$$\begin{tikzcd}
E\mathcal{F}_+ \wedge M \arrow[d, phantom, "\cong" rotate=90] & & & \\
\operatorname{hocolim}_{\mathcal{O}(G)_\mathcal{F}} \operatorname{Ind}_H^G \operatorname{Res}_H^G M \arrow[r, "\sim"] \arrow[d, "\cong\ (Ex)"] & M \arrow[r] \arrow[d, "\cong"] & \tilde{E}\mathcal{F} \wedge M^{\simeq *} \arrow[d] & \leftarrow \text{cofibre seqs} \arrow[dl, bend left] \\
E\mathcal{F}_+ \wedge F(E\mathcal{F}, M) \arrow[r, "\sim"] & \operatorname{holim} \operatorname{CoInd} \operatorname{Res} M \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \tilde{E}\mathcal{F} \wedge F(E\mathcal{F}_+, M)^{\simeq *} & \\
& F(E\mathcal{F}_+, M) & &
\end{tikzcd}$$

If M is $\mathcal{F}$-nil, $\tilde{E}\mathcal{F} \wedge M \simeq 0$ and $\tilde{E}\mathcal{F} \wedge F(E\mathcal{F}_+, M) \simeq *$ then annotations follow.

Lecture 7 – Faithful descent & applications (Balmer)

Yesterday: $\begin{tikzcd} \mathcal{K} \arrow[d, shift right, "F"'] \\ \mathcal{L} \arrow[u, shift right, "U"'] \end{tikzcd}$ $\quad \varphi = \operatorname{Spc} F \uparrow$ $\begin{matrix}\mathcal{K} \\ \operatorname{Spc}\mathcal{L}\end{matrix}$ $\quad \operatorname{Im}(\varphi) = \operatorname{supp}(\underbrace{U(\mathbb{1})}_{\text{ring object.}})$

Cor: $\operatorname{Im}(\operatorname{Res}^G_H) = \operatorname{supp}(A^G_H)$

$\mathcal{K}(G) = SH(G)^c$ (or any "Mackey tt-cat")

$A^G_H = \operatorname{coind}^G_H(\mathbb{1}) \underset{\uparrow \text{Wirth.}}{\cong} \operatorname{Ind}^G_H(\mathbb{1}) = \Sigma^\infty_G (G/H)_+$

Then $\operatorname{Im}(\operatorname{Spc}(\mathcal{K}(H)) \longrightarrow \operatorname{Spc}(\mathcal{K}(G))) = \operatorname{supp}(G/H_+)$

$$\begin{tikzcd}
\operatorname{Spc}(\mathcal{K}(G)^c) = \underbrace{\bigcup_{H<G} \operatorname{supp}(G/H_+)}_{\text{closed}} & P_G(K_{p,n}) \cup \underbrace{(\text{complement})}_{\text{open}} \\
\operatorname{Spc}(\mathcal{K}(H)) \arrow[u, "\operatorname{Spc}(\operatorname{Res}^G_H)"] & P_H(K_{p,n}) \arrow[u, mapsto]
\end{tikzcd}$$

By ind. on $|G|$, we need:

① understand the open complement ("Brauer open")

② understand when $P_1, P_2 \in \operatorname{Spc}(\mathcal{K}(H))$ have same image in $\operatorname{Spc}(\mathcal{K}(G))$.

This Brauer open $\operatorname{Spc}(\mathcal{K}(G)) \setminus \bigcup_{H<G} \operatorname{Im}(\operatorname{Spc}(\mathcal{K}(H)))$

$= \{ P \mid \forall H<G,\ P \ni G/H_+ \} = \{ P \mid \forall H<G : P \supset \operatorname{thick}^{\otimes}(G/H_+) \}$

$$\cong \mathrm{Spc}\left(\frac{\mathcal{K}(G)}{\langle G/H_+ \mid H < G\rangle}\right) =: \overline{\mathcal{K}(G)} \quad \text{"Brauer quotient"}$$

Example: $\cancel{k}\ \mathcal{K}(G) = D^b(kG\text{-mod})$ or $\mathrm{stmod}(kG)$

$\Rightarrow \overline{\mathcal{K}(G)} = 0$ unless G is elem. ab.

(Serre's thm / Chouinard's thm)

Remark: if $\exists$ $\mathcal{K}(1) \xrightarrow{\text{triv}} \mathcal{K}(G) \twoheadrightarrow \overline{\mathcal{K}(G)}$ (e.g. not stmod) then we can ask if this is an equivalence.

Thm: If $\mathcal{K}(-) = SH(-)^c$ then $SH \xrightarrow{\sim} \overline{SH(G)}$ (inverse is Φ^G)

Cor:

$$\begin{tikzcd}
\langle G/H_+ \mid H < G\rangle \arrow[dr, hook] & & \\
& SH(G)^c \arrow[r, two heads, "\Phi^G"] \arrow[d, "\pi"] & SH^c \arrow[dl, "\simeq"] \\
& \overline{SH(G)^c} &
\end{tikzcd}$$

$= - \wedge \tilde{E}\mathcal{P}$ in Justin's language

$E\mathcal{P}_+ \to \mathbb{1} \to \tilde{E}\mathcal{P} \to$

Then $\mathrm{Spc}(\cancel{\otimes}\mathcal{K}(G)) = \bigcup_{H<G} \mathrm{Im}(\mathrm{Spc}(\mathcal{K}(H))) \cup \{\pi^{-1}(Q) \mid Q \in \mathrm{Spc}(\overline{\mathcal{K}(G)})\}$

When $\mathcal{K}(G) = SH(G)^c$ then $\pi = \Phi^G$ & $\overline{\mathcal{K}(G)} = SH^c$ so

~~open Balmer open~~ $\pi^{-1}(Q)$ is $\underbrace{(\Phi^G)^{-1}(\mathcal{C}_{p,n})}_{\mathcal{P}_G(G,p,n)}$

$\mathcal{K}$ tt-cat, rigid & idempotent-complete. (e.g. $\mathcal{K} = \mathcal{T}^c$)

A a separable commutative ring in $\mathcal{K}$.

Degree: $A = A^{[1]} \to A^{[2]} \to \cdots \to A^{[n]} \to A^{[n+1]} \to \cdots$

$A^{[n]} \neq 0$, $A^{[n+1]} = 0$

$\Downarrow$

A is of degree n.

Fact: $A \otimes A \cong A \times A^{[2]}$ as rings, $A^{[2]}$ unique up to non-unique $\cong$.

$\underset{\mu}{\curvearrowright}$

$$A^{[n]} \otimes_{A^{[n-1]}} A^{[n]} \cong: A^{[n]} \times A^{[n+1]}$$

Example: $\deg(1^{\times n}) = n$

$\mathrm{supp}(A) \subseteq \mathrm{supp}(B)$ then $\deg_K(A) = \deg_{B\text{-Mod}_K}(F_B(A))$

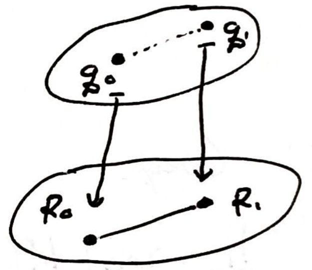
Hypothesis: $\deg(A) < \infty$ (prove things by induction on degree)

Thm: ("~~Going up~~") Let A sep. comm. of finite degree.

$$F_A : \mathcal{K} \longrightarrow A\text{-Mod}_{\mathcal{K}} \rightsquigarrow \varphi_A = \mathrm{Spc}(F_A) : \underbrace{\mathrm{Spc}(A\text{-Mod}_{\mathcal{K}})}_{= \mathrm{Spc}(A)} \to \mathrm{Spc}(\mathcal{K})$$

a) $\mathrm{Im}(\varphi_A) = \mathrm{supp}(A)$

"Going up" b) Let $\mathfrak{q}_0 \in \mathrm{Spec}\, A$ and $\mathfrak{p}_0 = \varphi_A(\mathfrak{q}_0)$. Let $\mathfrak{p}_1 \in \overline{\{\mathfrak{p}_0\}}$.
Then $\exists\, \mathfrak{q}_1 \in \overline{\{\mathfrak{q}_0\}}$ s.t. $\varphi_A(\mathfrak{q}_1) = \mathfrak{p}_1$



"Incomparability" c) If $\mathfrak{q}_1 \in \overline{\{\mathfrak{q}_0\}}$ and $\varphi_A(\mathfrak{q}_0) = \varphi_A(\mathfrak{q}_1)$
then $\mathfrak{q}_0 = \mathfrak{q}_1$.

d) For every $\mathfrak{p} \in \mathrm{Spc}(\mathcal{K})$ the fibres $\varphi_A^{-1}(\mathfrak{p})$ ~~are~~ is finite & discrete.

e) $\dim(\mathrm{Spc}(A)) = \dim(\mathrm{supp}(A))$.

~~Cor~~ **Thm**: Consider $A \overset{\eta\otimes 1}{\underset{1\otimes\eta}{\rightrightarrows}} A\otimes A$ (ring homs)

$$\begin{tikzcd} 1 \arrow[r, "\eta"] \arrow[d, "\eta"'] & A \arrow[d, "\eta\otimes 1"] \\ A \arrow[r, "1\otimes\eta"'] & A\otimes A \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathrm{Spc}(A\otimes A) \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \mathrm{Spc}(A) \arrow[d] \arrow[ddr, bend left, "\varphi_A"] & \\ \mathrm{Spc}(A) \arrow[r] \arrow[drr, bend right, "\varphi_A"'] & \mathrm{supp}(A) \arrow[dr, phantom, "\subseteq"] & \\ & & \mathrm{Spc}(K) \end{tikzcd}$$

We have a coequaliser of spaces:

$$\begin{tikzcd} \mathrm{Spc}(A\otimes A) \arrow[r, shift left, "\varphi_1"] \arrow[r, shift right, "\varphi_2"'] & \mathrm{Spc}(A) \arrow[r, two heads, "\varphi_A"] \arrow[dr, "\varphi_A"'] & \mathrm{supp}(A) \arrow[d, phantom, "\cap"] \\ & & \mathrm{Spc}(K) \end{tikzcd}$$

In particular, if $Q_1, Q_2 \in \mathrm{Spc}(A)$ s.t. $\varphi_A(Q_1) = \varphi_A(Q_2)$
then $\exists\, Q \in \mathrm{Spc}(A \otimes A)$ s.t. $\varphi_1(Q) = Q_1$
& $\varphi_2(Q) = Q_2$

<u>Often</u>: $\mathcal{K} = \mathcal{K}(G)$ and $A = A^G_H = \mathrm{CoInd}^G_H(\mathbb{1})$
and $A^G_H\text{-}\mathrm{Mod}_{\mathcal{K}(G)} \cong \mathcal{K}(H)$

i.e. φ_A induces by Res^G_H

$A \otimes A$ given by double coset formula

<u>Good news</u>: $A^G_H \otimes A^G_K \cong \prod_{H \backslash G / K} A^G_{H^x \cap K}$

$$
\begin{tikzcd}
 & \prod_{H \backslash G / K} A^G_{H^x \cap K} & \\
A^G_H \arrow[ur, "{}^x\mathrm{Res}^G_H"] & & A^G_K \arrow[ul, "\mathrm{Res}^G_K"']
\end{tikzcd}
$$

<u>Cor</u>: Coeq.

$$\coprod_{x \in H \backslash G / K} \mathrm{Spc}(\mathcal{K}(H^x \cap H)) \rightrightarrows \mathrm{Spc}(\mathcal{K}(H)) \twoheadrightarrow \mathrm{supp}(A^G_H) \subseteq \mathrm{Spc}(\mathcal{K}(G))$$

<u>Exercise</u>: $A \in \mathcal{K}$, $\mathrm{supp}(A) = \mathrm{Spc}(\mathcal{K}) \iff \langle A \rangle = \mathcal{K}$

$$\iff \begin{cases} \text{if } \bullet \xrightarrow{\xi} \mathbb{1} \xrightarrow{\eta} A \to \bullet \text{ exact} \\ \text{then } \exists\, n \text{ s.t. } \xi^{\otimes n} = 0 \end{cases}$$

$\iff f : x \to y$ s.t. $A \otimes f = 0$ implies $f^{\otimes n} = 0$ for $n \gg 0$

"Descent up to nilpotence".

Lecture 8 - TT ideals in genuine A-spectra (Noel)

$\mathrm{Spc}(SH_G^\omega)$ w/ pts the prime ideals given by
$\{X \in Sp_G^\omega \mid K(n-1)_* \Phi^H X = 0\} = P(H, p, n)$

Note $K(n)_*(X) = 0 \iff E_n^*(X) = 0$
$\Updownarrow$ — Morava E-theories
$K(n)^*(X) = 0$

$E_1 = KU_{\hat{p}}$ (we'll often omit p-completions from now on)

For a finite gp G, $KU_{G,\hat{p}}$ is an equivariant refinement of E_1. $\pi_*^G KU_G = R(G)_{\hat{p}}[\beta^{\pm}]$

Complete KU_G wrt the trivial family i.e. Borel complete
$KU_G \longrightarrow F(EG_+, KU_G)$
$\quad\quad\quad\quad \underline{\underset{KU}{\cong}}$

Thm (Atiyah-Segal)
On π_*^G this map induces $R(G)[\beta^{\pm}] \longrightarrow R(G)[\beta^{\pm}]_{\hat{I}}$
where $I = \ker(R(G) \to R(\{e\}))$
If G is a p-gp, then $KU_{G,\hat{p}} \xrightarrow{\sim} F(EG_+, KU_{G,\hat{p}})$

To identify the topology on $\mathrm{Spc}(SH^\omega(G))$ it suffices to identify the basis opens.
$$\{ P \in \mathrm{Spc}(SH^\omega(G)) \mid x \in P\}_{x \in SH^\omega(G)}$$
(Fix p.) For $X \in Sp^\omega$, set $\mathrm{type}(X) = \begin{cases} \infty & \text{if } H_*(X; \mathbb{F}_p) = 0 \\ 0 & \text{if } H_*(X; \mathbb{Q}) \neq 0 \\ n & \text{if } n \text{ maximal s.t. } K(n-1)_* X = 0 \end{cases}$

To identify the basic opens, we have to find (for each prime p) which functions
$$f : \mathrm{subgp}\, G/\mathrm{conj} \longrightarrow [0, \dots, \infty]$$

are of the form

$$f_Y : H \longmapsto \mathrm{Type}(\Phi^H Y)$$

for some $Y \in SH^\omega(G)$

$$\begin{tikzcd}
 & & P(C_{p^2}, p, n-1) \arrow[d] \\
G=C_{p^2};\ P(\{e\},p,n) \arrow[r, hook] \arrow[dr] \arrow[urr, dashed, "?"] & P(C_p,p,n-1) \arrow[r, hook] \arrow[d] & P(C_{p^2}, p, n-2) \\
 & P(C_p, p, n-2) &
\end{tikzcd}$$

In particular, if $X \in SH^\omega(C_p)$, non-equivariantly the type of X is 2, then $H_*(\Phi^{C_p}X, \mathbb{Q}) = 0$.

Suppose $f_X : \{e\} \longmapsto 2$ ①
$ C_p \longmapsto 0$ ② (for a contradiction).

Then ~~F(X∧EG+~~ $KU^*_{C_p}(X) \simeq KU^*(EC_{p+} \wedge X) = \pi_{-*} F(X, KU)^{hG}$

$\pi^{C_p}_{-*} F(X, KU_{C_p}) \overset{①}{=} 0$

$$F(X, KU_{C_p}) \simeq * \Rightarrow \Phi^{C_p} F(X, KU_{C_p}) \simeq *$$
$$\Rightarrow F(\Phi^{C_p}X, \Phi^{C_p}KU_{C_p}) \simeq *$$

~~Now Φ~~ Let ρ be a non-triv. irred. $\mathbb{C}$-rep. of C_p.
Then $\Phi^{C_p} KU_{C_p} \simeq (KU_{C_p,\hat{\rho}}[e_\rho^{-1}])^{C_p}$ $\qquad S^0 \to S^\rho$

$$KU_{C_p} \longrightarrow KU_{C_p} \wedge S^\rho \underset{\substack{\uparrow \\ \text{Bott periodicity.}}}{\simeq} KU_{C_p}$$

On $\pi_0^{C_p}$ induces

$$\underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z}_p[\rho]/(\rho^p - 1)}}{R(C_p)} \xrightarrow{(1-\rho)} R(C_p) \Rightarrow \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ \mathbb{Q}_p[\zeta_p]}}{\pi_0 \Phi^{C_p} KU_{C_p}}$$

$$\therefore\ F(\Phi^{C_p}X, \Phi^{C_p}KU_{C_p}) \simeq F(\Phi^{C_p}X, \prod_{i \in \mathbb{Z}} \Sigma^{2i} H\mathbb{Q}_p[\zeta_p]) \underset{\text{by cond ②}}{\neq} 0$$

$\Rightarrow$ no such X exists.

For C_{p^2}, $\pi_0 \Phi^{C_{p^2}} KU_{C_{p^2}} = R(C_{p^2})_p[(1-y)^{-1}]$

$$= \mathbb{Z}_p[x]/(x^{p^2}-1)\,[(1-x^p)^{-1}]$$

$$= \mathbb{Z}_p[x,y]/(x^p-y,\, y^p-1)\,[(1-y)^{-1}]$$

$$\cong \mathbb{Q}_p(\zeta_p)[x]/(x^p-\zeta_p)$$ a non-trivial $\mathbb{Q}_p$-mod.

Same argument as before shows that we do have

$$P(\{e\},p,n) \hookrightarrow P(C_{p^2},p,n-1)$$

Niko Naumann will talk about the extension to higher heights. for $G = A$ an abelian gp (WLOG p-gp).

$F(\Phi^A X, \Phi^A \underline{E}_n)$ is $0 \iff$ type $\Phi^A X > n - rk_p(A)$

$(rk_p(A) = \dim_{\mathbb{F}_p}(A \otimes \mathbb{F}_p))$

P_{p^N} geom. realisation of proper non-trivial partitions of the set $\{0,\dots,p^N\}$ (nb. $\Sigma_{p^N} \curvearrowright P_{p^N}$)

$$P_{p^N+} \to S^0 \xrightarrow{\text{cofib}} P_{p^N}^{\diamond}$$

Thm (Arone-Dwyer-Lesh, Arone-Mahowald)

$$\mathrm{Res}^{U(p^N-1)}_{\Delta}\, U(p^N-1)_+ \wedge_{\Sigma_{p^N}} F(P^{\diamond}_{p^N}, S^{\tilde{\rho}_{\Sigma_{p^N}}}) = X(p,N)$$

$C_p^{\times N} = \Delta \subset \Sigma_{p^N}$

$\Rightarrow \Phi^e X(p,N)$ is a type N spectrum at p and $\Phi^\Delta X(p,N)$ is a non-trivial wedge of spheres, hence of type 0.

⌈All of this work is j/w Tobias Barthel, Markus Hausmann, Niko Naumann, Thomas Nikolaus, Nat Stapleton & identifies $Spc(SH^{\omega}_G)$ as a space when $G \in \{\text{fin. abelian gps}\} + \mathcal{E}$. ⌋

If A is of p-rk k then

$$A \twoheadrightarrow C_p^{\times k} \longrightarrow C_p^{\times N}$$

& we can pull back ADLM spaces to show we can't lift our inclusions further up the tower.