Title: An explicit charge-charge correlation function at the edge of a two-dimensional Coulomb droplet.

Abstract: Consider a two-dimensional Coulomb droplet. It is expected that different charges at the edge should be correlated in a relatively strong way. The physical picture is that the screening cloud about a charge at the boundary has a non-zero dipole moment, which gives rise to a slow decay of the correlation function. This phenomenon was studied (on the "physical" level of rigor) by Forrester and Jancovici in a paper from 1995 for the elliptic Ginibre ensemble. Coincidentally a recent joint work between myself and Joakim Cronvall on reproducing kernels turned out to be closely related to this question.

Indeed, if there are n particles, we obtain that the order of magnitude of the correlation function $K_n(z,w)$ is proportional to $\sqrt{n}$ if z,w are on the boundary and $z \neq w$, while $K_n(z,z)$ is proportional to n. This gives the "slow decay" of correlations at the boundary. (For comparison, if one of the charges (say z) is in the bulk, then $K_n(z,w)$ decays quickly for $z \neq w$: $|K_n(z,w)| \lesssim e^{-c\sqrt{n}}$.)

In addition we find that in the limit as $n \to \infty$, there emerges the following correlation kernel $K(z,w)$ for z,w on the (outer) boundary:

$$K(z,w) = \frac{1}{\sqrt{2\pi}}(\Delta Q(z)\Delta Q(w))^{\frac{1}{4}} \frac{\sqrt{\phi'(z)}\overline{\sqrt{\phi'(w)}}}{\phi(z)\overline{\phi(w)} - 1}. \tag{0.1}$$

Here we assumed (for simplicity) that the droplet is connected and that z,w are on the outer boundary curve Γ; then ϕ is a Riemann mapping from $\operatorname{Ext}\Gamma$ to the exterior disc $\{|z| > 1\}$. (Thus it should be understood that $|\phi(z)| = |\phi(w)| = 1$ in (0.1).) Finally Q is the (rather arbitrary) external potential used to define the ensemble. For example: $Q(x+iy) = ax^2 + by^2$ in the case of elliptic Ginibre.

The kernel $S(z,w) = \frac{1}{2\pi}\frac{\sqrt{\phi'(z)}\overline{\sqrt{\phi'(w)}}}{\phi(z)\overline{\phi(w)}-1}$ appearing in (0.1) can be recognized as the so-called *Szegő kernel* of the boundary curve Γ. (That $S(z,z) = \infty$ reflects the fact that long-range vs. short-range interactions take place on different scales.)

Our method for deriving these results builds on the technique of full-plane orthogonal polynomials due to Hedenmalm and Wennman (work to appear in Acta Math). Using summation by parts and "tail-kernel approximation", we in fact obtain asymptotic results for the canonical correlation kernel in cases beyond the boundary-boundary case; in particular our results extend nicely to the exterior of the droplet.

In the basic case of the Ginibre ensemble, we obtain more precise asymptotics for $K_n(z,w)$ (an expansion in powers of $1/n$) by developing techniques found is Szegő's classical work on the distribution of zeros of partial sums $s_n(z) = 1 + z + \cdots + \frac{z^n}{n!}$ $(n \to \infty)$.

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Ameur, Y., Cronvall, J., *Szegő type asymptotics for the reproducing kernel in spaces of full-plane weighted polynomials.* Arxiv: 2107.11148.

(We are planning an update in a relatively near future, and we are therefore particularly grateful for comments.)