1. The Picard group of Sp_G

G – a finite group; $\forall H \leqslant G$: $\Phi^H : Sp_G \to Sp$ symm. mon. & left adjoints.

$$\Phi^G(G/H_+) = \begin{cases} * & H < G \\ S^0 & H = G \end{cases}$$

Ex. 1: $\forall H, H'$ compute $\Phi^H(G/H')$

Thm 2: (tom Dieck / Petrie; May et al., ...)

$$\begin{array}{ccccc} M & \longmapsto & \mathbb{1} \otimes_{\pi_0 \mathbb{1}} M & & \\ \rotatebox{90}{$\in$} & & & & \\ 0 \to \mathrm{Pic}(\underbrace{\pi_0(\underline{u})}_{\substack{\cong \\ A(G)}}) & \xrightarrow{i} & \pi_0 \mathrm{Pic}(Sp_G) & \xrightarrow{\varphi} & \mathrm{map}(\{H \leqslant G\}, \mathrm{Pic}(Sp) = \{S^n : n \in \mathbb{Z}\}) \\ & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & & X & \longmapsto & \{\Phi^H(X)\}_{H \leqslant G} \end{array}$$

is an exact sequence. There is a lot known about the image of φ.

Ex. 3: i) $\varphi i = 0$

ii) $\mathrm{Pic}(A(C_2)) = 0$

2. Thick $\otimes$-ideals

G finite gp. p prime, (interesting when $p \mid |G|$)

For $X \in Sp^{\omega}_{G,(p)}$ set

$$\mathcal{F}_X : \{H \subseteq G\} \to [0, \dots, \infty], \quad \mathcal{F}_X(H) := \mathrm{type}(\Phi^H(X))$$

Thm 4 (Balmer – Sanders):

i) For any $X, Y \in Sp^{\omega}_{G,(p)}$: $X \in \langle Y \rangle^{\otimes} \iff \mathcal{F}_X \geq \mathcal{F}_Y$ (pointwise)

ii) TFAE:

a) classify thick $\otimes$-ideals of $Sp^{\omega}_{G,(p)}$

b) determine topology on $\mathrm{Spec}(Sp^{\omega}_{G,(p)})$

c) determine for which functions $\mathcal{F} : \{H \leqslant G\} \to [0, \dots, \infty]$ there is some $X \in Sp^{\omega}_{G,(p)}$ s.t. $\mathcal{F} = \mathcal{F}_X$. ("$\mathcal{F}$ is admissible")

$(\mathcal{F}_X \longleftrightarrow D(X) \underset{\text{open}}{\subseteq} \mathrm{Spc}(Sp^{\omega}_{G,(p)}))$

Thm 5 (T. Barthel, M. Hausmann, T. Nikolaus, J. Noel, N. Stapleton, –)
Assume A is a finite abelian group, p prime, and $\mathcal{F}: \{A' \leq A\} \to [0, \dots, \infty]$. Then TFAE:

a) $\exists$ ~~finite cx~~ $X \in Sp^{\omega}_{A,(p)}$ s.t. $\mathcal{F}_X = \mathcal{F}$

b) $\forall A' \leq A'' \leq A$ s.t. A''/A' is a p-gp. have
$\mathcal{F}(A') \leq \mathcal{F}(A'') + rk_p(A''/A')$

Remark: $rk_p(B) := \dim_{\mathbb{F}_p}(B \underset{\mathbb{Z}}{\otimes} \mathbb{F}_p) \leq \log_p(|B|)$
$=$ iff $pB = 0$.

Cor. 6: $\forall X \in Sp^{\omega}_{A,(p)}$: $type(\Phi^A(X)) \geq type(\underbrace{\Phi^{\{0\}}(X)}_{\text{"underlying spectrum of X"}}) - rk_p(A)$

Ex. 7 (A. Mathew):
$Sp^{\omega}_{C_p,(p)} \ni X := \underset{\text{cofibre}}{\wedge}(t \, \mathbb{1} \to C_{p,+} \to \mathbb{1})$

Then $\Phi^{\{0\}}(t) = p$ but $\Phi^{C_p}(t) = 0$ because $\Phi^{C_p}(C_{p,+}) = *$
$\Rightarrow \Phi^{\{0\}}(X) = S^0/p$ has type 1 & $\Phi^{C_p}(X) = S^0 \vee S^1$ has type 0.

Proof of cor. 6:
Assume $X \in Sp^{\omega}_{A,(p)}$: $K(n)_*(\Phi^{\{0\}}(X)) = 0$

$\Rightarrow E_n^*(\Phi^{\{0\}}(X)) = 0$

$\Rightarrow \forall A' \leq A$: $E_n^*(EA'_+ \underset{A'}{\otimes} X) = 0$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel$
$\quad\quad\quad\quad\quad\quad\quad\quad \pi_{-*}(F(X, \underline{E_n})^{A'})$

$\Rightarrow F(X, \underline{E_n}) = 0$

where $F(-,-)$ is internal mapping object of Sp_A and
$\underline{E_n} := F(EA_+, \mathrm{Inf}^A_{\{0\}}(E_n))$

$$\Rightarrow * \simeq \Phi^A(F(X, E_n)) \simeq D(\Phi^A(X)) \otimes \underbrace{\Phi^A(E_n)}$$

$\hookrightarrow$ what are its finite acyclics?

$\{K \in Sp_{(p)}^{\omega} \mid K \otimes X = 0\}$

Remark: If G non-abelian then $\Phi^G(E_n) = *$

$\rightsquigarrow$ this case is hard!

Ex. 7:

i) $E_1 = KU_p^\wedge$; $\pi_0 E_1 = \mathbb{Z}_p$, $A = C_p$

$\Rightarrow \pi_0 E_1^{BA_+} \cong \mathbb{Z}_p[y]/(y^p - 1)$

$\Rightarrow \pi_0 \Phi^A(E_1) = \pi_0 E_1^{BA_+}[(1-y)^{-1}] \cong \mathbb{Q}_p(\zeta_p) \neq 0$ (rational)

Spec$(\pi_0 E_1^{BA_+})$: $\boxed{\mathbb{Q}_p - \mathbb{F}_p}$ — $\mathbb{Q}_p(\zeta_p)$

ii) $E_1 = KU_p^\wedge$: $A = C_p \times C_p$

$$\begin{array}{ccc} \pi_0 E_1^{BA_+} \cong \mathbb{Z}_p[x,y]/(x^p - 1, y^p - 1) & ; & e_i \in \{1-x, 1-y, x-y\} \\ \downarrow & & \\ \pi_0 E_1^{BA_+}[e_1^{-1}, e_2^{-1}] \overset{i)}{\cong} \mathbb{Q}_p(\zeta_p) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_p) \neq 0 & & \text{(rational)} \\ \Psi & & \\ e_3 \longmapsto \zeta_p \otimes 1 - 1 \otimes \zeta_p & & \end{array}$$

Exercise 8: $\pi_0 E_1^{BA_+}[e_1^{-1}, e_2^{-1}, e_3^{-1}] = 0$

Def^n: i) $X \in Sp_A$, $\mathcal{F}$ family : $\Phi^{\mathcal{F}}(X) := (\tilde{E}\mathcal{F} \otimes X)^A$

$(\Phi^{\{A' < A\}} = \Phi^A)$

ii) $\mathrm{cork}_p(\mathcal{F}) := \min\{\mathrm{rk}_p(A') \mid A' \leq A, A' \notin \mathcal{F}\}$

e.g. $\mathrm{cork}_p(\{A' < A\}) = \mathrm{rk}_p A$.

Thm 9 : ("blue shift")

The finite acyclics of $\Phi^{\mathcal{F}}(E_n)$ are those of $E_{n-\mathrm{cork}_p(\mathcal{F})}$ $(= *$ if $\mathrm{cork}_p \mathcal{F} > n)$