-completion of \( \lambda(C_c(G)) \) in \( \mathbb{B}(L^2(G)) \).

**Remark 3.2.3.** The full groupoid \( C^*-algebra \) is the completion of \( C_c(G) \) with respect to the norm \( \sup \|\pi(f)\| \) where the supremum is taken over all (cyclic) \( \ast \)-representations of \( C_c(G) \) which are bounded on \( C_c(X) \). We shall not discuss the full groupoid \( C^*-algebra \) here.

We end this section with a small but useful observation.

**Lemma 3.2.4.** Suppose \( f \in C_c(G) \) has support on a bisection \( S \subseteq G \). Then \( ||f|| = ||f||_\infty \).

**Proof.** Note that
\[
||f||^2 = \sup_{x \in X} \sum_{\beta \in G_x} |f(\beta)|^2 = \sup_{x \in X} |f(\gamma_x)|,
\]
where \( \gamma_x \) is the unique morphism in \( S \) with \( s(\gamma_x) = x \). Hence \( ||f|| = \sup_{\gamma \in S} |f(\gamma)| = ||f||_\infty \). \( \square \)

### 3.3 Pseudogroups

In this section we shall see that the \( C^*-algebra \) construction of an essentially principal étale LCH groupoid is a complete invariant. We follow [Ren08] and formulate the result as a theorem.

**Theorem 3.3.1.** Let \( (G_1, X_1) \) and \( (G_2, X_2) \) be étale LCH groupoids which are essentially principal. Then
\[
(C_r^*(G_1), C_0(X_1)) \cong (C_r^*(G_2), C_0(X_2)) \iff G_1 \cong G_2.
\]
That is, the \( C^*-algebra \) construction of Section 3.2 is a complete invariant.
The right-to-left implication follows directly from the $C^*$-algebra construction, so let’s focus our energy on the other implication. We shall follow the strategy of J.Renault in [Ren08] which involves the notions of (Weyl) pseudogroups and groupoids (of germs).

An inverse semigroup is a semigroup $\mathcal{S}$ with unique inverses in the sense that for every $s \in \mathcal{S}$ there is a $t \in \mathcal{S}$ such that $STS = S$ and $TST = T$. It is customary to write $T = S^{-1}$ or $T = S^*$ according to the context. Our prime example of an inverse semigroup is $\mathcal{P}(X)$, the collection of partial homeomorphisms on a topological space $X$. A partial homeomorphism on $X$ is a homeomorphism $f : U \rightarrow V$ between open subsets $U, V \subseteq X$ and the composition of $f_i : U_i \rightarrow V_i$, $i = 1, 2$, is defined as

$$f_2 \circ f_1 : f_1^{-1}(U_2) \rightarrow f_2(V_1 \cap U_2).$$

The domain of $f \in \mathcal{S}$ is denoted $D(f)$. Any such partial homeomorphism has an obvious inverse and we allow the empty homeomorphism between empty sets so that the composition of any two partial homeomorphisms makes sense.

**Definition 3.3.2.** A pseudogroup $\mathcal{S}$ on a space $X$ is a sub-inverse semigroup of $\mathcal{P}(X)$.

**Example 3.3.3.** Let $(G, X)$ be an etale LCH groupoid and let $\mathcal{S}$ be the collection of open bisections of $G$. The observations made in and around (3.1) shows that $\mathcal{S}$ is an inverse semigroup. Recall that if $S \in \mathcal{S}$, then $\zeta_S : s(S) \rightarrow r(S)$ is given as $\zeta_S(x) = r(\gamma_x)$, where $\gamma_x \in S$ uniquely satisfies $s(\gamma_x) = x$, cf. (3.2). The canonical action $\zeta : \mathcal{S} \rightarrow \mathcal{P}(X)$ given by

$$\zeta(S) = \zeta_S,$$

for $S \in \mathcal{S}$, is an inverse semigroup homomorphism. In particular, the image $\zeta(\mathcal{S}) \subseteq \mathcal{P}(X)$ is a pseudogroup on $X$.

A partial homeomorphism $f \in \mathcal{P}(X)$ belongs locally to $\mathcal{S}$ if every $x \in \text{dom}(f)$ has an open neighborhood $U \subseteq X$ around $x$ such that $f|_U = \varphi|_U$, for some $\varphi \in \mathcal{S}$. The ample pseudogroup, $[\mathcal{S}] \subseteq \mathcal{P}(X)$, of $\mathcal{S}$ is the smallest pseudogroup containing $\mathcal{S}$ and every partial homeomorphism which belongs locally to $\mathcal{S}$. We say that $\varphi_i \in \mathcal{S}$, $i = 1, 2$, have the same germ at $x \in X$ if there is an open neighborhood $U \subseteq X$ around $x$ such that $\varphi_1|_U = \varphi_2|_U$. We shall identify the triples $(\varphi_1(x), \varphi_1, x)$ and $(\varphi_2(x), \varphi_2, x)$ if and only if $\varphi_1$ and $\varphi_2$ have the same germ at $x \in X$. This is an equivalence relation and we denote the class of $(\varphi(x), \varphi, x)$ by $[\varphi(x), \varphi, x]$.

**Definition 3.3.4.** The groupoid of germs of a pseudogroup $\mathcal{S} \subseteq \mathcal{P}(X)$ is

$$G(\mathcal{S}) = \{ [y, \varphi, x] \mid \varphi \in \mathcal{S}, x \in D(\varphi), y = \varphi(x) \}$$

with product and inverse given by

$$[z, \varphi_2, y][y, \varphi_1, x] = [z, \varphi_2 \circ \varphi_1, x], \quad [y, \varphi, x]^{-1} = [x, \varphi^{-1}, y].$$

The range and source maps are given as

$$r[y, \varphi, x] = y, \quad s[y, \varphi, x] = x.$$

The topology on $G(\mathcal{S})$ is generated by the basic open sets of the form

$$\mathcal{U}(V, \varphi, U) = \{ [y, \varphi, x] \in G(\mathcal{S}) \mid \varphi \in \mathcal{S}, x \in U, y \in V \},$$

where $U, V \subseteq X$ are open.
The above operations are well-defined with respect to the identifications made and $G(\mathcal{Y})$ depends only on the ample pseudogroup $[\mathcal{Y}]$. Note also that the basic open sets $\mathcal{Y}(V, \varphi, U)$ are open bisections of $G(\mathcal{Y})$. This ultimately follows from the assumption that $\varphi$ is well-defined and injective. The following notion of morphisms between pseudogroups was introduced by A. Haefliger in [Hae88].

**Definition 3.3.5.** Let $\mathcal{G}$ and $\mathcal{G}'$ be pseudogroups on spaces $X$ and $X'$, respectively. An **étale morphism of pseudogroups** $\Phi: \mathcal{G} \to \mathcal{G}'$ is a maximal collection of homeomorphisms of open subsets of $X$ to open subsets of $X'$ subject to the conditions that

1. Domains of elements of $\Phi$ cover $X$,
2. $h' \circ \varphi \circ h \in \Phi$,
3. $\varphi' \circ \varphi^{-1} \in \mathcal{G}'$,

for every $h \in \mathcal{G}$, $h' \in \mathcal{G}'$ and $\varphi, \varphi' \in \Phi$. Furthermore, we say that $\Phi$ is an **equivalence** when $\Phi^{-1} = \{ \varphi^{-1} \mid \varphi \in \Phi \}$ is also an étale morphism.

Suppose $\Phi: \mathcal{G} \to \mathcal{G}'$ is an étale morphism of pseudogroups on spaces $X$ and $X'$, respectively. Let $(y, h, x)$ be a representative of $[y, h, x] \in G(\mathcal{G})$. For every $x \in X$, there is $\varphi_x \in \Phi$ with $x \in D(\varphi_x)$, cf. (1) in the above definition. Hence $(\varphi_y(y), \varphi_y \circ h \circ \varphi^{-1}_x, \varphi_x(x))$ represents an element in $\mathcal{G}'$. In addition, if $h$ and $\bar{h}$ have the same germ at $x \in X$, then $y = \bar{h}(x)$ and $\varphi_y \circ h \circ \varphi^{-1}_x$ and $\varphi_y \circ \bar{h} \circ \varphi^{-1}_x$ have the same germ at $\varphi_x(x)$. This shows that $G(\neg)$ is a contravariant functor from the category of pseudogroups to the category of topological groupoids.

In particular, equivalent groupoids induce the same groupoid of germs (up to isomorphism).

Having defined the basic notions of this section, let us show some properties.

**Lemma 3.3.6.** Let $\mathcal{G} \subseteq \text{Part}(X)$ be a pseudogroup and let $G(\mathcal{G})$ be its groupoid of germs. Let $\mathcal{S}$ be the inverse semigroup of open bisections of $G(\mathcal{G})$. Then $\zeta: \mathcal{S} \to \text{Part}(X)$ is an isomorphism onto $[\mathcal{G}]$.

**Proof.** Let us first see that $\zeta$ takes values in $[\mathcal{G}]$. If $S$ is an open bisection of $G(\mathcal{G})$, then we may write $S = \bigcup_i \mathcal{Y}(\text{im}(\varphi_i), \varphi_i, D(\varphi_i))$ for some $\varphi_i \in \mathcal{G}$, for every $i$. Then $\zeta_S$ is a partial homeomorphism $\bigcup_i D(\varphi_i) \to \bigcup_i \text{im}(\varphi_i)$ and if $x \in D(\varphi_i)$ for some specific $i$, then $\zeta_S(x) = \varphi_i(x)$. This is well-defined since $S$ is assumed to be a bisection. This shows that $\zeta_S$ belongs locally to $\mathcal{G}$ and hence $\zeta_S \in [\mathcal{G}]$. On the other hand, if $\varphi \in [\mathcal{G}]$, then $S_\varphi = \mathcal{Y}(\text{im}(\varphi), \varphi, D(\varphi))$ is an open bisection of $G(\mathcal{G})$ (this makes sense since $\mathcal{G}$ and $[\mathcal{G}]$ define the same groupoid of germs).

The correspondences $S \mapsto \zeta_S$ and $\varphi \mapsto S_\varphi$ are inverses of each other. Indeed, suppose $S = \bigcup_i \mathcal{Y}(\text{im}(\varphi_i), \varphi_i, D(\varphi_i))$ is an open bisection of $G(\mathcal{G})$. For every $x \in D(\varphi_i)$, we have $[\zeta_S(x), \zeta_S, x] = [\varphi_i(x), \varphi_i, x]$ since $\zeta_S$ and $\varphi_i$ agree on the open set $D(\varphi_i)$. Hence $S_{\varphi_i} = S$. Conversely, let $\varphi \in [\mathcal{G}]$ and form $S_\varphi = \mathcal{Y}(\text{im}(\varphi), \varphi, D(\varphi))$. Then $\zeta_{S_\varphi}(x) = r[\varphi(x), \varphi, x] = \varphi(x)$, for every $x \in s(S_\varphi) = D(\varphi)$ so that $\zeta_{S_\varphi} = \varphi$. This proves the lemma.

**Lemma 3.3.7.** Let $(G, X)$ be an étale LCH groupoid and let $\mathcal{S}$ be the inverse semigroup of open bisections of $G$. The canonical action $\zeta: \mathcal{S} \to \text{Part}(X)$ is injective if and only if $(G, X)$ is essentially principal.
Proof. Given an open bisection \( S \) of \( G \), the partial homeomorphism \( \zeta_\mathcal{S} \) is an identity map if and only if \( S \subseteq \Omega X \). Assuming \((G,X)\) is essentially principal this implies that \( S \subseteq X \) since \( S \) is open. So if \( \zeta_\mathcal{S} = \zeta_T \) for \( S,T \in \mathcal{S} \), then \( \zeta_{ST^{-1}} = \text{id}_{\mathcal{S}(S)} \) so \( ST^{-1} \subseteq X \). Hence \( S = T \). Consequently, we must show that \( \text{Int}(\Omega X) \subseteq X \) so let \( \gamma \in \text{Int}(\Omega X) \). Choose an open bisection \( S \subseteq \text{Int}(\Omega X) \) around \( \gamma \) and note that \( \zeta_\mathcal{S} = \text{id}_{\mathcal{S}(S)} \). Assuming that \( \zeta \) is injective, this implies that \( S = s(S) \subseteq X \). In particular, \( \gamma \in X \).

The following corollary gives a characterization of groupoids of germs.

**Corollary 3.3.8.** Let \((G,X)\) be an étale LCH groupoid and let \( \mathcal{S} \) be the inverse semigroup of open bisections of \( G \). The following are equivalent:

1. \((G,X)\) is (isomorphic to) the groupoid of germs of some pseudogroup.
2. \((G,X)\) is (isomorphic to) the groupoid of germs of \( \zeta(\mathcal{S}) \).
3. \((G,X)\) is essentially principal.

*Proof.* (1)\(\Rightarrow\)(3): Suppose \((G,X) = G(\mathcal{S})\) is the groupoid of germs of a pseudogroup \( \mathcal{S} \) on \( X \). By Lemma 3.3.6, the canonical action \( \zeta : \mathcal{S} \to [\mathcal{S}] \) is injective and so \((G,X)\) is essentially principal by Lemma 3.3.7.

(3)\(\Rightarrow\)(2): Assume that \((G,X)\) is essentially principal. By Lemma 3.3.7, this implies that the canonical action \( \zeta : \mathcal{S} \to \mathcal{P}art(X) \) is injective. We shall identify \( \mathcal{S} \) with its image \( \mathcal{S} = \zeta(\mathcal{S}) \) in \( \mathcal{P}art(X) \) and let \( H = H(\mathcal{S}) \) be its groupoid of germs. Define a groupoid homomorphism \( H \to G \) by

\[ [y,S,x] \mapsto \gamma_x, \]

where \( \gamma_x \in G \) is the unique morphism in \( S \) with \( s(\gamma_x) = x \). Surjectivity of this map follows from the fact that \( G \) can be covered by open bisections. If \([y_1,S_1,x_1]\) and \([y_2,S_2,x_2]\) both map to \( \gamma_x \in G \), then \( x_1 = x = x_2 \) and \( S_1 \) and \( S_2 \) have the same germ at \( x_1 = x_2 \). Hence \([y_1,S_1,x_1]\) = \([y_2,S_2,x_2]\), so the map is injective. Hence \((G,X)\) is isomorphic to the groupoid of germs of the pseudogroup \( \mathcal{S} = \zeta(\mathcal{S}) \). The implication (2)\(\Rightarrow\)(1) is clear.

In the following, let \((\mathcal{A},\mathcal{B})\) be a pair of \( C^* \)-algebras in which \( \mathcal{B} \subseteq \mathcal{A} \) is an abelian \( C^* \)-subalgebra containing an approximate unit of \( \mathcal{A} \). Let \( X \) be the locally compact Hausdorff space with \( \mathcal{B} = C_0(X) \). We shall now see how to construct a pseudogroup on \( X \) out of this data.

**Definition 3.3.9.** The **normalizer** of a \( C^* \)-subalgebra \( \mathcal{B} \) in a \( C^* \)-algebra \( \mathcal{A} \) is

\[ N(\mathcal{A},\mathcal{B}) = \{ a \in \mathcal{A} \mid a\mathcal{B}a^* \subseteq \mathcal{B}, \hspace{1em} a^*\mathcal{B}a \subseteq \mathcal{B} \}. \]

We say that \( \mathcal{B} \) is **regular** in \( \mathcal{A} \) if \( N(\mathcal{A},\mathcal{B}) \) generates \( \mathcal{A} \) as a \( C^* \)-algebra.

The expert will note that our notation differs slighty from the one of A. Kumjian in [Kum86] as we have chosen to make explicit the ambient algebra. The normalizer \( N(\mathcal{A},\mathcal{B}) \) is norm-closed and closed under addition, multiplication and taking adjoints. Clearly \( \mathcal{B} \subseteq N(\mathcal{A},\mathcal{B}) \). When \( \mathcal{B} \) contains an approximate unit \((1_n)_n \) of \( \mathcal{A} \) and \( a \in N(\mathcal{A},\mathcal{B}) \), then

\[ a^*a = \lim_{n \to \infty} a^*1_n a, \hspace{1em} aa^* = \lim_{n \to \infty} a1_n a^* \]
are both in \( \mathcal{B} \). On p. 21, we encountered the normalizing partial isometries of a masa in the Cuntz-Krieger algebras.

For \( a \in N(\mathcal{A}, \mathcal{B}) \), we shall consider the open subset (the domain of \( a \))

\[
d(a) = \{ x \in X \mid a^*a(x) > 0 \} \subseteq X
\]

(3.6)

together with the ideal \( C_0(d(a)) \subseteq C_0(X) \).

**Proposition 3.3.10.** Let \( a \in N(\mathcal{A}, \mathcal{B}) \). There is a unique homeomorphism \( \zeta_a : d(a) \rightarrow d(a^*) \) such that for every \( f \in C_0(X) \),

\[
a^*fa(x) = f(\zeta_a(x))a^*a(x),
\]

(3.7)

for \( x \in d(a) \).

**Proof.** Let \( a = v|a| \) be the polar decomposition in \( \mathcal{A}^* \), the double dual of \( \mathcal{A} \). Here, \(|a| = (a^*a)^{1/2} \). Any \( f \in C_0(d(a)) \) can be approximated by linear combinations of functions of the form \( |a|g|a| \) with \( g \geq 0 \) and \( g \in \mathcal{B} \). We have

\[
vfv^* = v|a|g|a|v^* = aga^* \in \mathcal{B},
\]

since \( g \in \mathcal{B} \) and \( a \in N(\mathcal{A}, \mathcal{B}) \). Furthermore, \( aga^* \leq ||g||aga^* \in C_0(d(a^*)) \) since \( g \geq 0 \). Hence \( \text{Ad}(v)(f) = aga^* \in C_0(d(a^*)) \). Similarly, one can show that \( \text{Ad}(v^*)(f) \in C_0(d(a)) \) for every \( f \in C_0(d(a^*)) \). Hence \( \text{Ad}(v) : C_0(d(a)) \rightarrow C_0(d(a^*)) \) is a \( C^* \)-isomorphism. We will let \( \zeta_a : d(a) \rightarrow d(a^*) \) be the unique homeomorphism induced by \( \text{Ad}(v) \). Then \( f(\zeta_a(x)) = \text{Ad}(v^*)(f)(x) \) for \( x \in d(a) \) and \( f \in C_0(d(a^*)) \). As \( a^*a \) is strictly positive on \( d(a) \), it follows that

\[
f(\zeta_a(x)) = \frac{a^*a(x)}{a^*a(x)}v^*fv(x) = \frac{|a|v^*fv|a|(x)}{a^*a(x)} = \frac{a^*fa(x)}{a^*a(x)},
\]

for every \( x \in d(a) \).

We have the following important corollary.

**Corollary 3.3.11.** If \( f \in \mathcal{B} \), then \( C_0(d(f)) = C_0(d(f^*)) \) and \( \zeta_f = \text{id}_{d(f)} \). Furthermore, if \( a, b \in N(\mathcal{A}, \mathcal{B}) \) then \( \zeta_{ab} = \zeta_a \circ \zeta_b \).

**Proof.** The first two assertions follow from the fact that \( d(f) = d(f^*) \) and the computation

\[
f^*(x)g(x)f(x) = g(x)|f(x)|^2.
\]

If \( f \in \mathcal{B} \), then for every \( x \in d(ab) \)

\[
f(\zeta_a(\zeta_b(x)))b^*(a^*a)b(x) = f(\zeta_a(\zeta_b(x)))a^*a(\zeta_b(x))b^*b(x)
\]

\[
= a^*fa(\zeta_b(x))b^*b(x)
\]

\[
= b^*a^*f(ab)(x)
\]

\[
= (ab)^*f(ab)(x)
\]

by several applications of (3.7). By uniqueness, we conclude that \( \zeta_{ab} = \zeta_a \circ \zeta_b \).
Definition 3.3.12. Inspired by Corollary 3.3.11 we shall refer to
\[ \mathcal{G}(A, B) = \{ \zeta_a \mid a \in N(A, B) \} \] (3.8)
as the Weyl pseudogroup on \( X \) of the pair \((A, B)\). The Weyl groupoid, \( G(A, B) \), is the groupoid of germs of \( \mathcal{G}(A, B) \).

It should be clear that the construction of the Weyl pseudogroup constitutes an invariant of isomorphism classes of pairs \((A, B)\). Indeed, let \( \Psi : (A, B) \rightarrow (A', B') \) be a \( C^* \)-isomorphism and write \( B = C_0(X) \) and \( B' = C_0(X') \). Then there is a unique homeomorphism \( \tau : X \rightarrow X' \) such that \( \Psi(f) = f \circ \tau^{-1} \) for \( f \in C_0(X) \). Fix \( a \in N(A, B) \). The image \( b = \Psi(a) \) is in \( N(A', B') \) and \( b^*b = \Psi(a^*a) = a^*a \circ \tau^{-1} \). Furthermore,
\[ \tau(d_X(a)) = \{ y \in X' \mid a^*a \circ \tau^{-1}(y) > 0 \} = \{ y \in X' \mid b^*b(y) > 0 \} = d_{X'}(b). \]

We may now construct an \( \mathcal{G} \) morphism
\[ \Phi : \mathcal{G}(A, B) \rightarrow \mathcal{G}(A', B') \]
by putting \( \Phi = \{ \tau|_{d_X(a)} \mid a \in N(A, B) \} \). This is clearly an equivalence.

Lemma 3.3.13. Let \((G, X)\) be an \( \mathcal{G} \) groupoid. Then \( C_0(X) \) is an abelian \( C^* \)-subalgebra of \( C^*_e(G) \) which contains an approximate unit of \( C^*_e(G) \).

Proof. Since \( X \subseteq G \) is locally compact and second countable, it is \( \sigma \) compact. Let \((K_i)_i\) be an increasing sequence of compact subsets exhausting \( X \). Choose a sequence \((h_i)_i\) in \( C_c(X) \) with \( \text{supp}(h_i) = K_i \) such that \( h_i(x) = 1 \) when \( x \in K_i \). For every \( f \in C_c(G) \) we see that
\[ f(h_i(\gamma)) = f(\gamma)h_i(s(\gamma)) \rightarrow f(\gamma) \]
for every \( \gamma \) as \( i \rightarrow \infty \).

Because of the above lemmas, it makes sense for us to consider the Weyl pseudogroup (and groupoid) of the pair \((A, B) = (C^*_e(G), C_0(X))\). However, as the next propositions show, we may also use the additional groupoid structure.

Given \( a \in C^*_e(G) \), we define its open support as \( \text{supp}'(a) = \{ \gamma \in G \mid a(\gamma) \neq 0 \} \).

Proposition 3.3.14. Let \((G, X)\) be an \( \mathcal{G} \) groupoid which is essentially principal. Put \( A = C^*_e(G) \) and \( B = C_0(X) \). Then the normalizer \( N(A, B) \) consists exactly of the elements of \( A \) whose open support is a bisection. Furthermore, \( \zeta_a = \zeta_{\text{supp}'(a)} \) when \( a \in N(A, B) \).

Proof. Let \( a \in A \) and suppose \( S = \text{supp}'(a) \) is an open bisection in \( G \). Recall that \( a^* \star f = a^*f \) when \( f \in B \), so
\[ (a^* \star f) \star a(\gamma) = \sum_{\beta \in G_2(\gamma)} a^*f(\gamma \beta^{-1})a(\gamma) = \sum_{\beta \in G_2(\gamma)} \overline{a(\beta \gamma^{-1})}f(r(\beta))a(\gamma). \]
The terms of the sum are non-zero only when both \( \beta \gamma^{-1} \) and \( \beta \) are contained in \( S \). As this is a bisection, this forces \( \gamma \in X \). Hence \( a^*fa \in B \) and similarly \( af a^* \in B \). Therefore, \( a \in N(A, B) \).
To prove the last assertion, take \( x \in X \) and let \( \gamma_x \in S \) be the unique morphism with \( s(\gamma_x) = x \). From the above computation, we see that

\[
a^* f a(x) = a^*(\gamma_x)a(\gamma_x)f(r(\gamma_x)) = a^*a(\gamma_x)f(\zeta_S(x)),
\]

for every \( f \in \mathcal{B} \). By uniqueness, we conclude that \( \zeta_a = \zeta_S \) as desired.

On the other hand, let \( a \in N(\mathcal{A}, \mathcal{B}) \) and put \( S = \text{supp}'(a) \). We show that \( S \) is a bisection by showing that the open sets \( SS^{-1} \) and \( S^{-1}S \) are contained in \( X \). Here, we need the assumption that \( G \) is essentially principal. Consider the set

\[
T = \{ \gamma \in G \mid s(\gamma) \in d(a), \ r(\gamma) = \zeta_a(s(\gamma)) \}.
\]

We will show that \( S \subseteq T \). If \( \gamma_x \in S \) with \( s(\gamma_x) = x \), then \( a(\gamma_x) \neq 0 \) and so \( a^*a(x) = \sum_{\beta \in G_x} |a(\beta)|^2 > 0 \). Hence \( x \in d(a) \). Since \( a \in N(\mathcal{A}, \mathcal{B}) \), we have \( a^*f a(x) = f(\zeta_a(x))a^*a(x) \) and so

\[
f(\zeta_a(x)) = \sum_{\beta \in G_x} |a(\beta)|^2 a^*a(x)f(r(\beta)), \tag{3.9}
\]

for every \( f \in \mathcal{B} \). Formula \( \tag{3.9} \) may be rephrased as expressing a convex combination of the pure state \( \zeta_a(x) \) (evaluation at \( \zeta_a(x) \)) in terms of the pure states \( r(\beta) \) (evaluation at \( r(\beta) \)). These are extreme points in the state space of \( \mathcal{B} \) and so it follows that \( a(\beta) = 0 \) exactly when \( r(\beta) \neq \zeta_a(x) \). By assumption, \( a(\gamma_x) \neq 0 \) and this establishes the inclusion \( S \subseteq T \). Furthermore, \( SS^{-1} \subseteq TT^{-1} \) and the latter is included in the loop bundle \( \Omega(X) \) since \( \zeta_a \) is a homeomorphism. The fact that \( SS^{-1} \) is open implies that \( SS^{-1} \subseteq X \) and similarly \( S^{-1}S \subseteq X \). Hence \( S \) is a bisection.

\[\square\]

**Theorem 3.3.15.** Let \( (G, X) \) be an \( \text{étale} \) LCH groupoid which is essentially principal. Put \( \mathcal{A} = C^*_v(G) \) and \( \mathcal{B} = C_0(X) \). Then the Weyl groupoid \( G(\mathcal{A}, \mathcal{B}) \) is canonically isomorphic to \( (G, X) \).

**Proof.** Let \( \mathcal{G}(\mathcal{A}, \mathcal{B}) \) be the Weyl pseudogroup of the pair \( (\mathcal{A}, \mathcal{B}) \) and let \( \mathcal{S} \) be the collection of open bisections of \( (G, X) \). In the above proposition, we saw that

\[
\mathcal{G}(\mathcal{A}, \mathcal{B}) = \{ \zeta_a \mid a \in N(\mathcal{A}, \mathcal{B}) \} = \{ \zeta_{\text{supp}'(a)} \mid a \in N(\mathcal{A}, \mathcal{B}) \} \subseteq \{ \zeta_S \mid S \in \mathcal{S} \} = \zeta(\mathcal{S}). \tag{3.10}
\]

We would like to have an equality so that \( \mathcal{G}(\mathcal{A}, \mathcal{B}) \) and \( \zeta(\mathcal{S}) \) generate the same groupoid of germs. So let \( S \) be an open bisection in \( G \) and choose \( h \in C_0(X) \) such that \( \text{supp}'(h) = s(S) \). Then \( \text{supp}'(h \circ s) = S \) though, alas, it is not clear whether \( h \circ s \) is in \( \mathcal{A} \).

We fix this by choosing a non-vanishing continuous function \( u : S \to \mathbb{C} \) whose image is contained in \( T \) (simply replace any \( u(\gamma) \) with \( u(\gamma)/|u(\gamma)| \)) together with a sequence \( (h_n)_n \) in \( C_c(G) \) with \( \text{supp}'(h_n) \subseteq s(S) \) such that \( h_n \to h \) uniformly. Then \( u \cdot h_n \in C_c(G) \) and

\[
\|u \cdot h_n - u \cdot h\|^2 = \sup_{x \in X} \sum_{\beta \in G_x} |u(\beta)h_n(s(\beta) - h(s(\beta)))|^2
\leq \|u\|^2 \sup_{x \in X} |h_n(s(\beta_x) - h(s(\beta_x)))|^2
= \|u\|^2 \|h_n - h\|_\infty^2 \to 0,
\]
as \( n \to \infty \). Here, \( \beta_x \) is the unique morphism in \( S \) with \( s(\beta_x) = x \). Hence \( u.h_n \) converges to \( u.h \) in \( A \). Since \( \text{supp}'(u.h) = S \), the inclusion in \((3.10)\) is, in fact, an equality. Also, Proposition \[ \text{3.3.14} \] shows that \( u.h \in N(A,B) \).

We conclude that \( \mathcal{G}(A,B) \) and \( \zeta(\mathcal{S}) \) generate the same groupoid of germs which is isomorphic to \((G,X)\), cf. Corollary \[ \text{3.3.8} \].

The assertion of Theorem \[ \text{3.3.1} \] follows directly.

\textbf{Proof of Theorem} \[ \text{3.3.1} \] Suppose \((G_1,X_1)\) and \((G_2,X_2)\) are étale LCH groupoids which are essentially principal and assume that their associated \( C^*\)-algebras are isomorphic. Invoking Theorem \[ \text{3.3.15} \] we obtain

\[
(G_1,X_1) \cong G(C^*_r(G_1),C_0(X_1)) \cong G(C^*_r(G_2),C_0(X_2)) \cong (G_2,X_2).
\]

The middle isomorphism follows from the functoriality of the groupoid of germs.

3.4 A groupoid view on \( \mathcal{O}_A \)

Let us now address an example of an étale LCH groupoid which will be quint-essential in our further analysis. The proofs are adapted from \[ \text{KPRR97} \]. Let \((X_A,\sigma_A)\) be a one-sided topological Markov shift which is irreducible in the sense that \( A \) is irreducible. We shall now construct a groupoid out of this data. As a set the groupoid will be

\[
G_A = \{(x,n,y) \in X_A \times \mathbb{Z} \times X_A \mid n = k - l, k,l \in \mathbb{Z_+}, \sigma_A^k(x) = \sigma_A^l(y)\}
\]

with unit space \( \{(x,0,x) \in G_A\} \). We shall identify \( X_A \) with the unit space via \( x \leftrightarrow (x,0,x) \).

The product and inversion operations are given as

\[
(x,n,y)(y,m,z) = (x,n+m,z), \quad (x,n,y)^{-1} = (y,-n,x),
\]

respectively, while the range and source maps are defined as

\[
r(x,n,y) = (x,0,x), \quad s(x,n,y) = (y,0,y),
\]

respectively. We shall often omit the explicit mention of the unit space \( X_A \) and simply refer to the groupoid \((G_A,X_A)\) as \( G_A \).

Let \( \mathcal{U} = \mathcal{U}_A \) be the collection of sets of the form

\[
N(\alpha,\beta) = \{(x,|\alpha| - |\beta|, y) \in G_A \mid x \in N_\alpha, \ y \in N_\beta\},
\]

indexed by the admissible words \( \alpha \) and \( \beta \) with \( r(\alpha) = r(\beta) \). Recall that \( N_\alpha \) and \( N_\beta \) are their corresponding basic open cylinder sets in \( X_A \). It is clear that the sets in \( \mathcal{U} \) cover \( G_A \). Note also that each \( N(\alpha,\beta) \) is a bisection (a formal argument is provided in the proof of Proposition \[ \text{3.4.1} \]). Furthermore, if \( \alpha,\beta,\alpha'\) and \( \beta' \) are admissible words, then

\[
N(\alpha,\beta) \cap N(\alpha',\beta') = \begin{cases} N(\alpha,\beta), & \alpha = \alpha'\delta, \beta = \beta'\delta, \\ N(\alpha',\beta'), & \alpha' = \alpha\delta', \beta' = \beta\delta', \\ \emptyset, & \text{otherwise}, \end{cases}
\]
for some admissible words $\delta$ and $\delta'$. In particular, $\mathcal{U}$ is stable under intersection and thus constitutes a basis for a second countable topology on $G_A$. It is Hausdorff and since $N(\alpha, \beta) = (N_\alpha \times \{|\alpha| - |\beta|\} \times N_\beta) \cap G_A$ is closed in $X_A \times \{|\alpha| - |\beta|\} \times X_A$, these sets are compact and so the topology on $G_A$ is locally compact and Hausdorff.

Let us now check that this topology renders inversion and composition continuous. It is clear that the inversion map takes $N(\alpha, \beta)$ to $N(\beta, \alpha)$. Next, suppose
\[ (x, n, y)(y, m, z) \in N(\alpha, \beta), \]
for some admissible words $\alpha$ and $\beta$. Take a positive integer $N > \max\{|\alpha|, n\}$ such that $N + (n + m) > |\beta|$ and consider the words
\[ \alpha' = x_{[1, N]} = \alpha_1 \cdots \alpha_{|\alpha| + 1} \cdots x_N, \]
\[ \beta' = z_{[1, N + (n + m)]} = \beta_1 \cdots \beta_{|\beta| + 1} \cdots z_{N + (n + m)}, \]
\[ \delta = y_{[1, N + n]}. \]
By construction, we have $(x, n, y) \in N(\alpha', \delta)$ and $(y, m, z) \in N(\delta', \beta')$. Furthermore, the composition map takes the open set $(N(\alpha', \delta) \times N(\delta', \beta')) \cap G_A^2$ into $N(\alpha, \beta)$. This shows that the operations of inversion and composition are continuous with respect to the topology generated by $\mathcal{U}$.

**Proposition 3.4.1.** The groupoid $G_A$ equipped with the topology defined by $\mathcal{U}_A$ is étale LCH and essentially principal.

**Proof.** We saw above that $G_A$ is a LCH groupoid, so it only remains to show that $G_A$ is étale. We show that the range map defines a homeomorphism between $N_{\alpha, \beta}$ and $N_\alpha$. Since $r$ is continuous and both the domain and range are compact Hausdorff, we only need to check that $r$ is a bijection.

If $x = r(x, |\alpha| - |\beta|, y) \in N_\alpha$, then $y = \beta x_{[|\alpha| + 1, \infty)}$ is unique. In particular, $r$ is injective. On the other hand, write $x = az$ for some $z \in X_A$ with $A(r(\alpha), z_1) = 1$. Since $r(\alpha) = r(\beta)$, the sequence $w = \beta z$ is in $N_\beta$ and $\sigma^{[a]}_A(z) = \sigma^{[\beta]}_A(w)$. Hence $(z, |\alpha| - |\beta|, w) \in N_{\alpha, \beta}$, so $r$ is surjective. A similar argument shows that $s$ is a local homeomorphism. This shows that $G_A$ is étale and that the basic open sets $N(\alpha, \beta)$ are bisections. In particular, the set $N(\alpha, \beta)$ contains at least two elements since it is in bijective correspondence with $N_\alpha$ and $A$ satisfies condition (I).

Finally, we verify that $X_A = \text{Int}(\Omega X_A)$. Since the left-to-right inclusion is always valid, take $\gamma \in \text{Int}(\Omega X_A)$ and let $N(\alpha, \beta) \subseteq \Omega X_A$ be a (basic) open neighborhood of $\gamma$. It follows that $\gamma = (x_0, |\alpha| - |\beta|, x_0)$ for some $x_0 \in X_A$. If $|\alpha| > |\beta|$, then $\alpha$ is an extension of $\beta$ and we can denote their difference by $\delta = \sigma^{[\beta]}_A(\alpha)$. It follows that $x_0 = \delta^\infty = \delta \delta \cdots$ and this implies that $N(\alpha, \beta)$ is the singleton $\{(x_0, |\alpha| - |\beta|, x_0)\}$. This contradicts the above observation and so $\gamma = (x_0, 0, x_0) \in X_A$. Hence $G_A$ is essentially principal.

The following theorem relates the groupoid determined by the topological Markov shift to the Cuntz-Krieger algebra studied in Chapter 2. The proof is borrowed from [KPRR97].

**Theorem 3.4.2.** Let $(X_A, \sigma_A)$ be an irreducible one-sided topological Markov shift and let $G_A$ be its associated étale groupoid. There exists an isomorphism $(\mathcal{O}_A, \mathcal{D}_A) \rightarrow (C^*_r(G_A), C(X_A))$. 

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Proof. We exhibit a CK family in $C^*_r(G_A)$ and show that it generates the reduced groupoid $C^*$-algebra. Since $\mathcal{O}_A$ is simple, this will prove the assertion of the theorem.

Let $(\mathcal{V}, \mathcal{E})$ be the directed graph determined by $A$ and consider the family $T_e = \chi_{N(e,r(e))}$ indexed over the edges $e \in \mathcal{E}$. From the convolution formula (3.3), we have

$$T_e^* T_f(x, k, y) = \sum_{\beta \in (G_A)_e} T_e(\beta) T_f(\beta(x, k, y)) = \sum_{\{z : (z,l,x) \in G_A\}} T_e(z,l,x) T_f(x+l, k, y)$$

which is non-zero only when $(z,l,x) \in N(e,r(e))$ and $(z,l+k,y) \in N(f,r(f))$. Hence $l = 1$ and $k = 0$ from which it follows that $e = f$ and $x = y$. Therefore,

$$T_e^* T_f(x,0,0) = \begin{cases} \chi_{D_e} & e = f, \\ 0 & \text{else,} \end{cases}$$

where $D_e = \{(x,0,x) \mid x_1 = r(e)\}$. As $\chi_{D_e}$ is a projection, $T_e$ is a partial isometry.

Next, note that $X_A = \coprod_{e \in \mathcal{E} \setminus \{\emptyset\}} N(e,e)$. We show that $T_e T_e^* = \chi_{N(e,e)}$ from which it follows that $\sum_e T_e T_e^* = 1$. In particular, the partial isometries have orthogonal ranges. Via the convolution formula (3.3), we see that

$$T_e T_e^*(x,k,y) = \sum_{\beta \in (G_A)_y} T_e((x,k,y) \beta^{-1}) (\beta^{-1}) = \sum_{\{z : (z,l,x) \in G_A\}} T_e(x,k-l,z) T_e(y,-l,z)$$

is non-zero only when $l = -1$ and $k = 0$, so $x = y \in N_e$. That is, $T_e T_e^* = \chi_{N(e,e)}$. Finally, we observe that $D_e = \bigcup_{f \in \mathcal{E}_r(e)} N(f,f)$ so that

$$\chi_{D_e} = \sum_{f \in \mathcal{E}_r(e)} \chi_{N(f,f)} = \sum_{f \in \mathcal{E}_r(e)} T_f T_f^*.$$ Hence $\{T_e\}_e$ is a CK family. Similarly, one can show that

$$T_\alpha T_\beta^* = \begin{cases} \chi_{N(\alpha,\beta)} & r(\alpha) = r(\beta), \\ 0 & \text{else,} \end{cases}$$

In order to show that the family $\{T_e\}_e$ generates the whole reduced groupoid $C^*$-algebra, we show that $\text{span}\{\chi_{N(\alpha,\beta)} \mid \alpha, \beta \in \mathcal{A}_{\mathbb{N}}\}$ is dense in $C^*_r(G_A)$, cf. Lemma 2.3.

By construction, $C_c(G_A)$ is dense in $C^*_r(G_A)$. Let $f \in C_c(G_A)$. By Lemma B.1.4, we may assume that $f$ has support on an open bisection of the form $N(\alpha,\beta)$. Here, the uniform norm dominates the $C^*$-norm, so it suffices to approximate $f$ in the uniform norm. This is possible by the Stone-Weierstrass theorem. Hence $\{\chi_{N(\alpha,\beta)} \mid \alpha, \beta \in \mathcal{A}_{\mathbb{N}}\}$ is dense in $C_c(G_A)$.

It should be clear that the induced $\ast$-homomorphism maps the diagonal $\mathcal{D}_A$ to $C(X_A)$. Indeed, $T_\alpha T_\alpha^*$ is mapped to $\chi_{N(\alpha,\alpha)}$ and

$$\chi_{N(\alpha,\alpha)}(x, k, y) = 1 \iff \chi_{N_{\alpha}}(x) = 1,$$

since the left hand side is the case if and only if $k = |\alpha| = l$ and $x = y$.

In this groupoid view on the Cuntz-Krieger algebras, the gauge action $\rho$ takes the form

$$\rho_t(f)(x,n,y) = e_k(t)f(x,n,y),$$

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for $f \in C_c(G_A)$. Since $\rho_t(\chi_{N(x,e)})((x,1,\sigma_A(x))) = \epsilon_t(x,1,\sigma_A(x))$, this definition agrees with the gauge action on $\mathcal{O}_A$. There is an interesting subgroupoid $(H_A, X_A)$ in $(G_A, X_A)$ given as

$$H_A = \{(x,0,y) \in G_A\}. \quad (3.13)$$

In Lemma 4.3.1 we show that $H_A$ is an example of an AF groupoid.

**Proposition 3.4.3.** The C*-algebra $C^*_r(H_A)$ is isomorphic to $\mathcal{F}_A$.

**Proof.** Recall that $\mathcal{F}_A$ is the fixed point algebra of the gauge action $\rho: T \cap \mathcal{O}_A$ and that there is a conditional expectation $E: \mathcal{O}_A \to \mathcal{F}_A$. If we consider $C_c(G_A)$ to be a dense *-subalgebra of $C^*_r(G_A)$, then the gauge action takes the form

$$\rho_t(f)(x,k-l,y) = \epsilon_{k-l}(t)f(x,k-l,y),$$

for $f \in C_c(G_A)$. It follows that $f$ is invariant under the gauge action when $\text{supp}(f) \subseteq H_A$. Hence $C_c(H_A) \subseteq \mathcal{F}_A$ and so $C^*_r(H_A) \subseteq \mathcal{F}_A$.

On the other hand, let us consider the sets

$$M_i = (X_A \times \{i\} \times X_A) \cap G_A = \{(x,i,y) \in G_A\}$$

which are clopen in $G_A$ for every $i \in \mathbb{Z}$. Note that $M_0 = H_A$. By a partition of unity argument (similar to that of Lemma 3.1.4) we can write any $f \in C_c(G_A)$ as a finite sum of maps $f_i$ supported on $M_i$. Since $\rho_t(f_i) = \epsilon_{i}(t)f$, we see that $E(f) = f_0 \in C_c(H_A)$. Therefore, $E(C_c(G_A)) \subseteq C_c(H_A)$. By continuity, we have $\mathcal{F}_A = E(\mathcal{O}_A) \subseteq C^*_r(H_A)$.

Let us now collect our main results from this and the previous chapter to obtain the following corollary.

**Corollary 3.4.4.** Let $(X_A, \sigma_A)$ be $(X_B, \sigma_B)$ be irreducible one-sided topological Markov shifts. The following are equivalent.

1. The shift spaces $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are continuously orbit equivalent,
2. The groupoids $G_A$ and $G_B$ are isomorphic.

**Proof.** We saw in Chapter 2 that the shift spaces $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are continuously orbit equivalent if and only if there is a diagonal preserving C*-isomorphism of the corresponding Cuntz-Krieger algebras. In light of Theorem 3.4.2 we then have

$$ (C^*_r(G_A), C(X_A)) \cong (\mathcal{O}_A, \mathcal{D}_A) \cong (\mathcal{O}_B, \mathcal{D}_B) \cong (C^*_r(G_B), C(X_B)).$$

Since the groupoid C*-algebra construction is a complete invariant by Theorem 3.3.1 this is equivalent to saying that $G_A \cong G_B$ as groupoids.
Groupoid (co)homology

Let $X$ be a locally compact Hausdorff (LCH) space and let $\Lambda$ be a topological abelian group. The collection of continuous maps $X \rightarrow \Lambda$ with compact support is denoted $C_c(X, \Lambda)$. This is an abelian group under point-wise addition. When $X$ is compact we suppress the subscript. If $\pi: X \rightarrow Y$ is a local homeomorphism between LCH spaces, then we have an induced group homomorphism $\pi_*: C_c(X, \Lambda) \rightarrow C_c(Y, \Lambda)$ given by

$$\pi_*(f)(y) = \sum_{\pi(x) = y} f(x),$$

for $f \in C_c(X, \Lambda)$. Furthermore, if $\pi': Y \rightarrow Z$ is another local homeomorphism between LCH spaces, then $(\pi' \circ \pi)_* = \pi'_* \circ \pi_*$. On the other hand, $\pi$ also induces a homomorphism $\pi^*: C_c(Y, \Lambda) \rightarrow C_c(X, \Lambda)$ given by $\pi^*(g) = g \circ \pi$. Similarly, $(\pi' \circ \pi)^* = \pi^* \circ (\pi')^*$.

4.1 Homology

We shall define a homology and cohomology theory for an étale LCH groupoid $(G, X)$ as is done in [Mat12]. For each $n \in \mathbb{N}$, we let $G^{(n)}$ be the set of composable strings of length $n$; that is,

$$G^{(n)} = \{(\gamma_1, \ldots, \gamma_n) \mid (\gamma_i, \gamma_{i+1}) \in G^{(2)}, i = 1, \ldots, n - 1\}.$$

In particular, $G^{(0)} = X$ and $G^{(1)} = G$. Each $G^{(n)}$ inherits the topology from the $n$-fold product $G^n$ and the family $\{G^{(n)}\}_n$ forms a simplicial set with face maps $d_i = d_i^{(n)}: G^{(n)} \rightarrow G^{(n-1)}$, $i = 0, \ldots, n$, defined as

$$d_i(\gamma_1, \ldots, \gamma_n) = \begin{cases} (\gamma_2, \ldots, \gamma_n) & i = 0, \\ (\gamma_1, \ldots, \gamma_i \gamma_{i+1}, \ldots, \gamma_n) & 0 < i \leq n - 1, \\ (\gamma_1, \ldots, \gamma_{n-1}) & i = n. \end{cases}$$

These are local homeomorphisms.

We define boundary operators $\delta_n: C_c(G^{(n)}, \Lambda) \rightarrow C_c(G^{(n-1)}, \Lambda)$ by

$$\delta_1 = s_* - r_*, \quad \delta_n = \sum_{i=0}^{n} (-1)^i (d_i)_*.$$

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By construction, the diagram
\[
C_c(G^{n+1}, \Lambda) = \cdots \to C_c(G^{(2)}, \Lambda) \xrightarrow{\delta_2} C_c(G, \Lambda) \xrightarrow{\delta_1} C_c(X, \Lambda) \xrightarrow{\delta_0} 0
\]  
(4.1)
forms a chain complex. An element \( f \in C_c(G^{(n)}, \Lambda) \) is a \textbf{cycle} if \( \delta_n(f) = 0 \) and a \textbf{boundary} if it is contained in the image of \( \delta_{n+1} \).

**Definition 4.1.1.** The \textbf{homology with constant coefficients} in \( \Lambda \) of an étale LCH groupoid \((G, X)\) is the homology of the chain complex (4.1). That is,
\[
H_n(G; \Lambda) = \frac{\ker \delta_n}{\text{im} \delta_{n+1}},
\]
for \( n \in \mathbb{Z}_+ \). We often abbreviate \( H_n(G) := H_n(G; \mathbb{Z}) \). Furthermore, we put
\[
H_0(G)^+ = \{ [f] \in H_0(G) \mid f \geq 0 \},
\]
where \([f]\) denotes the class of \( f \in C_c(X, \mathbb{Z}) \) in \( H_0(G) \).

**Remark 4.1.2.** The pair \((H_0(G), H_0(G)^+)\) need not be an ordered abelian group (see e.g., [RLL00]) since \( H_0(G)^+ \cap (-H_0(G)^+) \) need not be trivial. However, when the groupoid is AF, then \((H_0(G), H_0(G)^+)\) is an ordered abelian group, cf. Theorem 4.2.11.

Similarly, we can define a cohomology theory. The face maps induce \textbf{coboundary operators} \( \partial^{n+1} : C_c(G^{(n)}, \Lambda) \rightarrow C_c(G^{(n+1)}, \Lambda) \) given by
\[
\partial^1 = s^* - r^*, \quad \partial^n = \sum_{i=0}^n (-1)^i (d_i)^*,
\]
for every \( n \in \mathbb{N} \). On can check that \( \partial^{n+1} \circ \partial^n \) vanishes for all \( n \in \mathbb{Z}_+ \) and thus
\[
0 \xrightarrow{\partial^0} C_c(X, \Lambda) \xrightarrow{\partial^1} C_c(G, \Lambda) \xrightarrow{\partial^2} C_c(G^{(2)}, \Lambda) \xrightarrow{\partial^3} \cdots
\]  
(4.2)
is a chain complex.

**Definition 4.1.3.** The \textbf{cohomology} of the étale LCH groupoid \((G, X)\) with constant coefficients in \( \Lambda \) is the homology of (4.2). That is,
\[
H^n(G; \Lambda) = \frac{\ker \partial^{n+1}}{\text{im} \partial^n},
\]
for \( n \in \mathbb{Z}_+ \).

We immediately see that \( \partial^2(f)(\gamma_1, \gamma_2) = f(\gamma_2) - f(\gamma_1\gamma_2) + f(\gamma_2) \) for \( f \in C_c(G, \Lambda) \) and \((\gamma_1, \gamma_2) \in G^{(2)} \). The zeroth and first cohomology groups can therefore be characterized as
\[
H^0(G; \Lambda) = \ker \partial^1 = \{ f \in C_c(X, \Lambda) \mid f \circ r = f \circ s \}
\]
and
\[
H^1(G; \Lambda) = \frac{\{ f \in C_c(G, \Lambda) \mid f(\gamma_1\gamma_2) = f(\gamma_1) + f(\gamma_2), (\gamma_1, \gamma_2) \in G^{(2)} \}}{\{ \partial^1(f) \mid f \in C_c(X, \Lambda) \}} = \frac{\text{Hom}(G, \Lambda)}{\text{im} \partial^1}.
\]
This will be useful to us later.

In the case where \((G,X)\) is a group, the above construction coincides with the group (co)homology with \textit{constant} coefficients, that is, with a trivial action. Below, we spell out a particularly simple example in which this action need not be trivial, namely the group homology of the group of integers.

**Example 4.1.4.** Let \(\Gamma\) be a countable discrete group. A \(\Gamma\)-module \(M\) is a left \(\mathbb{Z}[\Gamma]\)-module specified by an action \(\Gamma \curvearrowright M\). When \(\Gamma = \mathbb{Z}\), we identify the group ring \(\mathbb{Z}[\Gamma]\) with the ring of Laurent polynomials \(\mathbb{Z}[t,t^{-1}]\). A free resolution of \(\mathbb{Z}\) (viewed as a trivial \(\mathbb{Z}[t,t^{-1}]\)-module) is given as

\[
0 \longrightarrow \mathbb{Z}[t,t^{-1}] \xrightarrow{1-t} \mathbb{Z}[t,t^{-1}] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0,
\]

where \(\epsilon\) is evaluation at 1, while the map \((1-t)\) is multiplication by \((1-t)\). Applying \(M \otimes_{\mathbb{Z}[t,t^{-1}]} \) to the above complex, we obtain the reduced complex

\[
0 \longrightarrow M \xrightarrow{1-t} M \longrightarrow 0.
\]

The group homology of \(\mathbb{Z}\) with coefficients in \(M\) is thus concentrated on degrees zero and one and given by

\[
H_0(\mathbb{Z}; M) = \ker (1-t) = M_{\mathbb{Z}}, \quad H_1(\mathbb{Z}; M) = \text{coker} (1-t) = M_{\mathbb{Z}}^*,
\]

the coinvariants and invariants of the action \(\mathbb{Z} \curvearrowright M\), respectively.

**Example 4.1.5.** Let \((X,X)\) be a trivial LCH groupoid. Then \(X^{(0)} = X^{(1)} = X\) and \(X^{(n)}\) is the diagonal inside \(X^n\). It is not hard to see that \(\delta_n = 0\) when \(n\) is odd and \(\delta_n = \text{id}\) when \(n \geq 2\) is even. Hence \(H_n(X; \Lambda) = 0\) when \(n \geq 1\) for any topological abelian group \(\Lambda\).

When \(\varphi: (G,X) \longrightarrow (H,Y)\) is a local homeomorphism, then there is an induced map \(H_n(\varphi): H_n(G; \Lambda) \longrightarrow H_n(H; \Lambda)\). Let us see how. For each \(n \in \mathbb{Z}_+\), we let \(\varphi^{(n)}: G^{(n)} \longrightarrow H^{(n)}\) be the coordinate-wise application of \(\varphi\). In particular, \(\varphi^{(0)} = \varphi|_X\). Note that \(\varphi^{(n)}\) is a local homeomorphism if and only if \(\varphi\) is a local homeomorphism. A straightforward computation shows that \(\varphi\) commutes with the structure maps and that \(\varphi^{(n-1)} \circ d_i^G = d_i^H \circ \varphi^{(n)}\), for each \(n \in \mathbb{N}\) and \(i = 0,\ldots,n\). For each \(f \in C_c(G^{(n)}, \Lambda)\), we then have

\[
\delta_n^H \circ \varphi_{!*}^{(n)}(f) = \sum_{i=0}^{n} (-1)^i (d_i^H \circ \varphi^{(n)})_{!*}(f)
\]

and

\[
\varphi_{!*}^{(n-1)} \circ \delta_n^G(f) = \sum_{i=0}^{n} (-1)^i (\varphi^{(n-1)} \circ d_i^G)_{!*}(f)
\]

since \(\varphi_{!*}^{(n)}\) is a group homomorphism. That is, \(\delta_n^H \circ \varphi_{!*}^{(n)} = \varphi_{!*}^{(n-1)} \circ \delta_n^G\) and so \(\{\varphi_{!*}^{(n)}\}_n\) is a chain map \(C_c(G^{(\bullet)}, \Lambda) \longrightarrow C_c(H^{(\bullet)}, \Lambda)\). Hence we have a well-defined group homomorphism\(^1\).

\(^1\)So, alas, a \(\mathbb{Z}\)-module is not just an abelian group.
\[ H_n(\varphi) : H_n(G; \Lambda) \to H_n(H; \Lambda). \]
In particular, any action on a groupoid induces an action on its homology groups.

We shall now consider a sufficient condition for two homomorphisms to induce the same map on homology.

**Definition 4.1.6.** Two homomorphisms \( \varphi, \psi : (G, X) \to (H, Y) \) between étale LCH groupoids are **similar** if there is a continuous map \( \vartheta : X \to H \) such that

\[
\vartheta(r(\gamma))\varphi(\gamma) = \psi(\gamma)\vartheta(s(\gamma)),
\]
for every \( \gamma \in G \). In the affirmative case, we write \( \varphi \simeq \psi \). The groupoids are **homologically similar** if there are local homeomorphisms \( \varphi : (G, X) \to (H, Y) \) and \( \psi : (H, Y) \to (G, X) \) such that \( \psi \circ \varphi \simeq \text{id}_G \) and \( \varphi \circ \psi \simeq \text{id}_H \).

**Lemma 4.1.7.** If local homeomorphisms \( \varphi, \psi : (G, X) \to (H, Y) \) between étale LCH groupoids are similar, then \( H_n(\varphi) = H_n(\psi) \).

**Proof.** Take a local homeomorphism \( \vartheta : X \to H \) satisfying (4.3). We can construct a chain homotopy \( h_n : C_c(G(n); \Lambda) \to C_c(H(n+1); \Lambda) \) by putting \( h_n = \sum_{j=0}^{n}(-1)^j(k_j(n))_* \), for \( n \in \mathbb{Z}_+ \). Here, \( k_j(n) : G(n) \to H(n+1), j = 0, \ldots, n \), is defined by

\[
k_j(n)(\gamma_1, \ldots, \gamma_n) = \begin{cases} (\vartheta(r(\gamma_1)), \varphi(\gamma_1), \ldots, \varphi(\gamma_n)) & j = 0, \\ (\varphi(\gamma_1), \ldots, \varphi(\gamma_j), \vartheta(s(\gamma_j)), \varphi(\gamma_{j+1}), \ldots, \varphi(\gamma_n)) & 0 < j < n-1, \\ (\varphi(\gamma_1), \ldots, \varphi(\gamma_n), \vartheta(s(\gamma_n))) & j = n. \end{cases}
\]

Then \( \delta^H \circ h_0 = (\varphi(0))_* - (\psi(0))_* \) and \( \delta^H_{n+1} \circ h_n + h_{n-1} \circ \delta^G_n = (\varphi(n))_* - (\psi(n))_* \) for \( n \in \mathbb{N} \). We will spare the reader the details of the computations. Evaluated on a cycle, the difference \( (\varphi(n))_* - (\psi(n))_* \) reduces to a boundary. It follows that \( \varphi \) and \( \psi \) induce the same map on homology.

The following immediate corollary explains the term «homological similarity».

**Corollary 4.1.8.** If \( (G, X) \) and \( (H, Y) \) are homologically similar étale LCH groupoids, then \( H_n(G; \Lambda) \cong H_n(H; \Lambda) \).

**Definition 4.1.9.** Let \((G, X)\) be a groupoid. A subset \( Y \subseteq X \) is **G-full** if \( r^{-1}(x) \cap s^{-1}(Y) \neq \emptyset \) for every \( x \in X \).

We give a sufficient condition for a groupoid to be homologically similar to its reduction (see Definition 3.1.14) to an open and full subset.

**Proposition 4.1.10.** Let \((G, X)\) be an étale LCH groupoid and let \( Y \subseteq X \) be an open G-full subset.

1. If there exists a continuous \( \vartheta : X \to G \) such that \( r(\vartheta(x)) = x \) and \( s(\vartheta(x)) \in Y \), for every \( x \in X \), then \( G \) is homologically similar to \( G|Y \).

2. If \( X \) is compact and totally disconnected, then such a map \( \vartheta : X \to G \) exists.
Proof. Note that $\vartheta$ is étale if it exists. Define étale homomorphisms $\varphi: G \to G[Y]$ and $\psi: G[Y] \to G$ by $\varphi(\gamma) = (\vartheta(r(\gamma)))^{-1} \gamma \vartheta(s(\gamma))$ and $\psi(\eta) = \eta$, for $\gamma \in G$ and $\eta \in G[Y]$. There are no surprises in checking that $\varphi \circ \varphi \simeq \text{id}_G$ and $\varphi \circ \psi \simeq \text{id}_{G[Y]}$. This proves (1).

In order to prove (2), let $\{U_i\}_i$ be a family of compact open bisections which cover $G$. Then $\{r(U_i)\}_i$ covers $X$ and we may assume that $s(U_i) \subseteq Y$, for every $i$. Indeed, if $\gamma \in U_i$ for some $i$ and $x = s(\gamma)$, then there exist $y \in Y$ and $\eta \in G_y$ such that $\eta \in G^x$. By replacing $\gamma$ by $\gamma \eta$ we have $r(\gamma \eta) = r(\gamma)$ and $s(\gamma \eta) = s(\eta) \in Y$. As $X$ is assumed compact, we may extract a finite subcollection $r(U_1), \ldots, r(U_n)$ covering $X$. If we recursively define

$$V_i = U_1, \quad V_i = U_i \setminus r \left( r^{-1} \left( \bigcup_{j=1}^{i-1} U_j \right) \right),$$

then $X$ is the disjoint union of the compact open $r(V_1), \ldots, r(V_n)$. For each $x \in X$, there is a unique index $\ell(x) = 1, \ldots, n$ such that $x \in r(V_{\ell(x)})$. We may therefore define a continuous map $\vartheta: X \to G$ by putting

$$\vartheta(x) = (r|_{V_{\ell(x)}})^{-1}(x),$$

for $x \in X$. That is, $x$ is mapped to the unique morphism in $V_{\ell(x)}$ with range $x$. Then $r(\vartheta(x)) = x$ by construction and $s(\vartheta(x)) \in s(V_{\ell(x)}) \subseteq Y$, so $\vartheta$ satisfies the hypotheses of (1). \qed

### 4.2 Kakutani equivalence

From now on, we shall restrict our attention to étale LCH groupoids whose unit space is compact and totally disconnected. We shall define a notion which is stronger than homological similarity called »Kakutani equivalence«. This is due to Matui in [Mat12].

**Definition 4.2.1.** Let $(G_i, X_i), i = 1, 2,$ be étale LCH groupoids with $X_i$ compact and totally disconnected. Then $G_1$ is **Kakutani equivalent** to $G_2$ if there are $G_i$-full clopen subsets $Y_i \subseteq X_i$ such that the reductions $G_1| Y_1$ and $G_2| Y_2$ are isomorphic.

It is obvious that this relation is both reflexive and symmetric. Lemma 4.2.5 asserts that it is also transitive. This is not immediately clear and we need two lemmas to prove this fact.

**Example 4.2.2.** Let $(G, X)$ be an étale LCH groupoid with $X$ compact and totally disconnected. Given $f \in C(X, \mathbb{Z})$ with $f \geq 0$, we define a groupoid $G_f$ by

$$G_f = \{(\gamma, i, j) \in G \times \mathbb{Z} \times \mathbb{Z} \mid 0 \leq i \leq f(r(\gamma)), \ 0 \leq j \leq f(s(\gamma))\}.$$  

Two elements $(\gamma, i, j)$ and $(\gamma', k, l)$ are composable if and only if $(\gamma, \gamma') \in G^{(2)}$ and $j = k$ in which case

$$(\gamma, i, j)(\eta, j, l) = (\gamma \eta, i, l)$$

and $(\gamma, i, j)^{-1} = (\gamma^{-1}, j, i)$. Hence $r(\gamma, i, j) = (r(\gamma), i, i)$ and $s(\gamma, i, j) = (s(\gamma), j, j)$. The unit space is $G_f^{(0)} = \{(x, i, i) \in G_f \mid x \in X, \ 0 \leq i \leq f(x)\}$ and the topology of $G_f$ is inherited from $G \times \mathbb{Z} \times \mathbb{Z}$. Note that the subset $\{(x, 0, 0) \in G_f^{(0)} \mid x \in X\} \subseteq G_f^{(0)}$ is clopen and $G_f$-full.
Lemma 4.2.3. Let \((G, X)\) be an étale LCH groupoid with \(X\) compact and totally disconnected and let \(Y \subseteq X\) be a clopen \(G\)-full subset. Then there exist \(f \in C(Y, \mathbb{Z})\) and an isomorphism \(\pi: (G[Y])_f \rightarrow G\) such that \(\pi(\gamma, 0, 0) = \gamma\) for every \(\gamma \in G[Y]\).

Proof. Take a countable family \(\{U_i\}_i\) of compact open bisections such that \(\{r(U_i)\}_i\) cover \(X \setminus Y\) and \(s(U_i) \subseteq Y\) for every \(i\). As \(X \setminus Y\) is compact, we may extract a finite subcollection \(U_1, \ldots, U_n\) such that \(r(U_1), \ldots, r(U_n)\) cover \(X \setminus Y\). Define compact open bisections recursively by putting

\[
V_1 = U_1, \quad V_i = U_i \setminus r\left(r^{-1}\left(\bigcup_{j=1}^{i-1} U_j\right)\right),
\]

for \(i = 2, \ldots, n\). Then \(X \setminus Y = \bigsqcup_{i=1}^n r(V_i)\) and \(s(V_i) \subseteq Y\) for every \(i\).

Given \(y \in Y\), let \(\lambda(y) = \{k \in \{1, \ldots, n\} \mid y \in s(V_k)\}\) and fix a bijection (a reordering)

\[
\alpha_y: \{k \in \mathbb{N} \mid k \leq |\lambda(y)|\} \rightarrow \lambda(y).
\]

Let us now define \(f \in C(Y, \mathbb{Z})\) by \(f(y) = |\lambda(y)|\). Next, we consider \(\theta: (G[Y])^{(0)}_f \rightarrow G\) given by

\[
\theta(y, i, j) = \left\{
\begin{array}{ll}
y & i = 0, \\
(s|V_{\alpha_y(i)})^{-1}(y) & i = 1, \ldots, f(i).
\end{array}
\right.
\]

The map \(\pi: (G[Y])_f \rightarrow G\) defined as

\[
\pi(\gamma, i, j) = \theta(r(\gamma), i, i) \cdot \gamma \cdot (s(\gamma), j, j)^{-1}
\]

is our sought isomorphism. It is straightforward to see that \(\pi\) is a groupoid homomorphism. To see that \(\pi\) is injective, suppose that

\[
\theta(r(\gamma), i, i) \cdot \gamma \cdot (s(\gamma), j, j)^{-1} = \theta(r(\gamma'), k, k) \cdot \gamma' \cdot (s(\gamma'), l, l)^{-1}.
\]

In particular, \(r(\theta(r(\gamma), i, i)) = r(\theta(r(\gamma'), k, k))\) is in some \(V_{i_0}\) from which it follows that \(i_0 = i = k\) and \(r(\theta(\gamma), i, i) = r(\theta(\gamma'), i, i)\). Hence \(r(\gamma) = r(\gamma')\). Similarly \(j = l\) and \(s(\gamma) = s(\gamma')\).

We see from \((4.5)\) that \(\gamma = \gamma'\) and so \(\pi\) is injective.

Let \(\gamma \in G\) and put \(x_1 = s(\gamma)\) and \(x_2 = r(\gamma)\). Let us choose \(\eta_1 \in r^{-1}(x_1) \cap s^{-1}(y_1)\) and \(\eta_2 \in r^{-1}(x_2) \cap s^{-1}(y_2)\), for some \(y_1, y_2 \in Y\). Then \(\eta = \eta_2^{-1} \gamma \eta_1 \in G_{y_1} \cap G^{y_2}\). Furthermore, \(x_1 \in r(V_{\alpha_1})\) and \(x_2 \in r(V_{\alpha_2})\) and so \(y_1 \in s(V_{\alpha_1})\) and \(y_2 \in s(V_{\alpha_2})\), for some unique \(a\) and \(b\). Hence, \(a = \alpha_{y_1}(j)\) and \(b = \alpha_{y_2}(i)\) for some unique \(i\) and \(j\). Now, \(\pi(\eta, i, j) = \gamma\) and so \(\pi\) is surjective. 

Lemma 4.2.4. Let \((G, X)\) be an étale LCH groupoid with \(X\) compact and totally disconnected and let \(Y_1, Y_2 \subseteq X\) be clopen \(G\)-full subsets. Then \(G[Y_1] = G\)-full.

Proof. Start by choosing \(f \in C(Y_1, \mathbb{Z})\) and an isomorphism \(\pi: (G[Y_1])_f \rightarrow G\) in accordance with Lemma 4.2.3. Consider the clopen subset of \(Y_1\) given by

\[
Z_1 = \{y \in Y_1 \mid \pi(y, k, k) \in Y_2, \text{ for some } k = 1, \ldots, f(y)\}.
\]

Given \(x \in X\), pick \(\gamma \in G_{y'} \cap G^x\), for some \(y' \in Y_2\). Then \(\pi^{-1}(y') = (y, k, k)\), for some \(y \in Z_1\) and \(k = 1, \ldots, f(y)\). Note that \(y' = r(\theta(y, k, k))\) and \(y = s(\theta(y, k, k))\) and so \(Z_1\) is \(G\)-full.

Define \(g \in C(Z_1, \mathbb{Z})\) by

\[
g(z) = \min\{k \in \{0, 1, \ldots, f(z)\} \mid \pi(z, k, k) \in Y_2\}
\]

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and put \( U = \{ \pi(z, g(z), 0) \mid z \in Z_1 \} \). This is a compact open bisection with \( s(U) = Z_1 \) and \( r(U) \subseteq Y_2 \). Indeed, \( s(\pi(z, g(z), 0)) = z \) and
\[
r(\pi(z, g(z), 0)) = r(\theta(z, g(z), g(z))) = \pi(z, g(z), g(z)) \in Y_2
\]
for every \( z \in Z_1 \), cf. \([4, 3]\). Now \( Z_2 := r(U) \) is \( G \)-full. To finish the proof, we note that there is an isomorphism \( G|Z_1 \to G|Z_2 \) given by
\[
\gamma \mapsto \pi(\gamma; g(z_1), g(z_2)) = \pi(z_2, g(z_2), 0) \gamma \pi(z_1, g(z_1), 0)^{-1}
\]
where \( z_1 = s(\gamma) \) and \( z_2 = r(\gamma) \) are in \( U \). \(\square\)

We are now ready to prove that Kakutani equivalence is a transitive relation.

**Lemma 4.2.5.** Kakutani equivalence is an equivalence relation.

**Proof.** Let \((G_i, X_i), i = 1, 2, 3\), be étale LCH groupoids with \( X_i \) compact and totally disconnected. Suppose \( G_1 \) is Kakutani equivalent to \( G_2 \) and that \( G_2 \) is Kakutani equivalent to \( G_3 \). Pick clopen and full subsets \( Y_1 \subseteq X_1 \), \( Y_2 \subseteq X_2 \) and \( Y_3 \subseteq X_3 \) together with isomorphisms \( \pi: G_1|Y_1 \to G_2|Y_2 \) and \( \pi': G_2|Y_2' \to G_3|Y_3 \). Note that \( \pi^{-1}(Y_2') \) is clopen and \( G_1 \)-full. Pick also clopen and full subsets \( Z \subseteq Y_2 \) and \( Z' \subseteq Y_2' \) such that \( G_2|Z \cong G_2|Z' \). Then \( \pi^{-1}(Z) \) and \( \pi'(Z) \) are clopen and full. As \( G_1|\pi^{-1}(Z) \cong G_3|\pi'(Z) \), we conclude that \( G_1 \) is Kakutani equivalent to \( G_3 \). \(\square\)

**Proposition 4.2.6.** Let \((G_i, X_i), i = 1, 2\), be étale LCH groupoids with \( X_i \) compact and totally disconnected. If \( G_1 \) is Kakutani equivalent to \( G_2 \), then the groupoids are homologically similar.

**Proof.** Start by choosing clopen \( G_i \)-full subsets \( Y_i \subseteq X_i \), \( i = 1, 2 \), such that \( G_1|Y_1 \cong G_2|Y_2 \). By Proposition \([4, 1, 10]\), \( G_1 \) is homologically similar to \( G_1|Y_1 \) and \( G_2 \) is homologically similar to \( G_2|Y_2 \). Hence \( G_1 \) is homologically similar to \( G_2 \). \(\square\)

**Proposition 4.2.7.** Let \((G, X)\) be an étale LCH groupoid with \( X \) compact and totally disconnected. The following are equivalent:

1. \((G, X)\) is compact and principal,
2. \((G, X)\) is Kakutani equivalent to a trivial groupoid.

**Proof.** (1) \(\Rightarrow\) (2): If \((G, X)\) is principal, we may consider \( G \subseteq X \times X \) to be an equivalence relation on \( X \). The only possible choice of a \( G \)-full subset \( Y \subseteq X \) rendering \( G|Y \) trivial is a complete list of representatives of the equivalences classes of \( G \), a transversal. However, a transversal need not be clopen in \( X \). By hypothesis, \((G, X)\) is a CEER, cf. Example \([3, 1, 9]\). If we let \( Y \) be the finite disjoint union of \( Y_1 \times \{1\}, \ldots, Y_n \times \{1\} \) (considered as a subset of \( X \)), we obtain a \( G \)-full and clopen subset such that \( G|Y \) is trivial. In particular, \( G \) is Kakutani equivalent to \( G|Y \).

(2) \(\Rightarrow\) (1): If \((G, X)\) is Kakutani equivalent to a trivial groupoid, then there is a clopen \( G \)-full subset \( Y \subseteq X \) such that \( G|Y \) is trivial. That is, \( G|Y = Y \). By Lemma \([4, 2, 3]\), there is an isomorphism \( G \cong (G|Y)_f = Y_f \) for some \( f \in C(Y, Z) \). In particular, \( G \) is compact. Next, let \( \gamma \in \Omega(X, x) \) be a loop at \( x \). Since \( Y \) is \( G \)-full there is a morphism \( \eta \in r^{-1}(x) \cap s^{-1}(y) \), for some \( y \in Y \), such that \( \eta^{-1} \gamma \eta = y \). It follows that \( \gamma = \eta \eta^{-1} = x \). Hence \((G, X)\) is principal. \(\square\)
4. Groupoid (co)homology

Combining Proposition 4.2.7, Example 4.1.5 and Proposition 4.2.6, we obtain the following.

Lemma 4.2.8. Let \((G, X)\) be an étale compact Hausdorff and principal groupoid with \(X\) totally disconnected. Then \(H_n(G; \Lambda) = 0\) for \(n \geq 1\) and any topological abelian group \(\Lambda\).

Definition 4.2.9. Let \((G, X)\) be an étale LCH groupoid with \(X\) compact and totally disconnected. A subgroupoid \((K, X) \subseteq (G, X)\) which is compact, open and principal is said to be elementary. Note that \(K^{(0)} = G^{(0)}\). The groupoid \((G, X)\) is an AF groupoid if it is an increasing union of elementary groupoids.

The meaning of the term »AF« stems from the fact that a \(C^\ast\)-algebra is AF (approximately finite dimensional) if and only if it is (isomorphic to) the reduced groupoid \(C^\ast\)-algebra of an AF groupoid, cf. Theorem 1.15 in [Ren80].

Theorem 4.2.10. Let \((G, X)\) be an AF groupoid and let \(\Lambda\) be any topological abelian group. Then \(H_n(G; \Lambda) = 0\) for \(n \geq 1\).

Proof. Let \(f \in C_c(G^{(n)}, \Lambda)\) be a cycle, that is, \(\delta_n(f) = 0\). It suffices to show that \(f \in \text{im}(\delta_{n+1})\). Since \((G, X)\) is AF, there exists an elementary subgroupoid \((K, X)\) such that the support of \(f\) is contained in \(K^{(n)}\). We may, therefore, consider \(f\) to be a cycle in \(C_c(K^{(n)}, \Lambda)\). By Lemma 4.2.8, the complex \(C_c(K^{(n)}, \Lambda)\) is exact at \(n\), so \(f \in \text{im}(\delta_{n+1}(C_c(K^{(n+1)}, \Lambda))) \subseteq \text{im}(\delta_{n+1})\). \(\square\)

We end the section by importing a classification result of AF groupoids. The reader is refered to [Mat12] for additional details and references.

Theorem 4.2.11. Let \((H, X)\) be an AF groupoid. Then there exists an isomorphism

\[
\pi: (H_0(H), H_0(H)^+) \longrightarrow (K_0(C^\ast_r(H)), K_0(C^\ast_r(H))^+)
\]

such that \(\pi([\chi_X]) = [1_{C^\ast_r(H)}]\).

4.3 Shift spaces revisited

Let us now restrict our attention even further. We shall return to the one-sided topological Markov shift \((X_A, \sigma_A)\) determined by the matrix \(A\) and its associated groupoid \(G_A\). As we have already seen, \(G_A\) is an example of an étale LCH groupoid which is essentially principal. Furthermore, \(X_A\) is a Cantor space (in particular, compact and totally disconnected) so we may apply the theory we have just established to these groupoids.

Lemma 4.3.1. Let \(G_A\) be the groupoid arising from the one-sided topological Markov shift \((X_A, \sigma_A)\). The subgroupoid \((H_A, X_A)\) inside \((G_A, X_A)\) given by

\[
H_A := \{(x, 0, y) \in G_A\}
\]

is an AF groupoid.

Proof. For each \(n \in \mathbb{Z}_+\), we put \(K_n = \{(x, 0, y) \in H_A \mid \sigma^n_A(x) = \sigma^n_A(y)\}\). Then \((K_n, X_A)\) is an elementary subgroupoid of \((H_A, X_A)\). The conclusion follows from the observation that \((K_n, X_A) \subseteq (K_{n+1}, X_A)\) for every \(n \in \mathbb{N}\) and that \(H_A = \bigcup_{n=1}^{\infty} K_n\). \(\square\)
Define a groupoid homomorphism \( \rho: G_A \to \mathbb{Z} \) by \( \rho(x, n, y) = n \) and consider the skew product \( G_A \times_\rho \mathbb{Z} \), cf. Example 3.1.11. The elements \((\gamma, n), (\gamma', m) \in G_A \times_\rho \mathbb{Z}\) are composable if and only if \((\gamma, \gamma') \in G_A^{(2)}\) and \(n + \rho(\gamma) = m\) in which case \((\gamma, n)(\gamma', m) = (\gamma\gamma', n)\). Inversion is given as \((\gamma, n)^{-1} = (\gamma^{-1}, n + \rho(\gamma))\) and so
\[
    s(\gamma, n) = (s(\gamma), n + \rho(\gamma)), \quad r(\gamma, n) = (r(\gamma), n),
\]
for \((\gamma, n) \in G_A \times_\rho \mathbb{Z}\). The unit space of \(G_A \times_\rho \mathbb{Z}\) is therefore \(X_A \times \mathbb{Z}\). There is an induced action \(\hat{\rho}: \mathbb{Z} \curvearrowright G_A \times_\rho \mathbb{Z}\) defined by
\[
    \hat{\rho}_m(\gamma, n) = (\gamma, n + m),
\]
for \((\gamma, n) \in G_A \times_\rho \mathbb{Z}\) and \(m \in \mathbb{Z}\). We put \(\hat{\rho} = \hat{\rho}_1\).

**Lemma 4.3.2.** The skew product \(G_A \times_\rho \mathbb{Z}\) is homologically similar to the AF groupoid \(H_A\).

**Proof.** Consider the clopen subset \(Y = X_A \times \{0\} \subseteq (G_A \times_\rho \mathbb{Z})^{(0)}\). If \((x, n) \in (G_A \times_\rho \mathbb{Z})^{(0)}\), then we can consider \(\gamma = (x, -n, y) \in G_A\) where \(y \in X_A\) is any sequence with \(\sigma_y^n(y) = x\). Then \(r(\gamma, n) = (x, n)\) and \(s(\gamma, n) = (y, 0) \in Y\) so \(Y\) is \(G_A\)-full. Observe that
\[
    (G_A \times_\rho \mathbb{Z})|Y = \{((\gamma, 0) \in G_A \times_\rho \mathbb{Z} \mid \rho(\gamma) = 0\} = \{((\gamma, 0) \in G_A \times_\rho \mathbb{Z} \mid \gamma \in H_A\},
\]
from which it follows that \((G_A \times_\rho \mathbb{Z})|Y\) is canonically isomorphic to \(H_A\). Hence \(G_A \times_\rho \mathbb{Z}\) is Kakutani equivalent to \(H_A\) and so the conclusion follows.

For skew products there is a Lyndon–Hochshild–Serre spectral sequence. The reader should consult [CM00] or Theorem 3.8 in [Mat12] for further details and proof.

**Theorem 4.3.3.** Let \((G, X)\) be an étale LCH groupoid and let \(\Gamma\) be a countable discrete group. Let \(\Lambda\) be a topological abelian group. If \(\rho: G \to \Gamma\) is a groupoid homomorphism, then there exists a spectral sequence
\[
    E^2_{p,q} = H_p(\Gamma; H_q(G \times_\rho \Gamma, \Lambda)) \Rightarrow H_{p+q}(G, \Lambda).
\]
Here, \(H_q(G \times_\rho \Gamma, \Lambda)\) is a \(\Gamma\)-module via the action on homology induced by \(\hat{\rho}: \Gamma \curvearrowright G \times_\rho \Gamma\).

The automorphism \(\hat{\rho}: G_A \times_\rho \mathbb{Z} \to G_A \times_\rho \mathbb{Z}\) induces an automorphism \(\hat{\rho}_*\) on \(C_c(X_A \times_\rho \mathbb{Z}, \mathbb{Z})\) given by
\[
    \hat{\rho}_*(f)(x, n) = \sum_{\hat{\rho}(y, m) = (x, n)} f(y, m) = f(x, n - 1),
\]
for \(f \in C_c(X_A \times \mathbb{Z}, \mathbb{Z})\) and \((x, n) \in X_A \times \mathbb{Z}\). What is the induced action \(H_0(\hat{\rho})\) on \(H_0(G_A \times_\rho \mathbb{Z})\)?

If \(V \subseteq X_A\) is a clopen subset, then
\[
    U = \{(x, 1, \sigma_A(x)) \mid x \in V\} \subseteq G_A
\]
is a compact open bisection. Furthermore, \(\sigma_A(V) \times \{0\} \subseteq X_A \times \mathbb{Z}\) is a clopen subset and \(\hat{\rho}(\sigma_A(V) \times \{0\}) = \sigma_A(V) \times \{1\}\). Since
\[
    s(U \times \{0\}) = s(U) \times \{1\} = \sigma_A(V) \times \{1\},
\]
\[
    r(U \times \{0\}) = r(U) \times \{0\} = V \times \{0\}
\]
a swift computation shows that
\[ s_\ast(\chi_{U \times \{0\}}) = \chi_s(U) \times \{1\} = \chi_{A(V) \times \{1\}}, \]
\[ r_\ast(\chi_{U \times \{0\}}) = \chi_r(U) \times \{0\} = \chi_{V \times \{0\}}. \]

Now, the fact that \( U \times \{0\} \) is a compact open \((G_A \times \Bbb Z)\)-bisection implies that the maps \( \chi_{A(V) \times \{1\}} \) and \( \chi_{V \times \{0\}} \) in \( C_c(X_A \times \Bbb Z, \Bbb Z) \) determine the same homology class in \( H_0(G_A \times \Bbb Z) \). Therefore, the automorphism \( H_0(\tilde{\rho}) \) takes the class of \( \chi_{V \times \{0\}} \) to the class of \( \chi_{A(V) \times \{0\}} \) and we shall identify this action with the shift action \( \tilde{\sigma} \) of Lemma 2.3.7.

**Theorem 4.3.4.** Let \( G_A \) be the étale LCH groupoid arising from the one-sided topological Markov shift \((X_A, \sigma_A)\). Its homology is given as
\[
H_n(G_A) \cong \begin{cases} 
K_0(C^*_r(G_A)) & n = 0, \\
K_1(C^*_r(G_A)) & n = 1, \\
0 & n \geq 2.
\end{cases}
\]

**Proof.** Let \( H_A \) be the AF subgroupoid of \( G_A \) discussed above and recall that \( H_n(H_A) = 0 \) whenever \( n \geq 1 \), cf. Theorem 4.2.10. By Theorem 4.2.11, we also have \( H_0(H_A) \cong K_0(C^*_r(H_A)) \) and we saw in Proposition 3.4.3 that \( C^*_r(H_A) \cong \mathcal{F}_A \). If \( \rho: G_A \to \Bbb Z \) is the groupoid homomorphism \( \rho(x, n, y) = n \), then the skew product \( G_A \times \rho \Bbb Z \) is homologically similar to \( H_A \). Therefore, \( H_n(G_A \times \rho \Bbb Z) \) vanishes for \( n \geq 1 \) and \( H_0(G_A \times \rho \Bbb Z) \cong H_0(H_A) \).

By Theorem 4.3.3, there is a spectral sequence
\[
E^2_{p,q} = H_p(\Bbb Z; H_q(G_A \times \rho \Bbb Z)) \implies H_{p+q}(G_A)
\]
and the group \( H_0(G_A \times \rho \Bbb Z) \cong K_0(C^*_r(H_A)) \) is a \( \Bbb Z \)-module specified by the shift automorphism \( \tilde{\sigma} \) of Lemma 2.3.7. From the spectral sequence and Example 4.1.4, we thus have
\[
H_0(G_A) \cong H_0(\Bbb Z; K_0(C^*_r(H_A))) \cong \operatorname{coker}(\text{id} - \tilde{\sigma}) \cong K_0(O_A),
\]
\[
H_1(G_A) \cong H_1(\Bbb Z; K_0(C^*_r(H_A))) \cong \ker(\text{id} - \tilde{\sigma}) \cong K_1(O_A).
\]
The last isomorphism follows from Theorem 2.3.8.
5.1 Flow equivalence

In this section, we give a brief account of the notion of flow equivalence between two-sided shift spaces. We provide the reader with relevant theorems which will be useful to us later. There are no proofs in this section.

Given a two-sided topological Markov shift \((\bar{X}_A, \bar{\sigma}_A)\), its suspension flow space is the quotient

\[
F_A = \frac{\bar{X}_A \times \mathbb{R}}{(x, s + 1) \sim (\bar{\sigma}_A(x), s)},
\]

which, equipped with the quotient topology, is a compact space. The class of an element \((x, s) \in \bar{X}_A \times \mathbb{R}\) is denoted \([x, s]\). A continuous flow on any compact space \(X\) is a family of homeomorphisms \((\phi_t)_{t \in \mathbb{R}}: X \to X\) in which each \(\phi_t\) is continuous in both variables and \(\phi_t \circ \phi_s = \phi_{s+t}\), for any \(s, t \in \mathbb{R}\). The set \(\{\phi_t(x) \mid t \in \mathbb{R}\}\) is called the orbit of \(x \in X\). Given \(t \in \mathbb{R}\), the map \(\phi_t: F_A \to F_A\) given by

\[
\phi_t([x, s]) = [x, s + t],
\]

is a well-defined homeomorphism on \(F_A\) and the family \((\phi_t)_{t \in \mathbb{R}}\) constitutes a continuous flow on \(F_A\) called the suspension flow over \(\bar{X}_A\).

**Definition 5.1.1.** Let \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_B, \bar{\sigma}_B)\) be two-sided topological Markov shifts and let \((\phi_t)_{t \in \mathbb{R}}\) and \((\psi_t)_{t \in \mathbb{R}}\) be their associated suspension flows. The shift spaces are flow equivalent if their suspension flows are equivalent. That is, if there exists a homeomorphism \(\tau: \bar{X}_A \to \bar{X}_B\) such that for each \(x \in \bar{X}_A\) there is a monotonically increasing map \(f_x: \mathbb{R} \to \mathbb{R}\) with the property that

\[
\tau(\phi_t(x)) = \psi_{f_x(t)}(\tau(x)),
\]

for every \(t \in \mathbb{R}\).

The above definition is usually phrased by saying that shift spaces are flow equivalent if there is a homeomorphism which takes orbits to orbits in an order preserving way. Note that conjugate shift spaces are flow equivalent but the converse is not true in general.

**Definition 5.1.2.** Given an \(N \times N\)-matrix \(A\) over \(\{0, 1\}\) the Bowen Franks group is defined as

\[
BF(A) = \frac{\mathbb{Z}^N}{(I - A)\mathbb{Z}^N}.
\]
We let \( u_A \) denote the class of \( (1, \ldots, 1) \in \mathbb{Z}^N \) in BF(A).

In their paper [BF77], the authors R. Bowen and J. Franks introduced the group above and showed that it is an invariant of flow equivalence. The reader will note that we have already encountered this group as the \( K_0 \)-group of the Cuntz-Krieger algebra \( \mathcal{O}_A \), see Section 2.3. Though the groups BF(A) and BF(A') are isomorphic, there is no canonical isomorphism between them and so we shall distinguish between them. In a notoriously short paper, W. Parry and D. Sullivan ([PS75]) showed that \( \det(I - A) \) is also an invariant of flow equivalence. Combining these two results, J. Franks has proved that the two constitute a complete invariant. See [Fra84] for a proof.

**Theorem 5.1.3.** Let \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_B, \bar{\sigma}_B)\) be irreducible two-sided topological Markov shifts. The following are equivalent.

1. The systems \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_B, \bar{\sigma}_B)\) are flow equivalent,
2. We have BF(A) \(\cong\) BF(B) and \(\det(I - A) = \det(I - B)\).

In their paper [CK80] (more specifically, Theorem 4.1), J. Cuntz and W. Krieger gave a computational proof of the fact that the pair \((\bar{\mathcal{O}}_A, \bar{\mathcal{D}}_A)\) is an invariant of flow equivalence. Here, \(\bar{\mathcal{O}}_A = \mathcal{O}_A \otimes \mathcal{K}\) and \(\bar{\mathcal{D}}_A = \mathcal{D}_A \otimes \bar{\mathcal{C}}\) and \(\mathcal{K}\) is the compact operators on \(\ell^2(\mathbb{N})\) while \(\bar{\mathcal{C}}\) is the diagonal compact operators.

**Theorem 5.1.4.** Suppose the two-sided shift spaces \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_B, \bar{\sigma}_B)\) are flow equivalent. Then there exists a diagonal preserving \(C^*\)-isomorphism \((\bar{\mathcal{O}}_A, \bar{\mathcal{D}}_A) \rightarrow (\bar{\mathcal{O}}_B, \bar{\mathcal{D}}_B)\).

The automorphism \(\bar{\sigma}_A\) on \(\bar{X}_A\) gives rise to an action \(\mathbb{Z} \curvearrowright \bar{X}_A\). The low-dimensional cohomology groups (with integral coefficients) of this topological dynamical system are

\[
H^0(\mathbb{Z}, \bar{X}_A) = \{ \xi \in C(\bar{X}_A, \mathbb{Z}) \mid \xi = \xi \circ \bar{\sigma}_A \}
\]

and

\[
\bar{H}^A(\bar{X}_A) := H^1(\mathbb{Z}, \bar{X}_A) = \frac{C(\bar{X}_A, \mathbb{Z})}{[\xi \circ \bar{\sigma}_A - \xi \mid \xi \in C(\bar{X}_A, \mathbb{Z})]}.
\]

If \([\xi]\) denotes the class of \(\xi \in C(\bar{X}_A, \mathbb{Z})\) in \(\bar{H}^A(\bar{X}_A)\), then we define the positive cone as

\[
\bar{H}^A_+ = \{ [\xi] \in \bar{H}^A \mid \xi \geq 0 \}.
\]

We shall refer to the pair \((\bar{H}^A, \bar{H}^A_+)\) as the ordered cohomology of the two-sided Markov shift \((\bar{X}_A, \bar{\sigma}_A)\). Next, we state the following remarkable theorem by M. Boyle and D. Handelman. A proof can be found in [BH96].

**Theorem 5.1.5.** Let \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_B, \bar{\sigma}_B)\) be irreducible two-sided topological Markov shifts. The following are equivalent:

1. The systems \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_B, \bar{\sigma}_B)\) are flow equivalent,
2. The ordered cohomology groups \((\bar{H}^A, \bar{H}^A_+)\) and \((\bar{H}^B, \bar{H}^B_+)\) are isomorphic.

A subset \(\bar{O} \subseteq \bar{X}_A\) is \(\bar{\sigma}_A\)-invariant if \(\bar{\sigma}_A(\bar{O}) = \bar{O}\). We record a characterization of elements in the positive cone in terms of finite \(\bar{\sigma}_A\)-invariant sets.
Proposition 5.1.6. Let $\tilde{X}_A, \tilde{\sigma}_A$ be a two-sided topological Markov shift and let $\xi \in C(\tilde{X}_A, \mathbb{Z})$. Then $[\xi] \in H^\perp_A$ if and only if $\sum_{y \in \tilde{O}} \xi(y) > 0$, for every finite $\tilde{\sigma}_A$-invariant set $\tilde{O} \subseteq \tilde{X}_A$.

See Proposition 3.13 in [BH96] for a proof.

5.2 $K$-theoretic considerations

In this section, we prove that an isomorphism $K_0(O_A) \longrightarrow K_0(O_B)$ sending $[1_A]$ to $[1_B]$ induces a diagonal preserving $C^*$-isomorphism $(O_A, D_A) \longrightarrow (O_B, D_B)$ provided that $\det(I - A) = \det(I - B)$ holds. We mainly follow the ideas of K. Matsumoto in [Mat13].

Let $\text{Proj}(D_A)$ be the collection of projections in $D_A$ and consider the normalizing partial isometries $N_s(O_A, D_A) = \{V \in O_A \mid V D_A V^* \subseteq D_A, V^* D_A V \subseteq D_A\}$. We define an equivalence relation on $\text{Proj}(D_A)$ in the following way: Given $E, F \in \text{Proj}(D_A)$, we declare that $E \sim F$ if and only if there is a normalizing partial isometry $V \in N_s(O_A, D_A)$ such that $V^* V = E$ and $V V^* = F$. Two projections $E, F \in \text{Proj}(D_A)$ are orthogonal if $EF = 0$ in which case we write $E \perp F$. Given two projections $P$ and $Q$ in any $C^*$-algebra $B$, we let $P \sim Q$ denote Murray-von Neumann equivalence in $B$.

In this section, a partial homeomorphism $\tau$ on $X_A$ is a homeomorphism between clopen subsets of $X_A$. We let $D(\tau)$ and $R(\tau)$ denote the domain and range of $\tau$, respectively. The collection of partial homeomorphisms $\tau$ with the property that $\tau(x) \in \text{orb}_{A}(x)$, for every $x \in D(\tau)$, is denoted $[\sigma_A]_s$. This has the structure of an inverse semigroup. Let $[\sigma_A]_{sc}$ be the sub-inverse semigroup of partial homeomorphisms $\tau \in [\sigma_A]_s$ such that there are continuous maps $k, l: X_A \longrightarrow \mathbb{Z}_+$ satisfying

$$\sigma_A^k(x)(\tau(x)) = \sigma_A^l(x)(x),$$

for $x \in D(\tau)$. The following result is Proposition 3.1 in [Mat13] (or Proposition 6.4 in [Mat09]).

Proposition 5.2.1. Given $\tau \in [\sigma_A]_{sc}$, there exists $U_\tau \in N_s(O_A, D_A)$ such that

$$\text{Ad}(U_\tau)(f) = f \circ \tau^{-1},$$

$$\text{Ad}(U_\tau)(g) = g \circ \tau,$$

for $f \in C(D(\tau))$ and $g \in C(R(\tau))$.

Remark 5.2.2. If $\alpha$ is a word, we shall write $\chi_{\alpha}$ instead of $\chi_{N_\alpha}$.

Lemma 5.2.3. Given $E_1, \ldots, E_n \in \text{Proj}(D_A)$ there exist $F_1, \ldots, F_n \in \text{Proj}(D_A)$ such that

$$E_i \sim_{D_A} F_i \perp F_j \sim_{D_A} E_j$$

for every $i \neq j$.

Proof. Fix $i = 1, \ldots, n$. We identify $D_A$ with $C(X_A)$ in which any projection is a finite sum of characteristic maps of cylinder sets. We may therefore assume that $E_i$ is of the form

$$E_i = \sum_{j=1}^{k_i} \chi_{\alpha^i(j)},$$

with $\alpha^i(j)$ ranging over all factors of $\alpha$. Then

$$E_i \sim_{D_A} F_i \perp F_j \sim_{D_A} E_j$$

for every $i \neq j$, as desired.
where \( \alpha^i(1), \ldots, \alpha^i(k_i) \) are admissible words of same length. Pick \( n \) distinct words \( \xi_1, \ldots, \xi_n \) of same length. Since \( A \) is assumed irreducible, there are words \( \eta^j(1), \ldots, \eta^j(k_i) \) of same length such that \( \xi_j \eta^j(i) \alpha^i(j) \) is admissible for every \( j = 1, \ldots, k_i \).

Define a partial homeomorphism \( \tau_\iota \) on \( X_A \) with \( D(\tau_\iota) = \bigcup_{j=1}^{k_i} N_{\alpha^i(j)} \) by

\[
\tau_\iota(x) = \xi_j \eta^j(i) x \in N_{\xi_j \eta^j(i) \alpha^i(j)},
\]

for \( x \in N_{\alpha^i(j)} \). Then \( R(\tau_\iota) = \bigcup_{j=1}^{k_i} N_{\xi_j \eta^j(i) \alpha^i(j)} \) and \( \tau_\iota \in [\sigma_A]_{sc} \).

Put \( F_i = \chi_{R(\tau_\iota)} \) and \( U_{\tau_\iota} \in N_{\alpha}(O_A, D_A) \) in accordance with Proposition 5.2.1 since the words \( \xi_1, \ldots, \xi_n \) are distinct, the projections \( F_1, \ldots, F_n \) are mutually disjoint. Furthermore,

\[
U_{\tau_\iota} U_{\tau_\iota}^* = \text{Ad}(U_{\tau_\iota})(\chi_{D(\tau_\iota)}) = \chi_{D(\tau_\iota)} \circ \tau_\iota^{-1} = E_i,
\]

\[
U_{\tau_\iota}^* U_{\tau_\iota} = \text{Ad}(U_{\tau_\iota})(\chi_{R(\tau_\iota)}) = \chi_{R(\tau_\iota)} \circ \tau_\iota = F_i
\]

and so \( E_i \sim F_i \).

**Lemma 5.2.4.** Given \( E, F \in \text{Proj}(D_A) \), there exists \( E' \in \text{Proj}(D_A) \) such that \( E \sim F' \).

**Proof.** Write

\[
E = \sum_{i=1}^{K} \chi_{\alpha(i)}, \quad F = \sum_{j=1}^{L} \chi_{\beta(j)},
\]

where \( \alpha(i) \) and \( \beta(j) \) are admissible words of length \( k \) and \( l \), respectively. We may assume that \( L > 1 \). Since \( A \) satisfies condition (1), there exist \( J \in \mathcal{A} \) together with words \( \xi(1), \ldots, \xi(K) \) of same length such that \( \beta(1) \xi(i) \) is admissible for every \( i \). As \( A \) is also irreducible there are words \( \eta(1), \ldots, \eta(K) \) such that \( J \eta(i) \alpha(i) \) is admissible for every \( i \).

Define a partial homeomorphism \( \tau \) on \( X_A \) with \( D(\tau) = \bigcup_{i=1}^{K} N_{\alpha(i)} \) by

\[
\tau(x) = \beta(1) \xi(i) \eta(i) x \in N_{\beta(1)},
\]

for \( x \in N_{\alpha(i)} \). Then \( R(\tau) = \tau(\bigcup_{i=1}^{K} N_{\alpha(i)}) \subset N_{\beta(1)} \) and \( \tau \in [\sigma_A]_{sc} \).

Pick \( U_{\tau} \in N_{\alpha}(O_A, D_A) \) as in Proposition 5.2.1 and note that \( (EU_{\tau}^*)(U_{\tau} E) = E \) while \( E' := U_{\tau} EU_{\tau}^* = \sum_{i=1}^{K} \chi_{\alpha(i)} \circ \tau^{-1} < F \).

Let \( [E]_{D_A} \) denote the equivalence class of \( E \in \text{Proj}(D_A) \) under the relation \( \sim_{D_A} \) and put

\[
K_0(O_A, D_A) = \{ [E]_{D_A} \mid 0 \neq E \in \text{Proj}(D_A) \}.
\]

(5.3)

This has a natural structure of an abelian semigroup via the addition

\[
[E]_{D_A} + [F]_{D_A} = [E' + F']_{D_A},
\]

(5.4)

where \( E', F' \in \text{Proj}(D_A) \) are chosen such that \( E \sim_{D_A} E' \perp F' \sim_{D_A} F \). It is readily verified that the addition is independent of the specific choice of orthogonal projections \( E' \) and \( F' \). The proof of the fact that \( K_0(O_A, D_A) \) is a group is practically identical to the proof of Theorem 4.1 in [Cun81b]. We include a proof only for completeness.
Lemma 5.2.5. Let $E, F \in \text{Proj}(\mathcal{D}_A)$ be non-zero orthogonal projections such that $E \sim F$. If $E, F < D$ for some $D \in \text{Proj}(\mathcal{D}_A)$, then $D - E \sim D - F$.

Proof. Take $V \in N_s(\mathcal{O}_A, \mathcal{D}_A)$ with $V^*V = E$ and $VV^* = F$ and put $U = (D - E - F) + V$. Then $U \in N_s(\mathcal{O}_A, \mathcal{D}_A)$ and $U^*U = D - F$ and $UU^* = D - E$. \hfill $\Box$

Theorem 5.2.6. The semigroup $K_0(\mathcal{O}_A, \mathcal{D}_A)$ is an abelian group.

Proof. The set $K_0(\mathcal{O}_A, \mathcal{D}_A)$ is a semigroup with addition given as in [5.4]. It remains to find a neutral element and inverses.

Let $E, F \in \text{Proj}(\mathcal{D}_A)$ be non-zero projections and take non-zero $E', F' \in \text{Proj}(\mathcal{D}_A)$ such that $E \sim E' < E$ and $F \sim F' < F$. We may assume that $F \leq E'$ (by Lemma 5.2.3) so that $(E - E') \perp (F - F')$ and

$$[E - E']_{\mathcal{D}_A} + [F - F']_{\mathcal{D}_A} = [E - (E' - F + F')]_{\mathcal{D}_A}.$$ 

Now, if $U \in N_s(\mathcal{O}_A, \mathcal{D}_A)$ satisfies $U^*U = E$ and $UU^* = F$, then $(E' - F) + U \in N_s(\mathcal{O}_A, \mathcal{D}_A)$ witnesses the relation $E' \sim (E - F)$. Now, if $E \sim E'' \leq (E - E')$ then $E'' \perp E'$ and $E'' \perp F'$. By the above lemma, we thus have $(E - E') \sim (E - E'') \sim E - (E' - F + F')$ so that $[E - E']_{\mathcal{D}_A} = [E - (E' - F + F')]_{\mathcal{D}_A}$. Hence

$$[E - E']_{\mathcal{D}_A} + [F - F']_{\mathcal{D}_A} = [E - E']_{\mathcal{D}_A}.$$ 

A similar argument shows that $[E - E']_{\mathcal{D}_A} + [F - F']_{\mathcal{D}_A} = [E - E']_{\mathcal{D}_A}$. It follows that $[E - E']_{\mathcal{D}_A} = [F - F']_{\mathcal{D}_A}$ is the neutral element in $K_0(\mathcal{O}_A, \mathcal{D}_A)$.

If $F \sim F', F'' < F$ with $F' \perp F''$, then $F - F' - F''$ is a (non-zero) projection in $\mathcal{D}_A$ and

$$[F]_{\mathcal{D}_A} + [F - F' - F'']_{\mathcal{D}_A} = [F - F' - F'' + F'']_{\mathcal{D}_A} = [F - F'']_{\mathcal{D}_A}$$

showing that $[F - F' - F'']_{\mathcal{D}_A}$ is the inverse element of $[F]_{\mathcal{D}_A}$. \hfill $\Box$

Any (non-zero) projection in $\mathcal{D}_A$ is a finite sum of projections of the form $S_\alpha S_\alpha^*$, for $\alpha \in \mathbb{A} \subset \mathbb{N}$. Furthermore,

$$S_\alpha S_\alpha^* \sim S_\alpha^* S_\alpha = S_{r(\alpha)}^* S_{r(\alpha)} \sim S_{r(\alpha)} S_{r(\alpha)}^*.$$ 

This proves the following lemma.

Lemma 5.2.7. The group $K_0(\mathcal{O}_A, \mathcal{D}_A)$ is generated by $[S_1 S_1^*]_{\mathcal{D}_A}, \ldots, [S_N S_N^*]_{\mathcal{D}_A}$.

Let $\epsilon_i = (0, \ldots, 1, \ldots, 0)$ be the standard basis of $\mathbb{Z}^N$ and let $[\epsilon_i]$ denote the class of $\epsilon_i$ in the quotient $\mathbb{Z}^N/(I - A')\mathbb{Z}^N$. In [Cum81b], J. Cuntz showed that $K_0(\mathcal{O}_A)$ is realized as the Murray-von Neumann classes of the non-zero projections in $\mathcal{O}_A$.

Theorem 5.2.8. The map $\eta: K_0(\mathcal{O}_A, \mathcal{D}_A) \rightarrow K_0(\mathcal{O}_A)$ given by $[S_i S_i^*]_{\mathcal{D}_A} \mapsto [S_i S_i^*]$ defines an isomorphism. In particular, $[E]_{\mathcal{D}_A} = [F]_{\mathcal{D}_A}$ in $K_0(\mathcal{O}_A, \mathcal{D}_A)$ if and only if $[E] = [F]$ in $K_0(\mathcal{O}_A)$.
5. Classification

Proof. The map $\delta : K_0(\mathcal{O}_A) \to \mathbb{Z}^N/(I - A^t)\mathbb{Z}^N$ given by $\delta : [S_iS_j^*] \mapsto [\epsilon_i]$ is an isomorphism. By Lemma 5.2.7, the map $\gamma : \mathbb{Z}^N \to K_0(\mathcal{O}_A, \mathcal{D}_A)$ defined by

$$\gamma : \epsilon_i \mapsto [S_iS_i^*]_{\mathcal{D}_A}$$

is a surjection. Let $\tilde{\gamma} : \mathbb{Z}^N/\ker(\gamma) \to K_0(\mathcal{O}_A, \mathcal{D}_A)$ denote the induced isomorphism. Since

$$\gamma(\epsilon_i) = [S_iS_i^*]_{\mathcal{D}_A} = [S_i^*S_i]_{\mathcal{D}_A} = \sum_{j=1}^N A(i, j)[S_jS_j^*]_{\mathcal{D}_A} = \sum_{j=1}^N A(i, j)\gamma(\epsilon_j)$$

we see that

$$0 = \gamma(\epsilon_i - \sum_{j=1}^N A(i, j)\epsilon_j) = \gamma(\epsilon_i - A^t\epsilon_i) = \gamma((I - A^t)\epsilon_i).$$

Hence $(I - A^t)\mathbb{Z}^N \subseteq \ker(\gamma)$. Let $\xi : \mathbb{Z}^N/(I - A^t)\mathbb{Z}^N \to \mathbb{Z}^N/\ker(\gamma)$ denote the induced surjection.

The composition $\delta \circ \eta : K_0(\mathcal{O}_A, \mathcal{D}_A) \to \mathbb{Z}^N/(I - A^t)\mathbb{Z}^N$ is a surjection with inverse given as $\tilde{\gamma} \circ \xi : \mathbb{Z}^N/(I - A^t)\mathbb{Z}^N \to K_0(\mathcal{O}_A, \mathcal{D}_A)$. In particular, $\delta \circ \eta$ is injective. It follows that $\eta$ is injective and thus an isomorphism. \hfill $\square$

Consider the Hilbert space $\ell^2(\mathbb{N})$ and let $\mathbb{K}$ denote the compact operators on $\ell^2(\mathbb{N})$. Let $\mathcal{C}$ be the diagonal compact operators. We put

$$\mathcal{O}_A := \mathcal{O}_A \otimes \mathbb{K}, \quad \mathcal{D}_A := \mathcal{D}_A \otimes \mathcal{C}$$

and make the identification $\operatorname{Proj}(\mathcal{D}_A) = c_{00}(\mathbb{N}, \operatorname{Proj}(\mathcal{D}_A))$, where the right hand side denotes the projection-valued sequences with finite support. Similar to the above construction, let $\mathcal{N}_s(\mathcal{O}_A, \mathcal{D}_A)$ be the collection of normalizing partial isometries. We introduce the relation $\sim$ on $\operatorname{Proj}(\mathcal{D}_A)$ by declaring $P \sim Q$ if and only if there exists $V \in \mathcal{N}_s(\mathcal{O}_A, \mathcal{D}_A)$ such that $V^*V = P$ and $VV^* = Q$.

Lemma 5.2.9. Given $P \in \operatorname{Proj}(\mathcal{D}_A)$, there exists $E \in \operatorname{Proj}(\mathcal{D}_A)$ such that $P \sim E$, where $P_E = (E, 0, \ldots) \in \operatorname{Proj}(\mathcal{D}_A)$.

Proof. Write $P = (P_1, \ldots, P_n, 0, \ldots)$ for some projections $P_i \in \operatorname{Proj}(\mathcal{D}_A)$. Choose projections $Q_1, \ldots, Q_n \in \operatorname{Proj}(\mathcal{D}_A)$ such that $P_i \sim_{\mathcal{D}_A} Q_i \perp Q_j \sim_{\mathcal{D}_A} P_j$ for $i \neq j$, cf. Lemma 5.2.3. Choose $V_i \in \mathcal{N}_s(\mathcal{O}_A, \mathcal{D}_A)$ such that $V_i^*V_i = P_i$ and $V_iV_i^* = Q_i$. Let $V = \operatorname{diag}(V_1, \ldots, V_n, 0, \ldots) \in \mathcal{O}_A$ and consider the matrix

$$A_n = \begin{pmatrix}
1_A & \cdots & 1_A & 0 & \cdots \\
0 & 0 & & & \\
& & & & \\
& & 0 & & \\
& & & & \\
& & & & \\
\end{pmatrix}.\$$

Put $E = Q_1 + \cdots + Q_n$ and observe that $(A_nV)(V^*A_n^*) = P_E$ while $(V^*A_n^*)(A_nV) = P$. Since the partial isometry $A_nV$ is in $\mathcal{N}_s(\mathcal{O}_A, \mathcal{D}_A)$ this proves the assertion. \hfill $\square$

The next proposition plays a key rôle in the theorem below.
Proposition 5.2.10. Let $P, Q \in \text{Proj}(\mathcal{D}_A)$ and suppose $P \sim Q$. Then $P \sim Q$.

Proof. Start by choosing projections $E, F \in \text{Proj}(\mathcal{D}_A)$ such that
\[
P \sim_{\mathcal{D}_A} (E, 0, \ldots), \quad Q \sim_{\mathcal{D}_A} (F, 0, \ldots),
\]
by the above lemma. Since $P \sim Q$, it follows that $(E, 0, \ldots) \sim (F, 0, \ldots)$ in $\mathcal{O}_A$. Pick a partial isometry $V \in \mathcal{O}_A$ witnessing this relation and let $1_1 = (1_A, 0, \ldots)$. By regarding $1_1V_1$ as an element in $\mathcal{O}_A$, this witnesses the relation $E \sim F$ in $\mathcal{O}_A$. By Theorem 5.2.8 we now have $E \sim F$. Consequently, $(E, 0, \ldots) \sim_{\mathcal{D}_A} (F, 0, \ldots)$. Indeed, if $U \in N_s(\mathcal{O}_A, \mathcal{D}_A)$ satisfies $U^*U = E$ and $UU^* = F$, then $\bar{U} = \text{diag}(U, 0, \ldots) \in N_s(\mathcal{O}_A, \mathcal{D}_A)$ and $\bar{U}^* \bar{U} = (E, 0, \ldots)$ and $\bar{U}U^* = (F, 0, \ldots)$. Finally,
\[
P \sim_{\mathcal{D}_A} (E, 0, \ldots) \sim_{\mathcal{D}_A} (F, 0, \ldots) \sim_{\mathcal{D}_A} Q
\]
from which the conclusion follows.

Theorem 5.2.11. Let $A$ and $B$ be irreducible $N \times N$-matrices over $\{0,1\}$ satisfying condition (I) such that $\det(I - A) = \det(I - B)$. If $\alpha : K_0(\mathcal{O}_A) \to K_0(\mathcal{O}_B)$ is an isomorphism with $\alpha([1_A]) = [1_B]$, then there exists a $C^*$-isomorphism $\Psi : (\mathcal{O}_A, \mathcal{D}_A) \to (\mathcal{O}_B, \mathcal{D}_B)$ such that $K_0(\Psi) = \alpha$.

Proof. Recall from Theorem 2.3.8 that $K_0(\mathcal{O}_A) = BF(A^t)$ and $K_0(\mathcal{O}_B) = BF(B^t)$. Since $\alpha : BF(A^t) \cong BF(B^t)$ and $\det(I - A) = \det(I - B)$, Theorem 5.1.3 implies that the two-sided topological Markov shifts $\langle X_A, \sigma_A \rangle$ and $\langle X_B, \sigma_B \rangle$ are flow equivalent. It follows from Theorem 5.1.4 that there is a diagonal-preserving $C^*$-isomorphism
\[
\varphi : (\mathcal{O}_A, \mathcal{D}_A) \to (\mathcal{O}_B, \mathcal{D}_B).
\]

We define the automorphism $\beta := \alpha \circ K_0(\varphi)^{-1}$ on $K_0(\mathcal{O}_B)$. By Huang’s theorem (see Theorem 2.15 in [Hua94]), any automorphism on $K_0(\mathcal{O}_B) = BF(B^t)$ is induced by a flow equivalence. This, in turn, gives rise to a diagonal-preserving $C^*$-automorphism $\psi$ on $(\mathcal{O}_B, \mathcal{D}_B)$ such that $K_0(\psi) = \beta$. Now, the composition $\psi \circ \varphi : (\mathcal{O}_A, \mathcal{D}_A) \to (\mathcal{O}_B, \mathcal{D}_B)$ satisfies
\[
[\psi \circ \varphi(1_A \otimes e_1)] = \beta([\varphi(1_A \otimes e_1)] = \alpha([1_A \otimes e_1]) = [1_B \otimes e_1]
\]
in $K_0(\mathcal{O}_B)$. Since $\psi \circ \varphi(1_A \otimes e_1)$ and $1_B \otimes e_1$ are projections in $\mathcal{O}_B$, Proposition 5.2.10 implies the existence of a partial isometry $V \in N_s(\mathcal{O}_B, \mathcal{D}_B)$ which satisfies
\[
V^*V = \psi \circ \varphi(1_A \otimes e_1), \quad VV^* = 1_B \otimes e_1.
\]

Put $P = 1_B \otimes e_1$. For $a \in \mathcal{O}_A$, we have
\[
\text{Ad}(V)(\psi \circ \varphi(a \otimes e_1)) = PV(\psi \circ \varphi(a \otimes e_1))V^*P \in P(\mathcal{O}_B \otimes \mathbb{C})P = \mathcal{O}_B \otimes \mathbb{C}e_1
\]
so that $\text{Ad}(V) \circ \psi \circ \varphi$ restricts to a $C^*$-isomorphism $\mathcal{O}_A \otimes \mathbb{C}e_1 \to \mathcal{O}_B \otimes \mathbb{C}e_1$. Furthermore – and more importantly – this isomorphism preserves the diagonal since $V \in N_s(\mathcal{O}_A, \mathcal{D}_A)$. This finishes the proof.
5. Classification

5.3 Five-term classification

Next, we consider the recent contributions of H. Matui and K. Matsumoto in [MM14]. Similar to the construction of the ordered cohomology, the authors introduce the pair \((H^A, H_+^A)\) as

\[
H^A := \frac{C(X_A, \mathbb{Z})}{\{\xi \circ \sigma_A - \xi \mid \xi \in C(X_A, \mathbb{Z})\}}, \quad H_+^A := \{[\xi] \in H^A \mid \xi \geq 0\}.
\]

Recall from Section 1.1 the map \(\rho: X_A \rightarrow X_A\) which we call the cut. We shall see that the pairs \((H^A, H_+^A)\) and \((H^A, H_+^A)\) are isomorphic.

**Lemma 5.3.1.** For every \(\zeta \in C(X_A, \mathbb{Z})\), there exist \(M \in \mathbb{N}\) and \(\xi \in C(X_A, \mathbb{Z})\) such that

\[
\zeta \circ \sigma_A^M = \xi \circ \rho.
\] (5.5)

**Proof.** The map

\[
d(\bar{x}, \bar{y}) = \frac{1}{\min\{n \in \mathbb{N} \mid x_n \neq y_n\} + 1}
\]

for \(\bar{x}, \bar{y} \in X_A\), defines a metric which induces the topology on \(X_A\). Any \(\zeta \in C(X_A, \mathbb{Z})\) is uniformly continuous and so there exists \(M \in \mathbb{N}\) such that \(\zeta(\bar{x}) = \zeta(\bar{y})\) whenever \(d(\bar{x}, \bar{y}) < 1/M\), since \(\mathbb{Z}\) is discrete. Hence the integer \(\zeta(\bar{x})\) is completely determined by the finite string \(x_{[-M, M]}\). Now, if \(x \in X_A\), then we may define a continuous map \(\xi: X_A \rightarrow \mathbb{Z}\) satisfying (5.5) by putting

\[
\xi(x) = \zeta(\sigma_A^M(\bar{x})),
\]

for any \(\bar{x} \in \rho^{-1}(x)\).

**Lemma 5.3.2.** There is an isomorphism

\[
\tilde{\rho}: (H^A, H_+^A) \rightarrow (\tilde{H}^A, \tilde{H}_+^A)
\]

induced by the cut \(\rho^*: C(X_A, \mathbb{Z}) \rightarrow C(X_A, \mathbb{Z})\).

**Proof.** Observe that if \(\xi \in C(X_A, \mathbb{Z})\), then \((\xi \circ \sigma_A - \xi) \circ \rho = (\xi \circ \rho) \circ \sigma_A - (\xi \circ \rho)\) since the cut intertwines the shift. Hence we may define a homomorphism \(\tilde{\rho}: H_A \rightarrow \tilde{H}_A\) by sending \([\xi] \mapsto [\xi \circ \rho]\).

Let \(\zeta \in C(X_A, \mathbb{Z})\). Via (5.5), there exist \(M \in \mathbb{N}\) and \(\xi \in C(X_A, \mathbb{Z})\) such that \(\zeta \circ \sigma_A^M = \xi \circ \rho\). In particular, \([\zeta] = [\zeta \circ \sigma_A^M] = [\xi \circ \rho] = \tilde{\rho}([\xi])\) so \(\tilde{\rho}\) is surjective. Furthermore, if \([\xi] \in H_+^A\), then \(\xi \circ \rho(\bar{x}) = \zeta \circ \sigma_A^M(\bar{x}) \geq 0\), for every \(\bar{x} \in X_A\). Hence \([\xi] \in H_+^A\), so \(H_+^A \subseteq \tilde{\rho}(H_+^A)\). The converse inclusion is clear.

It remains to show that \(\tilde{\rho}\) is injective. Take \(\xi \in C(X_A, \mathbb{Z})\) such that \(\tilde{\rho}([\xi]) = 0\), that is, \(\xi \circ \rho = \zeta \circ \sigma_A - \zeta\). For some \(\zeta \in C(X_A, \sigma_A)\). As we did above, we may take \(M \in \mathbb{N}\) and \(\eta \in C(X_A, \sigma_A)\) such that \(\zeta \circ \sigma_A^M = \eta \circ \rho\). The simple computation

\[
\xi \circ \sigma_A^M \circ \rho = \xi \circ \rho \circ \sigma_A^M = \zeta \circ \sigma_A^{M+1} - \zeta \circ \sigma_A^M = (\eta \circ \sigma_A - \eta) \circ \rho
\]

together with the fact that \(\rho\) is surjective implies that \(\xi \circ \sigma_A^M = \eta \circ \sigma_A - \eta\). Hence \([\xi] = [\xi \circ \sigma_A^M]\) vanishes in \(H^A\). \(\square\)
Five-term classification

Next, we prove a one-sided analog of Proposition 5.1.6. Similar to the two-sided case above, we say that a subset \( O \subseteq X_A \) is \( \sigma_A \)-invariant if \( \sigma_A(O) = O \).

**Lemma 5.3.3.** Let \( \xi \in C(X_A, \mathbb{Z}) \). Then \( [\xi] \in H^+_A \) if and only if \( \sum_{x \in O} \xi(x) \geq 0 \), for every finite \( \sigma_A \)-invariant set \( O \).

**Proof.** If \( O \subseteq X_A \) is a finite \( \sigma_A \)-invariant set, then every \( x \in O \) is periodic. Hence each such element can be extended uniquely to a bi-infinite sequence. Let \( \bar{O} \subseteq \bar{X}_A \) be the finite collection of such extensions and note that \( \rho \) restricts to a bijection between \( \bar{O} \) and \( O \). By the above lemma, we see that 
\[
\sum_{x \in \bar{O}} \xi(\rho(x)) \geq 0
\]
whenever \( [\xi] \in H^+_A \). Hence \( \sum_{x \in O} \xi(x) \geq 0 \).

Conversely, let \( \bar{O} \in \bar{X}_A \) be a finite \( \bar{\sigma}_A \)-invariant set. Then \( O = \rho(\bar{O}) \) is a finite \( \sigma_A \)-invariant set and \( \rho \) is injective on \( \bar{O} \). Therefore,
\[
\sum_{y \in \bar{O}} \xi(\rho(y)) = \sum_{x \in O} \xi(x) \geq 0
\]
showing that \( [\xi \circ \rho] \in \bar{H}^+_A \), cf. Proposition 5.1.6. Via the above lemma, we conclude that \( [\xi] \in H^+_A \).

Let \((G, X)\) be a fixed étale LCH groupoid. We shall briefly recall the canonical action of the collection of open bisections of \( G \) on \( X \) (see e.g., Section 3.3 or formula (3.2)). Given an open bisection \( S \subseteq G \), we consider the homeomorphism \( \zeta_S : s(S) \rightarrow r(S) \) given as
\[
\zeta_S = r \circ (s|_S)^{-1}.
\]
That is, if \( x \in s(S) \) then there is a unique \( \gamma_x \in S \) with \( s(\gamma_x) = x \) and \( \zeta_S(x) = r(\gamma_x) \).

**Definition 5.3.4.** A loop \( \gamma \in \Omega(X, x) \) is **attracting** on a compact open bisection \( S \) containing \( x \) if \( r(S) \subseteq s(S) \) and
\[
(\zeta_S)^n(y) \rightarrow x
\]
as \( n \rightarrow \infty \) for every \( y \in s(S) \).

We have the following two immediate observations.

**Lemma 5.3.5.** If \( X \) contains no isolated points, then \( x \in X \) cannot be attracting.

**Proof.** We may assume that any relevant compact open bisection is contained in \( X \) on which the structure maps \( r \) and \( s \) are identities. As \( X \) has no isolated points, no singletons are open, so \( x \) cannot be attracting.

**Lemma 5.3.6.** If \( \gamma, \gamma^{-1} \in G \) are attracting, then \( \gamma = \gamma^{-1} \).
5. Classification

Proof. Suppose $\gamma \in \Omega(X, x)$ is attracting on $S_\gamma$ and $\gamma^{-1}$ is attracting on $S_{\gamma^{-1}}$. In particular, $\gamma$ is attracting on $S = S_\gamma \cap S_{\gamma^{-1}}$. As $s$ is injective on $S$, we have

$$x \in s(S) = s(S_\gamma) \cap s(S_{\gamma^{-1}}) = s(S_\gamma) \cap r(S_{\gamma^{-1}}) \subseteq s(S_\gamma) \cap s(S_{\gamma^{-1}}).$$

Hence there is a morphism in $S_\gamma$ starting in $x$; this must be $\gamma$ since $S_\gamma$ is a bisection. Similarly, there is a morphism in $S_{\gamma^{-1}}$ starting in $x$ and this must be $\gamma^{-1}$. As $s$ is injective on $S$, this implies that $\gamma = \gamma^{-1}$. \qed

Next, we fix a one-sided topological Markov shift $(X_A, \sigma_A)$ and let $G_A$ be the corresponding groupoid.

A sequence $x \in X_A$ is periodic if $\sigma_A^k(x) = \sigma_A^{k+p}(x)$, for every $k \in \mathbb{N}$ and some $p \in \mathbb{N}$. We say that a sequence $x \in X_A$ is eventually periodic if $\sigma_A^k(x)$ is periodic for some $k \in \mathbb{Z}_+$. Equivalently, the set $\{\sigma_A^k(x) \mid k \in \mathbb{Z}_+\}$ is finite in which case we refer to the strictly positive integer

$$p = \min\{k - l \mid k, l \in \mathbb{Z}_+, k > l, \sigma_A^k(x) = \sigma_A^l(x)\}$$

as the period of $x$.\footnote{Note that the notions of periodic and eventually periodic coincide for the two-sided sequences.}

If $x \in X_A$ is eventually periodic and $\sigma_A^n(x)$ is periodic with $n$ minimal, we let $x' := x|_{[1,n]}$ denote the irregular part of $x$.

Note that the loop group at $x$ is trivial if and only if $x$ is not eventually periodic. Hence a loop $(x, k, x) \in \Omega X_A$ can be attracting only if $x \in X_A$ is eventually periodic. In the affirmative case, $\Omega(X_A, x) = \{(x, np, x) \mid n \in \mathbb{Z}_+\}$ is infinite cyclic. The following proposition characterizes the attracting elements in the groupoid $G_A$.

**Proposition 5.3.7.** Let $x \in X_A$ be eventually periodic with period $p$. Then $(x, np, x) \in G_A$ is attracting if and only if $n$ is strictly positive.

**Proof.** Fix a positive integer $n \in \mathbb{N}$ and let $x'$ denote the irregular part of $x$. Let $l = |x'|$ and choose $k \in \mathbb{N}$ such that $np = k - l$.

The sets

$$V = \{y = (y_j)_i \in X_A \mid x_j = y_j, j = 1, \ldots, k + 1\} = X|_{[1,k+1]},$$

$$W = \{z = (z_j)_i \in X_A \mid x_j = z_j, j = 1, \ldots, l + 1\} = X|_{[1,l+1]}$$

are basic open and compact neighborhoods of $x$ in $X_A$. Note that $V \subseteq W$ and $\sigma_A^k(V) = \sigma_A^l(W)$. Hence we may consider the open bisection

$$S = \{(y, np, z) \in G_A \mid z \in W, y \in V, \sigma_A^k(y) = \sigma_A^l(z)\}.$$ 

Clearly, $(x, np, x) \in S$. Note that $S$ is compact since it is a closed subset of $V \times \{np\} \times W$. Furthermore, $s(S) = W$ and $r(S) = V$, so $r(S) \subseteq s(S)$.

Let us now consider the action, $\zeta_S : s(S) \rightarrow r(S)$. Observe that $x_{k+1} = x_{l+1}$ since $x$ has period $np$ and consider the word $B = x_{[l+1,k]}$ of length $np$. Then we may write $x = x'B^\infty$. Now if

$$z = x'x_{l+1}z_{[l+1,\infty)} = x'x_{k+1}z_{[l+2,\infty)} \in s(U) = W,$$
then $\zeta_S(z)$ is the unique sequence in $r(U) = V$ with the property
\[
\sigma^1_A(z) = \sigma^k_A(\zeta_S(z)).
\]
Since $\sigma^1_A(z) = x_{l+1}z_{l+2,\infty} = x_{k+1}z_{l+2,\infty} = \sigma^k_A(x'B_{x_{k+1}z_{l+2,\infty}})$, we see that
\[
\zeta_S(z) = x'B_{x_{k+1}z_{l+2,\infty}}.
\]
That is, $\zeta^m_S(z) = x'B^m_{x_{k+1}z_{l+2,\infty}}$ whence
\[
(\zeta_S)^m(z) \to x'B^\infty = x
\]
for every $z \in W$ as $m \to \infty$. Hence $(x, np, x)$ is indeed attracting.

The fact that neither $(x, 0, x) = x \in X_A$ nor $(x, -np, x)$ are attracting now follows from Lemmas 5.3.5 and 5.3.6 since $X_A$ is a Cantor space. \qed

The next result relates the cohomology of the shift spaces to the cohomology of the corresponding groupoids. Recall from Section 4.1 that
\[
H^1(G_A) = \text{Hom}(G_A, \mathbb{Z})/\{f \circ s - f \circ r \mid f \in C(X_A, \mathbb{Z})\}
\]
To simplify the notation, we introduce
\[
H^1_+(G_A) = \{[\omega] \in H^1(G_A) \mid \omega(\gamma) \geq 0, \ \gamma \text{ attracting}\}.
\]

**Proposition 5.3.8.** There is an isomorphism $\Phi: (H^1(G_A), H^1_+(G_A)) \to (H^A, H^A_+)$. 

**Proof.** Define a map $\tilde{\Phi}: \text{Hom}(G_A, \mathbb{Z}) \to C(X_A, \mathbb{Z})$ by sending $\omega \mapsto \xi \in C(X_A, \mathbb{Z})$, where
\[
\xi(x) = \omega(x, 1, \sigma_A(x)).
\]
In particular, if $\omega = \partial(\eta)$, for some $\eta \in C(X_A, \mathbb{Z})$, then $\omega$ is mapped to
\[
\partial(\eta)((x, 1, \sigma_A(x)) = \eta(\sigma_A(x)) - \eta(x).
\]
That is, $\omega = \partial(\eta)$ is sent to $\eta \circ \sigma_A - \eta$. Hence we have a well-defined map $\Phi: H^1(G_A) \to H^A$. A similar argument shows that $\Phi$ is injective: Suppose $\omega \in \text{Hom}(G_A, \mathbb{Z})$ is mapped to $0 \in H^A$. Then
\[
\omega(x, 1, \sigma_A(x)) = \eta(\sigma_A(x)) - \eta(x) = \partial(\eta)(x),
\]
for some $\eta \in C(X_A, \mathbb{Z})$ and so $[\omega] = 0$ in $H^1(G_A)$. Next, let $\xi \in C(X_A, \mathbb{Z})$ and $(x, k-l, y) \in G_A$, for some $k, l \in \mathbb{N}$. Then $\sigma_A^k(x) = \sigma_A^l(y)$ and
\[
\omega(x, k-l, y) = \sum_{i=0}^{k-1} \xi(\sigma_A^i(x)) - \sum_{j=0}^{l-1} \xi(\sigma_A^j(y))
\]
defines an element in $\text{Hom}(G_A, \mathbb{Z})$. It is clear that $\omega$ is sent to $\xi$ via (5.6). Consequently, $\tilde{\Phi}$ is surjective from which it follows that $\Phi$ is surjective.

It remains to show that $\Phi(H^+_1(G_A)) = H^A_+$. Fix $\xi \in C(X_A, \mathbb{Z})$ and suppose $x \in X_A$ is eventually periodic with period $p$. Let $\gamma = (x, np, x)$ be attracting; as we have seen this forces
5. Classification

Let $\mathcal{O}_A$ and $\mathcal{O}_B$ be irreducible one-sided topological Markov shifts. The following are equivalent:

1. $(\mathcal{X}_A, \sigma_A)$ and $(\mathcal{X}_B, \sigma_B)$ are continuously orbit equivalent,
2. The étale LCH groupoids $G_A$ and $G_B$ are isomorphic,
3. There is a $C^*$-isomorphism $\Psi : (\mathcal{O}_A, \mathcal{D}_A) \rightarrow (\mathcal{O}_B, \mathcal{D}_B)$,
4. The Cuntz-Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are isomorphic and $\det(I - A) = \det(I - B)$,
5. There is an isomorphism $(\mathcal{B}F(A^i), u_A) \cong (\mathcal{B}F(B^i), u_B)$ and $\det(I - A) = \det(I - B)$.

Proof. The equivalences (2)$\iff$(1)$\iff$(3) were discussed in the first chapters of this thesis and are Theorem 2.2.1 and Corollary 3.4.4. If $\varphi : \mathcal{O}_A \rightarrow \mathcal{O}_B$ is a $C^*$-isomorphism, then $K_0(\varphi)$ is an isomorphism between $K_0(\mathcal{O}_A)$ and $K_0(\mathcal{O}_B)$ sending $[1_A]$ to $[1_B]$. The implication (4)$\Rightarrow$(5) now follows from the fact that the isomorphism $K_0(\mathcal{O}_A) \rightarrow \mathcal{B}F(A^i)$ of Theorem 2.3.3 takes $[1_A]$ to $u_A$. The implication (5)$\Rightarrow$(3) is Theorem 5.2.11.
Suppose now that \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are continuously orbit equivalent. By Theorem 5.3.9 above, the two-sided topological Markov shifts \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_B, \bar{\sigma}_B)\) are flow equivalent. It follows that \(\det(I - A) = \det(I - B)\), cf. Theorem 5.1.3 or [PS75]. Remembering the implication \((1) \Rightarrow (3)\), this proves the implication \((1) \Rightarrow (4)\) and thus the theorem. \(\square\)
Cuntz-Krieger algebras

A.1 Equivalent definitions

As we did in the very beginning of this thesis, we let $A$ be a fixed $N \times N$-matrix over $\{0, 1\}$. We recall the original definition of the Cuntz-Krieger algebra $O_A$ determined by $A$ in [CK80].

**Definition A.1.1.** The Cuntz-Krieger algebra $O_A$ is the universal $C^*$-algebra generated by $N$ partial isometries $S_1, \ldots, S_N$ subject to the CK-relations:

$$\sum_{j=1}^{N} S_j S_j^* = 1, \quad S_i^* S_i = \sum_{j=1}^{N} A(i,j) S_j S_j^*$$

for $i = 1, \ldots, N$.

Alternatively, the matrix $A$ uniquely determines a finite graph $(V_A, E_A) = (V, E)$ as follows: Let $V = \{1, \ldots, N\}$ be the set of (labeled) vertices. There is an edge from $i$ to $j$ if and only if $A(i,j) = 1$. With the notation of Section 1.1, we see that $V = \mathbb{A}$ and $E = \mathbb{A}_2$. In this graph-theoretic context, we shall refer to $A$ as the adjacency matrix of the graph. The graph is essential when $A$ is irreducible. We assume that $A$ is irreducible and satisfies condition (I).

**Definition A.1.2.** Let $(V, E)$ be the graph determined by the matrix $A$. The Cuntz-Krieger algebra $O_{\bar{A}}$ is the universal $C^*$-algebra generated by orthogonal projections $\{P_i\}_{i \in V}$ and orthogonal partial isometries $\{T_e\}_{e \in E}$ subject to the conditions

$$P_{r(e)} = T_e^* T_e, \quad P_i = \sum_{e \in E_i} T_e T_e^*$$

for every $i = V$.

The notation $O_{\bar{A}}$ for the Cuntz-Krieger algebra defined in this way is only tentative. The point of this section is to show that $O_A \cong O_{\bar{A}}$ when $A$ is irreducible and satisfies condition (I).

**Proposition A.1.3.** The $C^*$-algebras $O_A$ and $O_{\bar{A}}$ defined above are isomorphic.

**Proof.** Suppose $S_1, \ldots, S_N$ is a family of partial isometries generating $O_A$. Then

$$R_e = S_{s(e)} S_{r(e)} S_{r(e)}^*$$

for every $e \in E$.
defines a family of partial isometries in \( \mathcal{O}_A \) indexed by the edges \( e \in \mathcal{E} \). We start by showing that \( \{ R_e \}_e \) satisfies the relations (A.1). For each \( e \in \mathcal{E} \), we have

\[
R_e^* R_e = S_{r(e)} S_{r(e)}^* (S_{s(e)} S_{s(e)}^*) S_{r(e)} S_{r(e)}^*
\]

\[
= S_{r(e)} S_{r(e)}^* \left( \sum_{j=1}^{N} A(s(e), j) S_j S_j^* \right) S_{r(e)} S_{r(e)}^*
\]

\[
= S_{r(e)} S_{r(e)}^*
\]

and

\[
\sum_{f \in \mathcal{E}_r(e)} R_f R_f^* = \sum_{f \in \mathcal{E}_r(e)} S_{s(f)} (S_{r(f)} S_{r(f)}^*) S_{s(f)}^*
\]

\[
= S_{r(e)} \left( \sum_{f \in \mathcal{E}_r(e)} S_{r(f)} S_{r(f)}^* \right) S_{r(e)}^*
\]

\[
= S_{r(e)} \left( \sum_{j=1}^{N} A(r(e), j) S_j S_j^* \right) S_{r(e)}^*
\]

\[
= S_{r(e)} S_{r(e)}^*.
\]

By universality, there is a unique *-homomorphism \( \varphi: \mathcal{O}_A \rightarrow \mathcal{O}_A \) sending \( T_e \mapsto R_e \) for every \( e \in \mathcal{E} \). Since \( S_i = \sum_{e \in \mathcal{E}_i} R_e \) for every \( i \in \mathcal{V} \), \( \varphi \) is surjective. Furthermore, \( \varphi(T_e^* T_e) \neq 0 \) for every edge \( e \) and so \( \varphi \) is also injective by the Cuntz-Krieger uniqueness theorem (Theorem 2.1.5). \( \square \)
In this appendix, we briefly discuss the notion of left Haar systems on locally compact Hausdorff (LCH) groupoids. This section serves as a justification for the definition of the convolution product on the specific case of an étale LCH groupoid given in Section 3.2. Given the existence of a left Haar system on a LCH groupoid, there is a canonical groupoid $C^*$-algebra construction. The reader is referred to [Pat99]. In the end of this appendix, we give examples showing that Haar systems need not be unique and need not even exist.

### B.1 Haar systems

Let us first recall the definition of a locally compact groupoid.

**Definition B.1.1.** A topological groupoid $(G, X)$ is **locally compact** if

1. The unit space $X$ is LCH in $G$,
2. There exists a countable basis for the topology on $G$ consisting of relatively compact sets,
3. The fibers $G^x$ and $G_x$ are LCH in $G$.

We shall always assume that the groupoids in question are Hausdorff. As is the case with locally compact groups, we need some appropriate notion of measure on a LCH groupoid in order to construct a convolution algebra.

**Definition B.1.2.** A left **Haar system** on a locally compact groupoid $(G, X)$ is a family $\{\lambda_x\}_{x \in X}$ of positive regular Borel measures on $G_x$ subject to the conditions

1. $\text{supp}(\lambda_x) = G_x$,
2. For every continuous compactly supported $g : G \to \mathbb{C}$, the function $g^0 : X \to \mathbb{C}$ given by
   $$ g^0(x) = \int_{G_x} g \, d\lambda_x $$
   is continuous and compactly supported,
3. For every $\gamma \in G$ and continuous compactly supported $f : G \to \mathbb{C}$, we have
   $$ \int_{G_s(\gamma)} f(\gamma \beta^{-1}) \, d\lambda_s(\gamma)(\beta) = \int_{G_r(\gamma)} f(\alpha) \, d\lambda_r(\gamma)(\alpha). $$
Remark B.1.3. If \( \{\lambda_x\}_x \) is a Haar system, then \( \lambda^x = (\lambda_x)^{-1} \) defines a Haar system with support on \( G^x \).

It is well-known that any locally compact Hausdorff group admits a unique Haar measure. Unfortunately, this is too much to hope for in the case of groupoids. In fact, Section B.2 provides examples which show that Haar systems need not exist and need not be unique when they do exist. Some consolation is found in the fact that unique Haar systems do exist on étale LCH groupoids (see Proposition B.1.5). In order to show this, let us first consider a very useful lemma. Recall that a topological groupoid \((G, X)\) is said to be étale if the range and source maps \( r, s : G \rightarrow X \) are local homeomorphisms.

Lemma B.1.4. Let \((G, X)\) be a LCH groupoid and let \( f \in C_c(G) \). Then \( f \) is a finite sum of functions supported on basic open relatively compact subsets. If \((G, X)\) is étale, \( f \) is a finite sum of functions supported on open bisections.

Proof. Suppose \( f \in C_c(G) \) is supported on a compact subset \( K \). Cover \( K \) by finitely many basic open relatively compact subsets \( U_1, \ldots, U_n \). Choose a partition of unity \( \{h_i\}_{i=1}^n \) subordinate to the open cover \( \{U_i\}_{i=1}^n \). In particular, \( \text{supp}(h_i) \subseteq U_i \) and \( \sum_{i=1}^n h_i(\gamma) = 1 \), for every \( \gamma \in K \). Let us now define \( f_i \in C_c(K) \) by \( f_i(\gamma) = h_i(\gamma)f(\gamma) \), for \( \gamma \in K \). Then \( \text{supp}(f_i) \subseteq U_i \) and \( f = \sum_{i=1}^n f_i \). For the étale case, recall that \( G \) is covered by open bisections. Cover \( K \) by finitely many open bisections \( S_1, \ldots, S_n \) and apply the same procedure. \( \square \)

Proposition B.1.5. Let \((G, X)\) be an étale LCH groupoid. For every \( x \in X \) and continuous map \( F : X \rightarrow (0, \infty) \), the measures

\[
\lambda_x = \sum_{\gamma \in G_x} F(s(\gamma))\delta_\gamma
\]

define a left Haar system on \((G, X)\). Here, \( \delta_\gamma \) is the Dirac measure on \( \gamma \in G \).

Proof. We verify the axioms of Definition B.1.2 for \( \lambda_x \). It is clear that \( \lambda_x \) vanishes outside of \( G_x \). Conversely, \( \alpha \) is strictly positive, so \( \text{supp}(\lambda_x) \) is all of \( G_x \) and (1) holds. For (2) we may assume that \( g \in C_c(G) \) has support on an open bisection \( S \). As \( G_x \) is discrete, we see that

\[
g^0(x) = \int_{G_x} g \, d\lambda_x = \sum_{\gamma \in G_x} g(\gamma)\alpha(s(\gamma)) = g \circ (r|_S)^{-1}(x)\alpha \circ s \circ (r|_S)^{-1}(x),
\]

for every \( x \in X \). The last equality follows from the fact that \( S \) is a bisection. Hence \( g^0 \) has support on \( r(S) \subseteq X \). Given \( \gamma \in G \) and \( f \in C_c(G) \) we find that

\[
\int_{G_{s(\gamma)}} f(\gamma\beta^{-1}) \, d\lambda_{s(\gamma)}(\beta) = \sum_{\beta \in G_{s(\gamma)}} f(\gamma\beta^{-1})F(s(\gamma)) = \sum_{G_{r(\gamma)}} f(\alpha)F(s(\alpha)) = \int_{G_{r(\gamma)}} f(\alpha) \, d\lambda_{r(\gamma)}(\alpha)
\]

via the substitution \( \alpha = \gamma\beta^{-1} \). This shows (3). \( \square \)

Choosing \( F : X \rightarrow (0, \infty) \) to be the constant function 1, we obtain the counting measure. We shall refer to this system as the canonical Haar system on an étale LCH groupoid.
B. Haar systems

B.2 Exempli gratia

In this section, we provide the reader with two neat examples of LCH groupoids. Together they show the fact that Haar systems on LCH groupoids need not exist and if they do, they need not be unique.

**Lemma B.2.1.** If a LCH groupoid admits a left Haar system, it need not be unique.

**Proof.** We prove this assertion by providing a simple example which can be found on p. 32 in [Pat99]. Let $X$ be a locally compact, second countable and Hausdorff space and let $G = X \times X$ be a (maximal) equivalence relation on $X$. It is easily verified that $(G, X)$ is a principal LCH groupoid. Note that $G_x = X \times \{x\}$, for any $x \in X$.

Let $\mu$ be any positive regular Borel measure on $X$ with full support, that is, $\text{supp}(\mu) = X$. We put $\lambda_x := \mu \times \delta_x$. As usual, $\delta_x$ is the Dirac measure at $x$. Then $\{\lambda_x\}_{x}$ constitutes a left Haar system on $(G, X)$. It is clear that $\text{supp}(\lambda_x) = X \times \{x\} = G_x$. Next, if $g \in C_c(X \times X)$, then

$$g^0(x) = \int_{G_x} g \ d\lambda_x = \int_X g(y, x) \ d\mu(y)$$

is continuous and compactly supported for every $x \in X$. Finally, let $f \in C_c(X \times X)$ and $(x_0, y_0) \in X \times X$. Then

$$\int_{G_{y_0}} f((x_0, y_0))(y_0, z) \ d\lambda_{y_0}(z, y_0) = \int_X f(x_0, z) \ d\mu(z) = \int_{G_{x_0}} f(x_0, z) \ d\lambda_{x_0}(x_0, z).$$

This shows that any positive regular Borel measure with full support on $X$ gives rise to a Haar system on $(G, X)$.

Moving on, we give a simple example of LCH groupoid which does not admit a left Haar system. This example is borrowed from [Sed86]. Consequently, there exist LCH groupoids to which we cannot associate a convolution algebra nor a C*-algebra.

Let $T$ be the unit circle in the $yz$-plane and consider the product $H = [0, 1] \times T$ in $\mathbb{R}^3$. This is a groupoid with unit space $H^{(0)} = [0, 1] \times \{0\} \subseteq [0, 1] \times T$ which we may identify with $[0, 1]$. Composition and inversion are given component-wise. The range and source maps are both given as the projection onto the first coordinate. For each $x \in [0, 1]$, the sections are

$$H_x = \{x\} \times T \subseteq [0, 1] \times T$$

and we shall identify this with the circle $T$. Let $G \subseteq H$ be the subgroupoid with the same unit space $G^{(0)} = H^{(0)}$ but whose section $G_x = \{(x, 1, 0)\}$ is the trivial group when $x > 1/2$. Then $G$ is a compact Hausdorff groupoid.

**Lemma B.2.2.** The groupoid $(G, G^{(0)})$ defined above admits no left Haar system.

**Proof.** In order to reach a contradiction, we shall assume the existence of a left Haar system $\{\lambda_x\}_{x \in [0, 1]}$. We may assume that each measure $\lambda_x$ is normalized on $G_x$. Indeed, by definition of the Haar system the map

$$x \mapsto \int_{G_x} 1 \ d\lambda_x = \lambda_x(G_x)$$

1At least not in the sense described in this thesis.
is continuous and so the family \( \{ \lambda_x / \lambda_x(G_x) \} \) is again a left Haar system on \( G \). Take a continuous function \( f : H \to [0,1] \) satisfying \( f(x,1,0) = 1 \) and

\[
\int_{H_x} f(t) \, dt \leq 1/2,
\]

for all \( x \in [0,1] \). Here, we integrate with respect to the Lebesgue measure. When \( x > 1/2 \) we have

\[
\int_{G_x} f \, d\lambda_x = \int_{G_x} 1 \, d\lambda_x = \lambda_x(G_x) = 1.
\]

On the other hand, when \( x \leq 1/2 \), the Lebesgue measure and \( \lambda_x \) coincide by uniqueness of such measures on compact groups. So

\[
\int_{G_x} f \, d\lambda_x = \int_{G_x} f(t) \, dt \leq 1/2.
\]

Hence the map \( x \mapsto \int_{G_x} f \, d\lambda_x \) is not continuous at \( x = 1/2 \) violating the definition of the Haar system. \( \square \)
List of symbols

\[(G, X)\] A groupoid \(G\) with unit space \(X\), page 35

\[(X_A, \sigma_A)\] The one-sided topological Markov shift determined by \(A\), page 9

\[(\tilde{H}^A, \tilde{H}^A_+)\] The ordered cohomology of \((\tilde{X}_A, \tilde{\sigma}_A)\), page 62

\[(\tilde{X}_A, \tilde{\sigma}_A)\] The two-sided topological Markov shift determined by \(A\), page 9

\(A\) \(N \times N\) matrix over \(\{0, 1\}\), adjacency matrix., page 9

\(C^*_r(G)\) The reduced groupoid \(C^*\)-algebra of \((G, X)\), page 40

\(C_c(G)\) The convolution algebra of \((G, X)\), page 39

\(F_A\) The suspension flow space of \((\tilde{X}_A, \tilde{\sigma}_A)\), page 61

\(G(A, B)\) The Weyl groupoid, page 45

\(G(\mathcal{G})\) The groupoid of germs of \(\mathcal{G}\), page 41

\((G \times_{\rho} \Gamma)\) The skew product of a groupoid \(G\) with a group \(\Gamma\) via a homomorphism \(\rho: G \to \Gamma\), page 38

\(G_A\) The groupoid associated to \((X_A, \sigma_A)\), page 47

\(H^n(G; \Lambda)\) The cohomology of \((G, X)\) with (constant) coefficients in \(\Lambda\), page 52

\(H_A\) An AF subgroupoid of \(G_A\), page 49

\(H_n(G; \Lambda)\) The homology of \((G, X)\) with (constant) coefficients in \(\Lambda\), page 52

\(N(A, B)\) The normalizer of a \(C^*\)-subalgebra \(B\) in a \(C^*\)-algebra \(A\), page 43

\([\mathcal{G}]\) The ample pseudogroup, page 41

\([\sigma_A]_c\) The topological full group, page 10

\(\mathfrak{A}\) The alphabet \(\{1, \ldots, N\}\), page 9

\(\text{BF}(A)\) The Bowen-Franks group of the matrix \(A\), page 62

\(\mathcal{D}_A\) The diagonal in \(O_A\), page 18

\(\mathcal{F}_A\) The AF core in \(O_A\), page 17

\(\mathcal{G}\) A pseudogroup, page 41

\(\mathcal{G}(A, B)\) The Weyl pseudogroup of \((A, B)\), page 45

\(\Omega_X\) The isotropy bundle of \((G, X)\), page 36

\(\mathcal{O}_A\) The Cuntz-Krieger algebra, page 15

\(\mathcal{P}art(X)\) The inverse semigroup of partial homeomorphisms on \(X\), page 41

\(\mathcal{S}\) The collection of open bisections, page 41

\(\{\lambda_x\}_{x \in X}\) A left Haar system, page 76


