Invertible objects & descent — Stojanoska

j/w Heard and Mathew

$(\mathcal{C}, \otimes, \mathbb{1})$ symm. mon., $\exists N : M \otimes N \simeq \mathbb{1} \Rightarrow M$ invertible

$\operatorname{Pic} \mathcal{C} :=$ group of iso. classes of inv. objects w/ $\otimes$

Examples:

- Ideal class groups are Picard gps
- X scheme $\rightsquigarrow \operatorname{Pic}(X) = H^1(X, \mathcal{O}_X^\times)$
- $\operatorname{Pic}(SH) = \{ S^n = \Sigma^n \mathbb{1} \}_{n \in \mathbb{Z}}$

But ~~loca~~ chromatic localisations of $\overset{Sp}{\overset{\shortparallel}{SH}}$ have very interesting Picard groups

Notation: $SH := \mathrm{ho}\, Sp$ for this talk.

- $\operatorname{Pic}(SH(G)) \ni$ "$\dfrac{\Sigma^\infty A}{S^{\rho_G}}$" for A a "stable htpy rep'n"

  i.e. $\Phi^H(\Sigma^\infty A)^* = S^{n(H)} \quad \forall H \leq G$

$(\mathcal{C}, \otimes, \mathbb{1})$ sym. mon. ∞-cat.

$\rightsquigarrow \left\{ \begin{array}{l} \text{inv. objects \&} \\ \text{equivalences between them}, \otimes \end{array} \right\} =: \mathcal{P}ic\, \mathcal{C} \underset{\text{subgpd}}{\subseteq} \mathcal{C}$

a "grouplike E_∞-space" i.e. $\simeq \Omega^\infty \underbrace{\mathfrak{pic}\, \mathcal{C}}_{\text{connective spectrum}}$

$\Rightarrow \pi_0\, \mathcal{P}ic\, \mathcal{C} \cong \operatorname{Pic} \mathcal{C}$

We now have $\underbrace{\underset{\text{spectrum}}{\mathfrak{pic}\, \mathcal{C}}, \underset{\text{space}}{\mathcal{P}ic\, \mathcal{C}}}_{\text{nice!}}, \underset{\text{group}}{\operatorname{Pic} \mathcal{C}}$

$\Omega_{\mathbb{1}} \mathcal{P}ic\, \mathcal{C} \simeq \operatorname{Aut}_{\mathcal{C}}(\mathbb{1}) \subseteq \operatorname{End}_{\mathcal{C}}(\mathbb{1})$.

Example:

- $\mathcal{C} = \mathrm{Mod}_R$, R E_∞ ring spectrum

  $\operatorname{End}(\mathbb{1}) = \Omega^\infty R$

$$\begin{tikzcd} \operatorname{Aut}(\mathbb{1}) = GL_1 R \arrow[r, "\text{not an } \infty\text{-loop map}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega^\infty R \arrow[d] \\ (\pi_0 R)^\times \arrow[r] & \pi_0 R \end{tikzcd} \Longrightarrow \begin{array}{l} \pi_1\, \mathfrak{pic}\, R = \pi_0 GL_1 R = (\pi_0 R)^\times \\ \pi_t\, \mathfrak{pic}\, R = \pi_{t-1} R, \quad t \geq 2. \end{array}$$

<u>Descent theorem</u>: the functors $\mathcal{P}ic : Cat^{\otimes} \longrightarrow \mathcal{S}_*$
$\mathfrak{pic} : Cat^{\otimes} \longrightarrow Sp_{\geq 0}$

commute with limits.

We can use this for computations.

- <u>Example</u>: $\mathcal{U}_\bullet \longrightarrow X$ map of derived schemes w/ descent
  $\Rightarrow \mathcal{P}ic\, X \simeq \operatorname{holim} \mathcal{P}ic(\mathcal{U}_\bullet)$

<u>Example</u>: $A \xrightarrow{f} B$ map of E_∞-rings

$\rightsquigarrow$ Amitsur complex $A \to \{ B \rightrightarrows B \underset{A}{\otimes} B \rightrightarrows B \underset{A}{\otimes} B \underset{A}{\otimes} B \rightrightarrows \dots \}$ $= B^\bullet$

If f is faithfully flat, then
$\mathrm{Mod}(A) \simeq \mathrm{Tot}(\mathrm{Mod}(B^\bullet))$

But this is not very useful (faithfully flat a v. strong condition).

If f is faithful G-Galois extension (i.e. $G \curvearrowright B$,
$A \xrightarrow{\sim} B^{hG}$ & $B \underset{A}{\otimes} B \simeq \mathrm{Map}(G, B)$

$\Rightarrow \mathrm{Mod}(A) \simeq (\mathrm{Mod}(B))^{hG}$

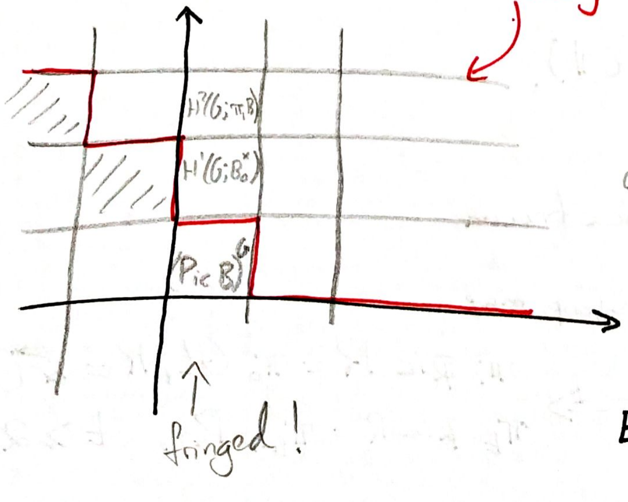
$\Rightarrow \mathcal{P}ic\, A \simeq (\mathcal{P}ic(B))^{hG}$ (or $\mathfrak{pic}\, A \simeq ((\mathfrak{pic}\, B)^{hG})_{\geq 0}$)

<u>Example</u>: $R \to T$ G-Galois of comm. rings
$\Leftrightarrow HR \longrightarrow HT \curvearrowleft G$

$\Rightarrow 0 \to H^1(G; T^\times) \to \mathrm{Pic}\, R \longrightarrow (\mathrm{Pic}\, T)^G \to 0$

G-$\mathcal{P}ic$-SS

$(\Sigma gl_1 B)^{hG}$



differentials are 1 left r up

Idea: compare diffs in SS for $(gl_1 B)^{hG}$ & $B^{hG} \simeq A$

<u>Example</u>: $KO \longrightarrow KU$ $\circlearrowleft C_2$

$\mathrm{Pic} = \mathbb{Z}/8$ $\qquad$ $\mathrm{Pic} = \mathbb{Z}/2$