The Balmer spectrum for compact Lie groups

(Hausmann)

Part 1: rationally (Greenlees)

Part 2: p-locally (BHNNNS + Greenlees)

G compact Lie group e.g. $O(n)$, $U(n)$, $\mathbb{T}^n$

G-space X $\Rightarrow$ $G \times X \to X$ cts in both variables

$f: X \to Y$ is a G-w.eq. if w. equivalence on fixed points for all closed subgroups of G.

$\mathrm{Top}_{G,*}^{fin}$ = ∞-cat of based finite G-CW complexes (cells are $\frac{G}{H} \times D^n$ with $H \leq G$ closed)

$\mathrm{Top}_{G,*}^{\omega}$ = closure of $\mathrm{Top}_{G,*}^{fin}$ under retracts.

Recall: G finite; $\mathrm{Sp}_G^{\omega} \simeq \underrightarrow{\mathrm{colim}}(\mathrm{Top}_{G,*}^{\omega} \xrightarrow[S^{\rho_G} \wedge -]{} \mathrm{Top}_{G,*}^{\omega} \to \dots)$

Choose a complete G-universe $\mathcal{U}_G$, i.e. an inner product G-rep. of countable dimension s.t. every f.d. rep'n embeds into $\mathcal{U}_G$. In addition, choose $V_1 < V_2 < \dots < \mathcal{U}_G$ with $\dim V_i < \infty$ and $\bigcup V_i = \mathcal{U}_G$

Now we can define

$$\mathrm{Sp}_G^{\omega} := \underrightarrow{\mathrm{colim}}\left(\mathrm{Top}_{G,*}^{\omega} \xrightarrow{S^{V_1} \wedge -} \mathrm{Top}_{G,*}^{\omega} \xrightarrow{S^{V_2 - V_1} \wedge -} \mathrm{Top}_{G,*}^{\omega} \to \dots\right)$$

$\rightsquigarrow$ objects are $S^{-V} \wedge \Sigma^{\infty} X$ for $X \in \mathrm{Top}_{G,*}^{\omega}$.

Def^n: $\mathrm{Sub}(G) = \{\text{closed subgroups } H \leq G\}$. Choose a metric d on G. $\rightsquigarrow$ $d_{sub}(H,K) := \sup\{d(h,K) \mid h \in H\} + \sup\{d(H,k) \mid k \in K\}$ defines a metric on $\mathrm{Sub}(G)$.

Example: $(g_i)_{i \in \mathbb{N}} \longrightarrow g \in G$

$\rightsquigarrow (g_i H g_i^{-1})_{i \in \mathbb{N}} \longrightarrow gHg^{-1}$ in $\mathrm{Sub}(G)$

Example: $H_1 < H_2 < \dots$ chain of subgroups

$H := \overline{\bigcup H_i} \Rightarrow (H_i)_{i \in \mathbb{N}} \longrightarrow H$ in $\mathrm{Sub}(G)$

Example: $(C_{2^i}) \longrightarrow \mathbb{T}$.

Properties:

- If $(H_i) \to H$ in $\mathrm{Sub}(G)$, then $H_i \leq_G H$ for almost all i.
- $\mathrm{Sub}(G)$ is compact.
- $\mathrm{Sub}(G)/G$ is compact, Hausdorff

Example: $G = \mathbb{T} \Rightarrow \mathrm{Sub}(G) = \begin{matrix} \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ C_1 & C_2 & & & & \mathbb{T} \end{matrix}$

Example: $G = O(2) \Rightarrow \mathrm{Sub}(G)/G = \begin{matrix} D_1 & D_2 & & & & \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \ O(2) \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \ \mathbb{T} \\ C_1 & C_2 & & & & \end{matrix}$

Burnside ring of G:

$$A(G) := [\mathbb{S}_G, \mathbb{S}_G]^G = \operatorname*{colim}_{\substack{V \leq \mathcal{U}_G \\ \text{f.d.}}} [S^V, S^V]^G$$

$H \leq G$ closed $\Rightarrow$ 'mark homom.' $\Phi^H : A(G) \longrightarrow \mathbb{Z}$

$$[f : S^V \to S^V] \longmapsto \deg(f^H : S^{V^H} \to S^{V^H})$$

$\rightsquigarrow \Phi : A(G) \longrightarrow \mathrm{Fun}(\mathrm{Sub}(G)/G, \mathbb{Z})$

Theorem (tom Dieck):

- Φ factors through $\underset{\text{cts funs}}{C}(\mathrm{Sub}(G)/G, \underset{\text{w/ discrete top}}{\mathbb{Z}})$
(in particular, image is compact hence finite)
- If $K \leq H$ is a cotoral inclusion (i.e. $K \trianglelefteq H$, H/K a torus) then $\Phi(x)(K) = \Phi(x)(H)$ $\forall x \in A(G)$
- Φ is injective & $\Phi \otimes \mathbb{Q} : A(G) \otimes \mathbb{Q} \xrightarrow{\cong} C_{ct}(\mathrm{Sub}(G)/G, \mathbb{Q})$ is isom

TTG
of comm. rings (where C_{ct} = cts funs with cotoral inclusion property)

Part 1: rational

For $H \leq G$ define $\mathcal{P}_H = \{ X \in Sp^{\omega}_{G,\mathbb{Q}} \mid \Phi^H(X) \simeq * \}$

Prop: We have $\mathcal{P}_K \subseteq \mathcal{P}_H \iff K$ is G-conjugate to a cotoral subgp of H.

⌜ **Example**: $G = \mathbb{T}$, every subgp is cotoral so $A(G) \cong \mathbb{Z}$ ⌟

Proof sketch: $\Leftarrow$) Based on $\Phi^{\mathbb{T}}(\underline{H\mathbb{Q}}) \not\simeq *$

$\Rightarrow$) uses idempotents in $A(H) \otimes \mathbb{Q}$ and induction Ind_H^G. □

Theorem (Greenlees):

- The assignment $\mathrm{Sub}(G)/_G \longrightarrow \mathrm{Spc}(Sp^{\omega}_{G,\mathbb{Q}})$

$$H \longmapsto \mathcal{P}_H$$

is a bijection. Under this bijection, $V \subseteq \mathrm{Sub}(G)/_G$ is Zariski-closed $\iff$ V is Hausdorff-closed (i.e. closed in topology of $\mathrm{Sub}(G)/_G$) and closed under cotoral subgps (= ct closed)

- tt-ideals $\xleftrightarrow{\cong}$ {open ct-closed subsets of $\mathrm{Sub}(G)/_G$}
- f.g. tt-ideals $\xleftrightarrow{\cong}$ {clopen ct-closed —— " ——}

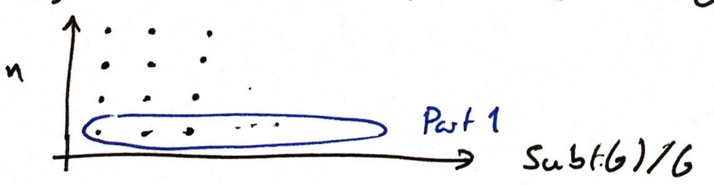
Example: $G = \mathbb{T}$, $\underset{C_1}{\bullet}\quad \underset{C_2}{\bullet}\quad \bullet\quad \bullet\bullet\bullet\bullet \cdots \underset{\mathbb{T}}{\bullet}$

tt-ideals $\xleftrightarrow{\cong}$ {subsets of $\{C_i\}_{i\in\mathbb{N}}$} $\cup$ {$\mathrm{Sub}(\mathbb{T})$}

f.g. " $\xleftrightarrow{\cong}$ {finite subsets of $\{C_i\}_{i\in\mathbb{N}}$} $\cup$ {$\mathrm{Sub}(\mathbb{T})$}

Part 2: p-locally

$H < G$, $n \in \mathbb{N} \rightsquigarrow P(H,n) := \{ X \in Sp^{\omega}_{G,\mathbb{Z}_{(p)}} \mid \Phi^H(X) \wedge K(n-1) \simeq * \}$



Thm • The assignment

$$\mathrm{Sub}(G)/_G \times (\mathbb{N}_{\geq 0} \cup \{\infty\}) \longrightarrow \mathrm{Spc}(\mathrm{Sp}^{\omega}_{G,(p)})$$
$$(H, n) \longmapsto P(H, n)$$

is a bijection

• The Zariski topology can be described in terms of the topology on $\mathrm{Sub}(G)/G$ and the poset structure.

• If $G = A$ abelian, then:

Thm: We have $P(K, n+i) \subseteq P(H, n)$
$\Leftrightarrow K \leq H$ and $\pi_0(H/K)$ a p-group and $i \geq \mathrm{rk}_p(\pi_0(H/K))$.

In particular, if $K \leq H$ cotoral then $P(K, n) \leq P(H, n)$

Def^n: $f: \mathrm{Sub}(G)/_G \longrightarrow \mathbb{N} \cup \{\infty\}$ is admissible if whenever $K \leq H$ with $\pi_0(H/K)$ a p-group, then ~~f(H)~~
$$f(H) + \mathrm{rk}_p \pi_0(H/K) \geq f(K)$$

Thm: The Zariski topology has as a basis the closed sets $V_f = \{P(H, n) \mid n \geq f(H)\}$ where f ranges through all continuous admissible functions.

• tt-ideals $\overset{\simeq}{\longleftrightarrow}$ {admissible functions}
• f.g. tt-ideals $\overset{\simeq}{\longleftrightarrow}$ {continuous admissible functions}