Lecture 3 - The spectrum & classification of tt-ideals (Balmer)

$\mathcal{K}$ essentially small (rigid) tt-cat. (e.g. $\mathcal{K} = \mathcal{T}^c$ for $\mathcal{T}$ Δ-ed)

Remark: R comm. ring, $\mathcal{K} = D^{perf}(R) \cong K^b(R\text{-proj})$.
Then R local $\Leftrightarrow$ $\mathcal{K}$ is "local" in the sense that

Def^n: $\mathcal{K}$ is local if whenever $x, y \in \mathcal{K}$ are s.t. $x \otimes y = 0$ then $x = 0$ or $y = 0$

Q: How to go from general $\mathcal{K}$ to a local one?

 If $F : \mathcal{K} \xrightarrow[tt]{} \mathcal{L}$ with $\mathcal{L}$ local then

$$\underset{\substack{\cap \\ \mathcal{K}}}{\mathrm{Ker}(F)} = \{x \in \mathcal{K} \mid F(x) = 0\} =: \mathcal{P} \text{ is a } \underset{\substack{\Vert \\ \text{thick triangulated} \\ \otimes\text{-ideal subcat}}}{\text{tt-ideal}}$$

$$\Rightarrow \begin{tikzcd} \mathcal{K} \arrow[rr, "F"] \arrow[dr, "Q"'] & & \mathcal{L} \\ & \mathcal{K}/\mathcal{P} \arrow[ur, "\bar{F}"'] & \end{tikzcd}$$

$\bar{F}$ conservative

Def^n: A tt-ideal $\mathcal{P} \subset \mathcal{K}$ is a prime if it is proper and $x \otimes y \in \mathcal{P}$ implies $x \in \mathcal{P}$ or $y \in \mathcal{P}$.

Notation: $\mathrm{Spc}(\mathcal{K}) := \{\mathcal{P} \subseteq \mathcal{K} \text{ prime}\}$

Def^n: For $x \in \mathcal{K}$ let ~~Supp~~ $\mathrm{supp}(x) \subseteq \mathrm{Spc}(\mathcal{K})$ be

$$\mathrm{supp}(x) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid x \notin \mathcal{P}\}$$
$$= \left\{\mathcal{P} \;\middle|\; \begin{array}{l} \mathcal{K} \to \mathcal{K}/\mathcal{P} \\ x \mapsto Q_{\mathcal{P}}(x) \neq 0 \end{array}\right\}$$

Prop^n: 0) $\mathrm{supp}(0) = \emptyset$
$\mathrm{supp}(\mathbb{1}) = \mathrm{Spc}(\mathcal{K})$

1) $\mathrm{supp}(x \oplus y) = \mathrm{supp}(x) \cup \mathrm{supp}(y)$
2) $\mathrm{supp}(\Sigma x) = \mathrm{supp}(x)$
3) $\forall$ ex. Δ $x \to y \to z \to \Sigma x$, have $\mathrm{supp}(z) \subseteq \mathrm{supp}(x) \cup \mathrm{supp}(y)$
4) $\mathrm{supp}(x \otimes y) = \mathrm{supp}(x) \cap \mathrm{supp}(y)$ (since $\mathcal{P}$ prime)

Def^n: We equip $\mathrm{Spc}(\mathcal{K})$ with the top. which has $\{\mathrm{supp}(x) \mid x \in \mathcal{K}\}$ as a basis of closed sets.

For each $S \subseteq \mathrm{ob}\,\mathcal{K}$,

$$Z(S) = \bigcap_{x \in S} \mathrm{supp}(x) = \{P \mid P \cap S = \emptyset\}$$

is a general closed set.

Theorem 1: The pair $(\mathrm{Spc}(\mathcal{K}), \mathrm{supp})$ is the universal support data: if X a top. space and $\sigma(x) \subseteq X$ closed $\forall x \in \mathcal{K}$ satisfying (0) – (4) above then $\exists!$ cts map $f: X \longrightarrow \mathrm{Spc}(\mathcal{K})$ s.t. $\sigma(x) = f^{-1}(\mathrm{supp}(x))$ $\forall x \in \mathcal{K}$

Proof: (Ex) $f(x) = \{y \in \mathcal{K} \mid x \notin \mathrm{supp}(y)\} \in \mathrm{Spc}(\mathcal{K})$ $\square$

~~Thm~~ **Not^n**: For each $Y \subseteq \mathrm{Spc}(\mathcal{K})$ can consider

$$\mathcal{K}_Y = \{x \in \mathcal{K} \mid \mathrm{supp}(x) \subseteq Y\}$$

This is a tt-ideal.

Remark: $\mathcal{K}_Y$ is radical: $x^{\otimes n} \in \mathcal{K}_Y \Rightarrow x \in \mathcal{K}_Y$.

If $\mathcal{K}$ is rigid $(\Rightarrow \underset{\text{direct summand}}{x \leq x \otimes x^\vee \otimes x})$ any tt-ideal J is automatically radical $(x^2 \in J \Rightarrow x \otimes x^\vee \otimes x \in J \Rightarrow x \in J)$

Thm: The assignment $Y \longmapsto \mathcal{K}_Y$ is an inclusion-preserving bijection, between

$$\left\{ \begin{array}{l} Y \subseteq \mathrm{Spc}(\mathcal{K}),\ Y = \bigcup Y_\alpha \text{ with} \\ \mathrm{Spc}(\mathcal{K}) \setminus Y_\alpha \text{ q-compact open} \end{array} \right\} \xrightarrow{\sim} \{J \subseteq \mathcal{K} \text{ tt-ideals}\}$$

$$\mathrm{supp}(J) := \bigcup_{x \in J} \mathrm{supp}(x) \longleftarrow J$$

$$Y \longrightarrow \mathcal{K}_Y$$

Def^n: $Y \subseteq \mathrm{Spc}(\mathcal{K})$ s.t. $Y = \bigcup_\alpha Y_\alpha$ with Y_α closed & $\mathrm{Spc}(\mathcal{K}) \setminus Y_\alpha$ q-compact are called Thomason subsets (or "dual-open" in the sense of Hochster)

Remark: A space S is spectral (Hochster) if it is q-compact & has a basis of q-compact open, and every irred. closed non-empty $Z \subseteq S$ has a unique generic pt.

i.e. $z \in Z$ s.t. $\overline{\{z\}} = Z$. | **Prop** $\mathrm{Spc}(\mathcal{K})$ is spectral

Ex: a) $P \in \mathrm{Spc}(\mathcal{K}) \Rightarrow \overline{\{P\}} = \{Q \in \mathrm{Spc}(\mathcal{K}) \mid Q \subseteq P\}$

b) Explain why $\subseteq$, not $\supseteq$, compared w/ definition of Zariski topology.

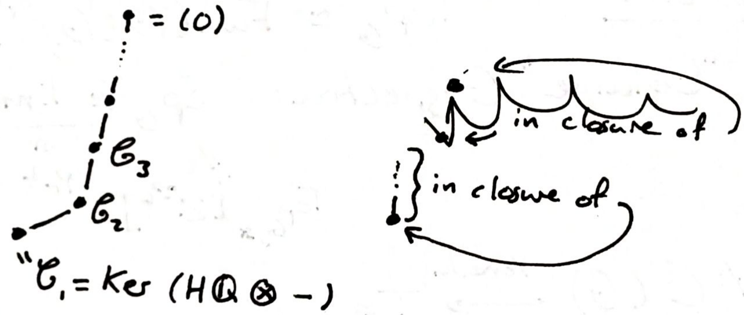
Thm: (B. / Buan - Krause - Solberg)
Let $\mathcal{K}$ be tt-cat. If (S, σ) are support data s.t.
1) S is spectral, and
2) The assignment $Y \mapsto \mathcal{K}^{\sigma}_{Y} := \{x \in \mathcal{K} \mid \sigma(x) \subseteq Y\}$ is a bijection as in the previous thm,

then the universal map $f: S \longrightarrow \mathrm{Spc}(\mathcal{K})$ is a homeomorphism.

Ex: $\mathcal{K}$ local $\Leftrightarrow$ $\mathrm{Spc}(\mathcal{K})$ has a unique closed pt.

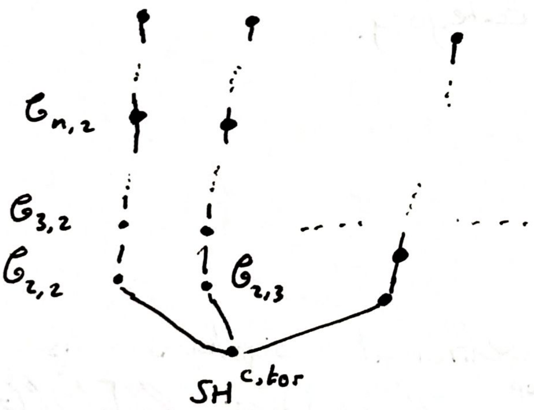
Thm: (Hopkins - Smith)

$\mathrm{Spc}(SH^{c}_{(p)}) =$



where $\mathcal{C}_n = \mathrm{Ker}(K(n-1) \otimes -)$, Morava K-theory (at p).

Cor: $\mathrm{Spc}(SH^c)$



Thm: (Thomason)

X a q.c. & q.sep scheme then $\mathrm{Spc}(D^{perf}(X)) = X$

$$P_{\xi} := \mathrm{Ker}\,(D^{perf}(X) \to D^{perf}(\mathcal{O}_{X,\xi})) \leftarrow\!\shortmid\ \xi$$

Cor: $\mathrm{Spc}(K^b(R\text{-proj})) = \mathrm{Spec}(R)$

Thm: (BCR / BFIKP)

$\mathrm{Spc}(\mathrm{stmod}(kG)) = \mathrm{Proj}\, H^*(G;k)$
finite gp (scheme)

$\mathcal{U} \subseteq \mathrm{Spc}(K)$ w/ compl. Z
q.c

idem. compl.

$$K_Z \rightarrowtail K \twoheadrightarrow (K/K_Z)^{\natural} = K(\mathcal{U})$$

$\rightsquigarrow$ sheaf of categories

$\rightsquigarrow \mathcal{U} \mapsto \mathrm{End}_{K(\mathcal{U})}(\mathbb{1})$ sheaf of rings on $\mathrm{Spc}(K)$

Lecture 4 – Nilpotence & descent (Noel)

Last time: $\mathrm{Top}_G \simeq \mathrm{Fun}(\mathcal{O}(G)^{op}, \mathrm{Top})$

"Genuine" G-spectra $\mathrm{Sp}_G \simeq \underset{\in \mathrm{Pr}^L}{\varinjlim}(\mathrm{Top}_{G,*} \xrightarrow{\Sigma^{\rho_G}} \mathrm{Top}_{G,*} \to \dots)$

$= \mathrm{Top}_{G,*}[\Sigma^{-\rho_G}]$

$Y: \mathcal{O}(G) \xhookrightarrow{\text{Yoneda}} \mathrm{Top}_G$

Thm (Dugger): Y exhibits Top_G as the homotopy cocompletion of the orbit category.

$X \in \mathcal{C} \in \{\mathrm{Top}_G, \mathrm{Top}_{G,*}, \dots\}$ is compact when the canonical map

$$\varinjlim_{\text{filtered}} \mathcal{C}(X, Y_i) \to \mathcal{C}(X, \varinjlim_{\text{filtered}} Y_i)$$ is an equiv.

Def^n: Category of coefficient systems is

$$\mathrm{Fun}(\mathcal{O}(G)^{op}, \mathbb{Z}\text{-mod}) =: \mathbb{Z}[\mathcal{O}(G)]$$

Also a chain complex version

$$\mathrm{Fun}(\mathcal{O}(G)^{op}, \mathrm{Ch}(\mathbb{Z}))$$

Given a G-space X, coeff. system M, define Bredon cohomology of X w/ coeffs in M as

$$H^*(\mathrm{Hom}_{\mathbb{Z}[\mathcal{O}(G)]}(C_*(X), M))$$

where $C_*(X): G/H \mapsto C_*(X^H)$ e.g. CW/singular chains.

Ex: • There is the constant coefficient system $\underline{\mathbb{Z}}$ (constant functor at $\mathbb{Z}$)

• Burnside ring coeff. system: $\frac{G}{H} \mapsto A(H) :=$ gp completion of ($\cong$-classes of finite H-sets, $\amalg$)

Exercise: Calculate the C_p-Bredon cohomology of $S(V)$ (unit sphere) where V is an irred. $\mathbb{R}-C_p$-rep. with coeffs in $\underline{\mathbb{Z}}$ and $A(-)$

Prelim: calculate $H^*_G(\frac{G}{H} \times S^N, \underline{M})$

We had $\mathrm{Res}^G_H : \mathrm{Top}_G \to \mathrm{Top}_H$ closed symm. monoidal

$(\mathrm{Res}^G_H(X\times Y) \simeq \mathrm{Res}^G_H X \times \mathrm{Res}^G_H Y$ & $\mathrm{Res}^G_H \mathrm{Top}(X,Y) \simeq \mathrm{Top}(\mathrm{Res}^G_H X, \mathrm{Res}^G_H Y))$

$\Rightarrow \mathrm{Ind}^G_H(X \times \mathrm{Res}^G_H Y) \xrightarrow{\sim} \mathrm{Ind}^G_H X \times Y$

In particular, $\mathrm{Ind}^G_H \mathrm{Res}^G_H Y \xrightarrow{\sim} \frac{G}{H} \times Y$.

We had $i_{\mathcal{F}} : \mathcal{O}(G)_{\mathcal{F}} \longrightarrow \mathcal{O}(G)$

$\rightsquigarrow i^* : \mathrm{Fun}(\mathcal{O}(G)^{op}, \mathrm{Top}) \to \mathrm{Fun}(\mathcal{O}(G)_{\mathcal{F}}^{op}, \mathrm{Top})$

again closed symm. monoidal

and $i_{\mathcal{F}!} i_{\mathcal{F}}^* X \simeq i_{\mathcal{F}!} i_{\mathcal{F}}^*(*) \times X \simeq E\mathcal{F} \times X$

Recall: $(E\mathcal{F})^H = \begin{cases} \emptyset & \text{if } H \notin \mathcal{F} \\ * & \text{if } H \in \mathcal{F} \end{cases}$

and $E\mathcal{F} \simeq \operatorname{hocolim}_{\mathcal{O}(G)_{\mathcal{F}}} \frac{G}{H}$

Bousfield-Kan formula:

Bousfield-Kan formula for $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Top}) \underset{i_! = \mathrm{Lan}}{\overset{i^*}{\rightleftarrows}} \mathrm{Fun}(\mathcal{D}^{op}, \mathrm{Top})$, $i : \mathcal{C} \to \mathcal{D}$

Idea: a bit like base-change; if M a right R-mod & $i: R \to S$ ring map then the induced S-mod of M along i is $M \otimes_R S$ (derived setting $\rightsquigarrow$ derived $\otimes^{\mathbb{L}}_R$)

For $d \in \mathcal{D}$, $i_! F(d) = F(-) \otimes^{\mathbb{L}}_{\mathcal{C}} \mathcal{D}(d, i(-))$

$$= \left| \coprod_{c \in \mathcal{C}} F(c) \times \mathcal{D}(d, i(c)) \leftleftarrows \coprod_{c_1, c_2} F(c_2) \times \mathcal{C}(c_1, c_2) \times \mathcal{D}(d, i(c_1)) \leftleftarrows \cdots \right|$$

(where F is objectwise cofibrant)

As a special case, when $\mathcal{D} = \star$ then $i_! F \simeq \underrightarrow{\text{hocolim}}\, F$

$$\underleftarrow{\quad}$$

$$E\mathcal{F} = i_! i^*(\underbrace{c(*)}_{\text{constant functor}})$$

$$= i^*(*) \otimes^L_{\mathcal{O}(G)_{\mathcal{F}}} \mathcal{O}(G)(-, i_{\mathcal{F}}(\ ))$$

$$= * \otimes^L_{\mathcal{O}(G)_{\mathcal{F}}} i_{\mathcal{F}}(\ )$$

$$= \underrightarrow{\text{hocolim}}_{G/H \in \mathcal{O}(G)_{\mathcal{F}}} G/H$$

so $E\mathcal{F}$ is uniquely characterised by its fixed point data, i.e. if $X^H \simeq \begin{cases} \emptyset & H \notin \mathcal{F} \\ * & H \in \mathcal{F} \end{cases}$ then $EF \xrightarrow{\sim} X$ ($\uparrow$ contractible space of such maps.)

(this comes from $i^*_{\mathcal{F}}(*) \to i^*_{\mathcal{F}}(X)$
$\rightsquigarrow i_{\mathcal{F}!}\, i^*_{\mathcal{F}}(*) \to X$)

$$\underleftarrow{\quad}$$

$$\mathrm{Top}_G \xrightarrow[(\cdot)_+]{} \mathrm{Top}_{G,*} \quad \Rightarrow \quad E\mathcal{F}_+ \to S^0 \xrightarrow[\text{cofibre}]{} \tilde{E}\mathcal{F}$$

where $(\tilde{E}\mathcal{F})^H = \begin{cases} * & H \in \mathcal{F} \\ S^0 & H \notin \mathcal{F} \end{cases}$

Let P be family of proper subgps. Then claim

$$\tilde{E}P \simeq \underrightarrow{\text{colim}}\,(S^0 \to S^{\tilde{\rho}_G} \to S^{2\tilde{\rho}_G} \to \dots)$$

$\tilde{\rho}_G = \mathbb{R}G/\Delta$. Note $\Delta \cong (\mathbb{R}G)^G$

$$(\tilde{E}P)^G \simeq \underrightarrow{\text{colim}}\,((S^0)^G \to (S^{\tilde{\rho}_G})^G \to \dots)$$

$$\simeq \underrightarrow{\text{colim}}\,(S^0 \xrightarrow{\sim} S^0 \xrightarrow{\sim} \dots)$$

$$\mathrm{Res}^G_H \rho_G = \mathrm{Res}^G_H (\tilde{\rho}_G + 1)$$

$$\Rightarrow |G/H|\, \rho_H = \mathrm{Res}^G_H \tilde{\rho}_G + 1$$

$$\Rightarrow |G/H|\, \rho_H - 1 = \mathrm{Res}^G_H \tilde{\rho}_G$$

$\Rightarrow \mathrm{Res}^G_H S^{k\tilde{\rho}_G}$ is increasingly connected, so

$(S^{\infty\tilde{\rho}_G})^H$ is contractible

For any rep V get $S^0 \xrightarrow{e_V} S^V$

$\rightsquigarrow X \wedge \tilde{E}P \simeq X[e_{\tilde{\rho}_G}^{-1}]$

The map $X \longrightarrow X \wedge \tilde{E}P$ is a smashing localisation.
If $X \xrightarrow{\sim} X \wedge \tilde{E}P$ then $X \xrightarrow{\sim} \Sigma^{\tilde{\rho}_G} X$

G/H_+

$$\mathrm{Top}_{G,*}[\Sigma^{-\rho_G}] \simeq \mathrm{Top}_{G,*}[\Sigma^{-\tilde{\rho}_G}][\Sigma^{-1}]$$

$$\downarrow \wedge \tilde{E}P$$

$$\begin{aligned}
\mathrm{Top}_{G,*}[\tilde{E}P][\Sigma^{-\rho_G}] &\simeq (\mathrm{Top}_{G,*}[\tilde{E}P][\Sigma^{-\tilde{\rho}_G}])[\Sigma^{-1}] \\
&\simeq \mathrm{Top}_{G,*}[\tilde{E}P][\Sigma^{-1}] \\
&\simeq \mathrm{Top}_*[\Sigma^{-1}] \\
&\simeq Sp.
\end{aligned}$$

$$\begin{cases} * & \text{if } H < G \\ S^0 & \text{if } H = G \end{cases}$$

i.e. this agrees with $(-)^G$

$\Rightarrow \Phi^G X \simeq (\tilde{E}P \wedge X)^G$