What is a II_1 factor?

Outline:

- What is a von Neumann algebra?
- What is a factor?
- What is type II_1?
- Why do we care mostly about II_1 factors?
- What do we know about II_1 factors?

von Neumann algebras:

H = Hilbert space (over $\mathbb{C}$) with inner product $\langle\ ,\ \rangle\colon H \times H \to \mathbb{C}$ and norm $\|x\| = \sqrt{\langle x, x\rangle}$.

$$B(H) = \{\text{bounded (i.e. continuous) linear operators } H \to H\}.$$

Strong operator topology: for operators $T_n, T \in B(H)$

$$T_n \to T \text{ in strong operator topology} \iff T_n(x) \to T(x) \text{ in } H \text{ for every } x \in H.$$

Strong operator convergence = pointwise convergence.

Adjoints: Every bounded operator T admits a unique bounded *adjoint operator* T^* defined by the property that

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle$$

for every $x, y \in H$.

A *von Neumann algebra* is a subalgebra $M \subseteq B(H)$ which contains the identity operator I, is closed under taking adjoints, and is closed in the strong operator topology.

Abelian von Neumann algebras look like $L^\infty(X, \mu)$ for some measure space (X, μ). For example $L^\infty([0,1])$ and $\ell^\infty(\mathbb{N})$. Other examples: $B(H)$. $n \times n$ complex matrices.

Factors:

A *factor* is a simple von Neumann algebra, that is, a von Neumann algebra with no ideals. Equivalently, the *center*

$$Z(M) = \{T \in M \mid ST = TS \text{ for every } S \in M\}$$

is as small as possible:

$$Z(M) = \{\lambda I \mid \lambda \in \mathbb{C}\}.$$

By a *normalized trace* on M we mean a linear functional $\tau \colon M \to \mathbb{C}$ such

$$\tau(T^*T) \geq 0, \quad \tau(TS) = \tau(ST), \quad \tau(I) = 1.$$

Example: the usual normalized trace on $n \times n$ matrices

$$\operatorname{tr} \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} = \frac{1}{n} \sum_i x_{ii}.$$

Theorem/Definition. For a von Neumann algebra M the following are equivalent.

1. M is a II_1 factor.
2. M is a factor, M admits a normalized trace, $M \not\cong n \times n$ complex matrices.
3. M admits a unique normalized trace which is moreover faithful (explain) and normal (do not explain), and its values on projections (explain?) are $[0, 1]$.

Examples of II_1 factors are not so easy to come by, but we will return to that later. They are, however, very important. This will be explained now.

Reduction theory: Given two von Neumann algebras M and N one can construct their direct sum $M \oplus N = \{(m, n) \mid m \in M, n \in N\}$ with pointwise operations. More generally, given a sequence of von Neumann algebras $M_1, M_2, \ldots$ one constructs its ℓ^∞-direct sum

$$\bigoplus_{n=1}^{\infty} M_n = \{(m_n)_{n=1}^{\infty} \mid m_n \in M_n \text{ and } \sup_n \|m_n\| < \infty\}.$$

Is every von Neumann algebra a sum of factors? Almost yes. Replace sum by integral, and the answer becomes yes.

Example:

$$L^\infty([0,1]) = \int_{[0,1]} \mathbb{C}, \qquad \ell^\infty(\mathbb{N}) = \bigoplus_{n=1}^{\infty} \mathbb{C} = \int_{\mathbb{N}} \mathbb{C}$$

So basically, everything about von Neumann algebras comes down to understanding factors.

Types of factors (type I, II, III, finite and infinite):

- type I $= B(H)$
- type II_1
- type II_∞ = type $\mathrm{II}_1 \otimes B(H)$
- type III = the rest

Type I factors are completely understood. Type II_∞ basically reduces to type II_1.

If M is type III, then a crossed product $M \rtimes \mathbb{R}$ is type II_∞, and taking the crossed product once more essentially gets you back:

$$(M \rtimes \mathbb{R}) \rtimes \mathbb{R} = M \otimes B(H).$$

Type III basically reduces to type II_∞. So basically, everything about von Neumann algebras comes down to understanding factors of type II_1.

How to construct factors of type II_1? Using groups.

A group G acts on $\ell^2(G)$ by translation λ_g:

$$(\lambda_g f)(h) = f(g^{-1}h), \qquad f \in \ell^2(G),\ g, h \in G$$

λ = *left regular representation*, extends by linearity to a representation of the group algebra $\mathbb{C}[G]$.

The *group von Neumann algebra* $\mathrm{vN}(G)$ is the strong operator closure of $\mathbb{C}[G]$ when we view $\mathbb{C}[G] \subseteq B(\ell^2(G))$.

Theorem. $\mathrm{vN}(G)$ *is a* II_1 *factor* $\iff$ *every non-trivial element* g *of* G *has an infinite conjugacy class* $\{hgh^{-1} \mid h \in G\}$ *(and* G *is not the trivial group) . We call such groups ICC.*

Example of (non-isomorphic) II_1 factors: $\mathrm{vN}(\mathbb{F}_2)$ and $\mathrm{vN}(S_\infty)$.

For a permutation $\pi \in \mathrm{Perm}(\mathbb{N})$, its support is

$$\mathrm{supp}(\pi) = \{n \in \mathbb{N} \mid \pi(n) \neq n\}.$$

$$S_\infty = \{\pi \in \mathrm{Perm}(\mathbb{N}) \mid \mathrm{supp}(\pi) \text{ is finite}\} = \bigcup_{n \in \mathbb{N}} S_n.$$

General problem in the theory of von Neumann algebras: for which groups G and H is $\mathrm{vN}(G) \simeq \mathrm{vN}(H)$? How much about the group G does $\mathrm{vN}(G)$ remember? This is very hard!

Whenever G is ICC and amenable, then $\mathrm{vN}(G) \simeq \mathrm{vN}(S_\infty)$. Conversely, if G is ICC and not amenable, then $\mathrm{vN}(G) \not\simeq \mathrm{vN}(S_\infty)$.

Big open question: is $\mathrm{vN}(\mathbb{F}_2) \simeq \mathrm{vN}(\mathbb{F}_n)$ for some $3 \leq n \leq \infty$?

This is the free group factor problem. If true, then $\mathrm{vN}(\mathbb{F}_2) \simeq \mathrm{vN}(\mathbb{F}_n)$ for any $3 \leq n \leq \infty$.

Approximation properties of groups are remembered by the von Neumann algebra. It is known that

$$\mathrm{vN}(\mathrm{PSL}(2,\mathbb{Z})) \not\simeq \mathrm{vN}(\mathrm{PSL}(3,\mathbb{Z})),$$

because $\mathrm{vN}(\mathrm{PSL}(2,\mathbb{Z}))$ has the completely bounded approximation property, whereas $\mathrm{vN}(\mathrm{PSL}(3,\mathbb{Z}))$ does not.

Another open question: if $m, n \geq 3$ and $m \neq n$, is

$$\mathrm{vN}(\mathrm{PSL}(m,\mathbb{Z})) \not\simeq \mathrm{vN}(\mathrm{PSL}(n,\mathbb{Z}))?$$