(Affine) Kac-Moody symmetric spaces

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**Introduction:**

We have seen finite dimensional Riemannian symmetric spaces:

- **classification:** $G$ simple Lie group

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<td>$K$ compact real</td>
<td>$G_e/K$</td>
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<td>$K/\text{Fix}(G)$</td>
<td>$G_{nc}/\text{Fix}(G) = \text{Fix}(\omega)$</td>
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<td>$\text{sect}(X,Y) \geq 0$</td>
<td>$\text{sect}(X,Y) \leq 0$</td>
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<td>Cartesian involution</td>
<td>diffeomorphic to a VS</td>
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- **structure:**
  - flat
  - Weyl group action
  - 'spheres' of flats around 'fixed' hyperplanes

**Aim:** replace $G$ by some affine KMG

$\Rightarrow$ construct symmetric spaces

$\Rightarrow$ yields:
  - similar classification
  - similar structure
  - Lorentzian
Affine Kac-Moody algebras

Let $G$ be complex reductive, $\beta$ diagram automorphism

\[
\begin{align*}
A_n & \quad \alpha_1, \alpha_2, \ldots, \alpha_n \\
D_n & \quad \alpha_1, \ldots, \alpha_n \\
D_4 & \quad \text{triviality}
\end{align*}
\]

Definition:

\[
L(G, \beta) := \{ f: \mathbb{R} \rightarrow G \mid f(t + 2\pi \alpha) = \beta f(t), f \text{ satisfies some regularity condition} \}
\]

Example: $f$ is smooth
- $f$ is differentiable
- $f$ is holomorphic on $\mathbb{C}^*$
- $f$ has finite Fourier expansion

\[
\hat{L}(G, \beta) = L(G, \beta) + \mathbb{C} c + \mathbb{C} d
\]

\[
[f, g](t) = [f(t), g(t)] + \omega_t(f, g)c
\]

antisym 2-form, satisfying cocycle $1d$

i.e., $\omega(f, g) = \frac{1}{2\pi} \int_0^{2\pi} \langle fg', \alpha \rangle \ dt$

\[
\text{Res}_0 (fg')
\]

\[
\begin{align*}
[d, f](t) &= f'(t) \\
[c, d] &= [c, f] = 0
\end{align*}
\]

$d$ is a derivation
$c$ is central
Holomorphic realization (only non-twisted \(\rightarrow\) twisted is more complicated)

\[ \mathcal{M}_{f_0} = \{ f: \mathbb{C}^* \rightarrow \mathcal{O}_f \mid f \text{ is holomorphic} \} \]

Remark:

\[ g = \text{Id} \Rightarrow f(t+2\pi) = f(t) \Rightarrow \text{we can study functions on } S^1 \]

Embedding \( S^1 \hookrightarrow \mathbb{C}^* \) as unit circle

we can study the Fourier series \( f(z) = \sum a_n z^n = \sum a_n e^{int} \)

from \( z = e^{it} \) we find

\[ [d, f(z)] = [d, f(e^{it})] = \frac{1}{i} e^{it} \cdot ie^{it} = iz f'(z) \]

Remark

\( g \) is simple and \( f(z) \) polynomial (i.e. \( f(z) = \sum_{n=0}^{n_0} a_n e^{int} \))

\[ \Rightarrow \mathcal{L}_{alg} g = g \otimes \mathbb{C}[t, t^{-1}] \text{ and } \]

\[ \mathcal{L}_{alg} g = g(A) \text{ where } A \text{ is the affinization of } g. \]

Remark:

Twisted KMA's can be realized as fixed point algebras complemented of non-twisted algebras. As they are closed subspaces all concepts, we study carry over.
Tame Fréchet structures:

Aim:

Theorem:

$\hat{\mathfrak{g}}$ is a tame Fréchet Lie algebra.

Definition:

A locally convex topological vector space is Fréchet iff it is Hausdorff, metrizable. (or equivalently: the topology is generated by a countable family of norms).

Ex:

- $C^\infty(S^1, \mathbb{R})$, choose $\| f \|_n : = \sup_{x \in S^1} |\frac{\partial^n}{\partial x^n} f(x)|$

- $C^\infty(M, \mathbb{R}^k)$, $M$ compact.

- Hol(\mathbb{C}, \mathbb{C})$ choose $\| f \|_n = \sup_{z \in B(0,n)} |f(z)|$

completeness follows from Montel's theorem.

- Hol(\mathbb{C}^*, \mathbb{C})$ choose $\| f \|_n = \sup_{z \in A_n} |f(z)|$

where $A_n = \{ z \mid e^{-n} \leq |z| \leq e^n \}$ annulus
Problem:
1. no inverse function theorem
   no uniqueness theorem for Dgl's
   (possibly geodesics, with 'switches')
2. dual spaces are not Frechet
   \[ \overrightarrow{\text{no usual tensor calculus}} \]

Explanation

Let \((F, \| \cdot \|_n)\) Frechet. Define \(B_n := \text{completion of } F \text{ with respect to } \| \cdot \|_n\)

\[ \implies B_n \text{ is Banach and } F = \varprojlim B_n \]

\[ F^* = \varinjlim B_n^* \text{ (inverse limit)} \]

Solution
1. write component functions, Christoffel symbols
2. Tameness \(\Rightarrow\) some kind of quasi isometry!
Definition:
Grading on \( F \): \( \| f \|_1 \leq \| f \|_2 \leq \cdots \)

From now on \( F, G \) Graded

Definition:
\* \( \rho : F \to G \) is tame linear map \( \iff \| \rho(f) \|_n \leq \| f \|_{n+k} \cdot C(n) \) for all \( n > b \) for some \( (b, k, C(n)) \)

\* \( \phi : F \to G \) is tame isomorphism:
\( \phi \) is isomorphism and \( \hat{\phi}, \hat{\phi}^{-1} \) are tame.

\* \( \phi : F \to G \) is tame direct summand \( \iff \exists \psi : F \to G \) and \( \psi : G \to F \) tame such that \( \psi \circ \phi = \text{Id}_F \) ("one-sided inverse")

(equivalently: \( F \) is closed, complemented subspace of \( G \))

Model space:
Let \( B \) Banach.
Put \( \Sigma'(B) := \{ (b)_n, n \in \mathbb{N} | \sum_{n=1}^{\infty} |b_n| e^{nk} < \infty \ \forall k \} \)

"space of exponentially decreasing sequences"

Definition:
\( F \) is tame Fréchet \( \iff F \) is tame direct summand of \( \Sigma'(B) \) for some \( B \).
Let us look at these concepts more closely:

**Example:** $B := \mathbb{C}$

Then $\Sigma'(B) = \{(b_n)_{n \in \mathbb{N}} \mid \sum_{n=1}^{\infty} |b_n| e^{\alpha n} < \infty \forall \alpha \in \mathbb{N}\}$

**Lemma:**

$\Sigma'(B) = \text{Hol}(B, \mathbb{C})$

**Proof:**

$\Rightarrow$

Let $(b_n)_{n \in \mathbb{N}} \in \Sigma'(B)$. Claim: $f = \sum_{n=1}^{\infty} b_n z^n : \mathbb{C} \to \mathbb{C}$ is holomorphic.

$$|f|_{B(0,e^\alpha)} = \left| \sum_{n=1}^{\infty} b_n z^n \right|_{B(0,e^\alpha)} \leq \sum_{n=1}^{\infty} |b_n| |z^n| = \sum_{n=1}^{\infty} |b_n| e^{\alpha n} < \infty$$

$\Leftarrow$ Use equivalent description of $\Sigma'(B)$ as follows:

$\Sigma'(B) := \{(b_n)_{n \in \mathbb{N}} \mid \sup_{n} |b_n| e^{\alpha n} < \infty \forall \alpha \in \mathbb{N}\}$

$\Rightarrow \sup_{n} |b_n| e^{\alpha n} = \sup_{n} \left| \frac{f^{(n)}(0)}{n!} e^{\alpha n} \right| \leq \sup_{n} \frac{e^{\alpha n}}{n!} \left| \frac{f^{(n)}(0)}{n!} \right|

\[\text{use } b_n = \frac{1}{n!} f^{(n)}(0)\]

$$\leq \sup_{n} \frac{e^{\alpha n}}{n!} \left| \frac{f^{(n)}(0)}{n!} \right| \leq \sup_{z \in B(0,e^\alpha)} \frac{|f(z)|}{e^{\alpha n}} = \sup_{z \in B(0,e^\alpha)} \frac{\sup_{z \in B(0,e^\alpha)} |f(z)|}{e^{\alpha n}} \leq \infty$$

$|f^{(n)}(0)| \leq n! \frac{\sup_{z \in B(0,e^\alpha)} |f(z)|}{e^{\alpha n}}$
Definition:

If $g$ is a tame Fréchet–li algebra $\iff g$ is a tame $F$-space

- $\text{ad}(X)$ is tame $\forall X \in g$

Proof that $\hat{g}$ is a tame Fréchet lie algebra

1. Do it for $\text{Hol}(\mathbb{C}^*, \mathbb{C})$:
   - choose $B = \mathbb{C}^2$, cut $\sum_{n=0}^{\infty} a_n z^n$ into $(\sum_{n=0}^{\infty} a_n z^n, \sum_{n=-\infty}^{-1} a_n z^n)$
   - $\Rightarrow$ use this to get: $\text{Hol}(\mathbb{C}^*, \mathbb{C}^n) \simeq \text{Hol}(\mathbb{C}^*, \mathbb{C}_0)$ is tame
   - prove; tame structure is independent of norm on $\mathbb{C}_0$

2. $\text{ad}(f)$ can be estimated on each compact $K \subset \mathbb{C}$ by $|f|$
   - $\text{ad}(1)$ more difficult $K=0$ is impossible.

Definition:

$\phi: U \subset F \rightarrow G$ is $(k, b, C(n))$-tame iff

$|\phi(f)|_n \leq C(n)(1+|f|) \forall n > b$

Example: $\phi: U \rightarrow V$ for $F, G$ Banach tame differs

$\Rightarrow |\phi(f)| \leq C_1 (1+|f|)$ and $|\phi^{-1}(g)| \leq C_2 (1+|g|)$

$\Rightarrow \frac{1}{C_2} (1|f|-1) \leq |\phi(f)| \leq C_1 (1+|f|)$

$\Rightarrow$ quasi-isometry!
Theorem: (Nash–Moser)

\[ F, G \text{ tame Fréchet, } \phi: U \subseteq F \to G \text{ smooth, tame} \]

Assume: \( D \phi(f) h = k \) has a unique solution \( h = V \phi(f) k \) for \( f \in U \)

and \( V \phi: U \times G \to F \) is smooth

\[ \Rightarrow \phi \text{ is locally invertible and } \phi^{-1} \text{ is smooth tame}. \]

Important remark:

Need invertibility in open subset!!

Explanation

Construct for \((F, 1, 1_n)\) and \((G, 1, 1_m)\) a sequence of Banach spaces, by completing with respect to all norms:

\[ F = \lim_{\leftarrow} B_n \quad \text{and} \quad G = \lim_{\leftarrow} C_m \]

and \( \phi: U \to V \) induces maps \( \phi_n: U_n \to U_{n+k} \)

\[ \phi_n = U_n \]

\[ F \quad G \]

and \( d\phi_n \) is invertible at \( x_0 \) \( \Rightarrow F \cap O_n(x_0) \)

\[ \phi: F \to G \text{ is then invertible on } \cap_n O_n(x_0) \text{ which might be non-open.} \]
Loop groups of holomorphic maps

\[ MG := \{ f: \mathbb{C}^* \rightarrow G \mid f \text{ holomorphic} \} \]

\[ \hat{MG} := \mathbb{C}^* \rightarrow MG \times \mathbb{C}_w^* \text{ \quad } w \cdot f(z) = f(wz) \]

Theorem:

\( MG \) and \( \hat{MG} \) are tame Fréchet manifolds.

Usual idea for proof:

use: \( \exp: g \rightarrow G \) is a local diffeomorphism

\[ \Rightarrow \exists U, V \text{ s.t. } \exp: U \rightarrow V \text{ diffeo} \]

\[ \cong \text{ Lexp: } LU \rightarrow LV \text{ is diffeo} \]

where \( LU := \{ f: \mathbb{S}^1 \rightarrow U \mid \text{regularity} \} \)

\[ LV := \{ f: \mathbb{S}^1 \rightarrow V \mid \text{regularity} \} \]

Problem: in general: \( U, V \) bounded \( \Rightarrow \) bounded holomorphic functions are constant

\[ \Rightarrow \text{ Holomorphic functions are non-local!} \]
Can we use the exponential map?

**Theorem:**
\[ \hat{\text{M}} \gamma = T_e \hat{\text{M}} G \text{ and } \text{M} \gamma = T_e \text{M} G \]

(Here the tangential space is defined as germs of curves.

**Theorem:**
\[ G \text{ simple: } \text{M} \exp \text{ is no local diffeomorphism} \]
\[ \text{Im}(\text{M} \exp) \text{ has infinite codimension!} \]

Proof by example:
\[ G = \text{SL}(2, \mathbb{C}): \text{ Fact: } \left( \begin{array}{cc} -1 & b \\ 0 & -1 \end{array} \right) \notin \text{Im}(\exp(\text{SL}(2, \mathbb{C}))) \]

\[ \Rightarrow \text{Find sequence } f_n(z) \text{ s.t.: } \lim_{n \to \infty} f_n(z) = \text{Id} \]
\[ \left( \begin{array}{cc} -1 & b \\ 0 & -1 \end{array} \right) = f_n(\text{Id}) \]

Choose \[ f_n(z) := \left( \begin{array}{cc} e^{i \theta_n} b^{\frac{z_n}{n}} & \frac{z_n}{n} \\ 0 & e^{-i \theta_n} \end{array} \right) \]
Topology on $MG$:

Lemma:
- Fretchet topology
- Compact open topology
- Inverse limit topology

\[
\Rightarrow 0 \quad f_n(z) = \begin{pmatrix} e^{i\alpha} & b \\ 0 & e^{-i\alpha} \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix} \Rightarrow f_n \notin \text{Im} \text{Mer}(MG)
\]

(2) On any compact subset $K$ we have

\[
\lim_{n \to \infty} f_n(z) = \begin{pmatrix} e^{i\frac{z}{n}} & b \frac{z}{n} \\ 0 & e^{-i\frac{z}{n}} \end{pmatrix} \Rightarrow \begin{pmatrix} z & \to 0 \\ 0 & \to 1 \end{pmatrix}
\]

Idea: larger setting i.e.: $C^\infty$ loops ...

Problem:

Lemma

$MG$ is the maximal group allowing for the semi-direct product with $C^*$, corresponding to $d$. 

\[\text{44}\]
Proof:

1. The two dimensional bundle makes no problem, as Banach bundles over tame spaces are tame.

\[ MG \xrightarrow{\phi} \text{Map} \times G \xrightarrow{\pi_2} \text{Map} \]

\[ f \mapsto (\phi^{-1} f', f(1)) \]

\[ \pi_1 \rightarrow G \]

2. \( \phi \) is injective: assume: \( f^{-1} f' = g^{-1} g' = \alpha \)

\[ \Rightarrow f' = f \alpha, \quad f(1) = g(1) \]

\[ g' = g \alpha \]

\[ \Rightarrow \text{Uniqueness for } D6L \Rightarrow f = g \]

3. Charts for \( \phi(MG) \) are constructed as products for charts in \( G \times \text{Map} \).

4. \( \pi_1 \) is surjective, as \( f(1) \) can take any value in \( G \).

\[ \text{Im}(\pi_2) = \{ \alpha \in \text{Map} \mid e^{\frac{\alpha}{s_1}} = \text{Id} \} \]

Monodromy map:

\[ \Rightarrow \text{Idea: Show that 1 is regular value (based on Nash-Moser)} \]

and prove implicit function. Inverse of regular value is sub manifold.
Compact forms:

\[ (\text{Mo}_c^f) := \{ f \in \text{Mo}_f \mid f(x) \in \text{c o}_c \} \]

where subscript c denotes the compact real form.

Similar for loop groups.

- All spaces are tame, as fixed point spaces of involutions are closed \( \Rightarrow \) tame.

- Similarly quotients are tame as in all cases, if we consider the space \( K \) is closed.
Theorem (Popescu)

Any tame Fréchet Lie group admits a unique left invariant connection, such that $\nabla_x Y = \frac{1}{2} [x, Y]$

If the group admits a metric, then this is the LC-connection

Curvature

$$\text{sect}(x, y) = \frac{1}{4} \frac{||[x, y]||^2}{||x \wedge y||^2}$$

Theorem (generalized Vuk Karni)

1) $\text{sect}(M) < a$
2) $\text{sect}(M) > b$
3) $\text{sect}(M)$ constant

equivalent for tame Fréchet Lorentzian mfg with metric and LC.
Adjoint action:

\[ x = (w, g, z) \quad \text{and} \quad (u \in Mg) \]

\[ C_d \overset{\mathcal{M}}{\leftrightarrow} \overset{\mathcal{C}}{\mathcal{G}} \overset{\mathcal{C}^2}{\mathcal{B}undle} \]

\[ Ad(x) u = g \omega(u) g^{-1} + \langle g(w(u) g^{-1}, g' g^{-1} \rangle c \]

\[ Ad(x) c = c \]

\[ Ad(x) d = d - g' g^{-1} + \frac{1}{2} \langle g' g^{-1}, g' g^{-1} \rangle c \]

Note for tomorrow: \( d \) coefficients \( \nu_1 \) are preserved.

Ad invariant scalar product:

\[ \langle u, v \rangle = \frac{1}{2\pi} \int_0^{2\pi} \langle u(t), v(t) \rangle dt \]

\[ \langle c, d \rangle = -1 \]

\[ \langle c, c \rangle = \langle d, d \rangle = \langle f, c \rangle = \langle f, d \rangle = 0 \]

Theorem:

On the unitary form, this scalar product is Lorentzian.
Definition:
A KM-sym space is a Lorentzian tame Frechet symmetric space, whose translation group is KM.

Remark:
There should be some characterization, but problems compare finite dim Lorentz spaces.

Classification of irreducible ones
via OSAKAs

Definition
An OSAKA is a pair consisting of a KMA and some involution, whose fixedpoint algebra is a loop algebra of the compact type.
\[(\hat{\mathcal{M}}G_0)_{/\text{Fix}(6)} \quad \hat{\mathcal{M}}G_0/\mathcal{M}G_0 \quad \hat{\mathcal{M}}G_0/\mathcal{M}G_0 \quad IV\]

\[\langle R(x, y) x, y \rangle \geq 0 \quad \langle R(x, y) x, y \rangle < 0 \quad \text{diffeomophic to VS} \]

- Affine WMA's are classified by Kac
- Involutions by Levenstein, Rousseau, Heitze et al.

Remark:
As a consequence of Vulkarni, we do not get curvature bounds, the problem is an exploding denominator; but Jacobi equation shows, that the numerator is the important part.