$\Rightarrow \mathrm{Res}^G_H \, S^{k\tilde{\rho}_G}$ is increasingly connected, so

$(S^{\infty\tilde{\rho}_G})^H$ is contractible

For any rep V get $S^0 \xrightarrow{e_V} S^V$

$$\rightsquigarrow X \wedge \tilde{E}P \simeq X[e_{\tilde{\rho}_G}^{-1}]$$

The map $X \longrightarrow X \wedge \tilde{E}P$ is a smashing localisation.
If $X \xrightarrow{\sim} X \wedge \tilde{E}P$ then $X \xrightarrow{\sim} \Sigma^{\tilde{\rho}_G} X$

$G/H_+ \wedge$

$$\begin{tikzcd}
\mathrm{Top}_{G,*}[\Sigma^{-\tilde{\rho}_G}] \simeq \mathrm{Top}_{G,*}[\Sigma^{-\tilde{\rho}_G}][\Sigma^{-1}] \arrow[d, "\wedge \tilde{E}P"] \\
\mathrm{Top}_{G,*}[\tilde{E}P][\Sigma^{-\tilde{\rho}_G}] \simeq (\mathrm{Top}_{G,*}[\tilde{E}P][\Sigma^{-\tilde{\rho}_G}])[\Sigma^{-1}]
\end{tikzcd}$$

$$\simeq \mathrm{Top}_{G,*}[\tilde{E}P][\Sigma^{-1}]$$

$$\simeq \mathrm{Top}_*[\Sigma^{-1}]$$

$$\simeq Sp.$$

$$\begin{cases} * & \text{if } H < G \\ S^0 & \text{if } H = G \end{cases}$$

i.e. this agrees with $(-)^G$

$\Rightarrow \Phi^G X \simeq (\tilde{E}P \wedge X)^G$

Lecture 5 - Separable extensions - (Balmer)

$\mathcal{K}$ tt-cat $\rightsquigarrow \mathrm{Spc}(\mathcal{K}) := \{ \mathcal{P} \subset \mathcal{K} \mid \text{prime}: \begin{smallmatrix} x \otimes y \in \mathcal{P} \\ \Rightarrow x \in \mathcal{P} \text{ or } y \in \mathcal{P} \end{smallmatrix} \}$

$\downarrow F$ tt-functor (ex. + $\otimes$-fnctr) $\qquad \Rightarrow \mathcal{K}_{\mathcal{P}} := (\mathcal{K}/\mathcal{P})^{\natural}$ local

$\mathcal{L}$

$$\rightsquigarrow \mathrm{Spc}(\mathcal{L}) \xrightarrow{\varphi = \mathrm{Spc}(F)} \mathrm{Spc}(\mathcal{K})$$
$$Q \longmapsto F^{-1}(Q)$$

What is $\mathrm{Im}(\varphi)$?

Prop: Suppose F has a right adjoint $\begin{smallmatrix} \mathcal{K} \\ F \downarrow \dashv \uparrow \mathcal{U} \\ \mathcal{L} \end{smallmatrix}$ and let

$A = \mathcal{U}(\mathbb{1})$, a ring object.

Suppose $\mathcal{K}$ rigid (or F & $\mathcal{U}$ satisfy a projection formula:

$\mathcal{U}(F(x) \otimes y) \simeq x \otimes \mathcal{U}(y) \quad \forall x \in \mathcal{K}, y \in \mathcal{L}$)

⌈**Ex:** $\mathcal{K}$ rigid $\Rightarrow$ projection formula. Hint: consider $\mathrm{Hom}_{\mathcal{K}}(x, -)$ ~~both~~ applied to both sides.⌋

Then $\mathrm{Im}(\mathrm{Spc}(F)) = \mathrm{supp}(A)$

Proof: • $\mathcal{P} \in \mathrm{Im}(\varphi) \Rightarrow \mathcal{P} \in \mathrm{supp}(A)$:

$\mathcal{P} = F^{-1}(Q)$, Q a prime. Suppose $\mathcal{P} \notin \mathrm{supp}(\mathcal{U}(\mathbb{1}))$

$\Rightarrow \mathcal{U}(\mathbb{1}) \in \mathcal{P}$

$$\Rightarrow F\mathcal{U}(\mathbb{1}_{\mathcal{L}}) \in Q$$
$$\overset{\shortparallel}{F\mathcal{U}F(\mathbb{1}_{\mathcal{K}})} \geq F(\mathbb{1}_{\mathcal{K}}) = \mathbb{1}_{\mathcal{L}} \quad ↯$$

• Conversely:

Suppose $\mathcal{P} \in \mathrm{supp}(A)$. RTP: $\exists Q \in \mathrm{Spc}(\mathcal{L})$ s.t. $\mathcal{P} = F^{-1}(Q)$

Trick: consider $\mathcal{J} = \langle F(\mathcal{P}) \rangle$ ← t.t. ideal gen. by...

$S = \{ F(x) \mid x \notin \mathcal{P} \}$ $\otimes$-mult. $\subseteq \mathcal{L}$

Suffices to find Q prime s.t. $\mathcal{J} \subseteq Q$ & $Q \cap S = \varnothing$

$\qquad\qquad \Downarrow \qquad\qquad\qquad \Downarrow$

$\qquad F^{-1}(Q) \supseteq \mathcal{P} \qquad F^{-1}(Q) \subseteq \mathcal{P}$

Exercises last time $\Rightarrow$ enough to show $\mathcal{J} \cap S = \varnothing$.

Rem: $\langle F(P)\rangle = \mathrm{thick}(\mathcal{L}\otimes F(P))$

Suppose for a contradiction that $J \cap S \neq \emptyset$, i.e. $\exists x \notin P$ s.t. $F(x) \in J = \mathrm{thick}(\mathcal{L}\otimes F(P))$

$$\begin{aligned} \mathcal{U}F(x) &\in \mathcal{U}(\mathrm{thick}(\mathcal{L}\otimes F(P))) \\ &\subseteq \mathrm{thick}(\underbrace{\mathcal{U}(\mathcal{L}\otimes F(P))}_{\subseteq\, \mathcal{U}(\mathcal{L})\otimes P \,\subseteq\, P}) \subseteq P \end{aligned}$$

$\mathcal{U}F(x) \cong \mathcal{U}(\mathbb{1})\otimes x = A\otimes x$

proj. formula

$\xRightarrow[x\notin P]{P \text{ prime}} \quad A \in P \quad \Rightarrow \quad P \notin \mathrm{supp}(A)$

$\square$

Def^n: A ring object $A = (A, \mu: A\otimes A \to A, \eta: \mathbb{1}\to A)$ in $\mathcal{T}$ is __separable__ if A is proj. as $A^e := A\otimes A^{op}$-module, i.e. μ admits an (A,A)-linear section.

i.e. $A\otimes A \underset{\sigma}{\overset{\mu}{\rightleftarrows}} A$ s.t. $\mu\sigma = \mathrm{id}$ &

$$\begin{tikzcd} & A\otimes A \arrow[dl, "1\otimes\sigma"'] \arrow[d, "\mu"] \arrow[dr, "\sigma\otimes 1"] & \\ A^{\otimes 3} \arrow[dr, "\mu\otimes 1"'] & A \arrow[d, "\sigma"] & A^{\otimes 3} \arrow[dl, "1\otimes\mu"] \\ & A\otimes A & \end{tikzcd}$$

~~$A\otimes A \cong A\times A$~~

Example:
- $A = 0$, $A = \mathbb{1}$
- A idempotent, i.e. $\mu: A\otimes A \xrightarrow{\cong} A$, then $\sigma = \mu^{-1}$.

  $\underset{\mathrm{Id}}{\mathbb{1}\otimes -} \xrightarrow{\eta\otimes 1} \underset{L}{A\otimes -}$ then $L^2 = L$. This is exactly a smashing localisation on $\mathcal{T}$.

  e.g. $\mathcal{K}_Y \subseteq \mathcal{K} = \mathcal{T}^c$, $Y \subseteq \mathrm{Spc}(\mathcal{K}) \rightsquigarrow \mathrm{Loc}(\mathcal{K}_Y) =: \mathcal{T}_Y = e_Y \otimes \mathcal{T}$

  $$\begin{tikzcd} \mathcal{T}_Y \arrow[d, hook] \\ \mathcal{T} \arrow[d, shift right] \\ f_Y\otimes\mathcal{T} \arrow[u, hook, shift right] \end{tikzcd} \quad f_Y \otimes -$$

  $e_Y \to \mathbb{1} \to f_Y \to$

  $\otimes$-idemp.

- X Noeth. scheme, $U \overset{j}{\hookrightarrow} X$ open then $j_* \mathcal{O}_U =: A$ is an idempotent ($\Rightarrow$ sep.) ring in $D(X)$, s.t.

$$A\text{-mod}_{D(X)} = \{X \mid A \otimes X \simeq X\}$$
$\uparrow$ "A-local objects"

⌜Think: idemp. sep. as "Zariski" case⌟

- Galois $\quad A \otimes A = \prod_G A \qquad G \curvearrowright A$

- ℓ/k is a finite sep. ext. of fields

⌜Ex: find σ for $\mathbb{C}/\mathbb{R}$⌟

A-modules:
$$\begin{tikzcd} \mathcal{T} \arrow[d, shift right, "F_A"'] \\ A\text{-Mod}_{\mathcal{T}} \arrow[u, shift right, "U_A"'] \end{tikzcd} \quad \begin{cases} \text{obs } (X \in \mathcal{T},\ \rho : A \otimes X \to X) \\ \text{mors: } A\text{-linear } X \to X' \end{cases}$$

Thm: The category $A\text{-Mod}_{\mathcal{T}}$ is Δ^{ed} so that F_A & U_A is exact. (A separable). If A is commutative then $A\text{-Mod}_{\mathcal{T}}$ inherits a $\otimes_A$ making it tt & F_A a $\otimes$-functor.

⌜Uses: every A-Mod X is $F_A U_A(X) \to X$ ∃ nat. section⌟

$\mathcal{T} \overset{\text{f.f.}}{\leftrightarrows} \mathcal{S}$ ($\uparrow$ localisation) $\longrightarrow$ Here: $\mathcal{T} \overset{U_A \text{ faithful}}{\leftrightarrows} A\text{-Mod}_{\mathcal{T}}$

(so we've dropped "full")

Thm: Let $f : Y \to X$ étale morphism of separated schemes then
$$\begin{tikzcd} D(X) \arrow[d, shift right, "Lf^*"'] \\ D(Y) \arrow[u, shift right, "Rf_*"'] \end{tikzcd}$$
$A = Rf_*(\mathcal{O}_Y)$ is separable in $D(X)$ and

$$\begin{tikzcd} & D(X) \arrow[dl, "F_A"'] \arrow[dr, "Lf^*"] & \\ A\text{-Mod}_{D(X)} \arrow[rr, phantom, "\cong"] & & D(Y) \end{tikzcd}$$
$\uparrow$ ∃

Thm (Neeman): For X Noeth. these are essentially the only ones.

Think: separable is "étale" case.

Equivalent versions:

Ex: Let G a finite group, $H \leq G$. Consider in ~~G-sets~~ kG-modules, k ~~comm. ring~~ field. $A^G_H = k[G/H]$ w/ mult. $\mu(\gamma \otimes \gamma') = \begin{cases} \gamma & (\gamma = \gamma') \\ 0 & (\text{else}) \end{cases}$ a comm. ring object.

$$\eta : k \longrightarrow A \ ; \ 1 \longmapsto \sum_{\gamma \in G/H} \gamma .$$

Separable: $A \longrightarrow A \otimes A ; \ \gamma \longmapsto \gamma \otimes \gamma$

Thm:
$$\begin{tikzcd}
A^G_H\text{-}\mathrm{Mod}_{T(G)} \arrow[rr, phantom, "\cong"] & & T(H) \\
& T(G) \arrow[ul, "F_A"] \arrow[ur, "\mathrm{Res}^G_H"'] &
\end{tikzcd}$$
where $T(G) = D(kG)$ or StMod_{kG} or Mod_{kG}

⌜ Really $\mathrm{Map}_H(G, \mathbb{1})$ separable; we replace w/ the isomorphic $k[G/H]$ ⌟

Thm (B + Dell'Ambrogio + Sanders):

Works for $T(G) = SH(G)$ with $A^G_H = \Sigma^\infty_G (G/H)_+$

Cor: Since A^G_H is compact in $SH(G)$, we have: $\mathrm{Spc}(\mathrm{Res}^G_H) : \mathrm{Spc}(SH(H)^c) \to \mathrm{Spc}(SH(G)^c)$ has image equal to ~~Z~~ $\mathrm{supp}(\Sigma^\infty_G (G/H)_+)$

Lecture 6 – $\mathcal{F}^e$-nilpotence & applications

In Sp_G: $\quad \mathrm{Ind}_H^G X \xrightarrow{\sim} \mathrm{CoInd}_H^G X \quad$ Wirthmuller isom

When X is S^0:

$$(G/H)_+ \xrightarrow{\sim} F(G/H_+, S^0)$$

$\uparrow$ spectral version of $k[G/H]$

$$\underset{\substack{\text{SI} \\ \pi_*^G \mathrm{Ind}_H^G X}}{\pi_*^G \mathrm{CoInd}_H^G X} = \pi_*(\mathrm{CoInd}_H^G X)^G \cong \pi_*^H X$$

$\Rightarrow$ we have maps

$$\underset{\overset{\shortparallel}{G/H_+ \wedge X}}{\mathrm{Ind}_H^G \mathrm{Res}_H^G X} \longrightarrow X \longrightarrow \underset{\overset{F(G/H_+, X)}{\shortparallel}}{\mathrm{CoInd}_H^G \mathrm{Res}_H^G X}$$

$$\Downarrow$$

$$\pi_*^H X \xrightarrow{\mathrm{Ind}} \pi_*^G X \xrightarrow{\mathrm{Res}} \pi_*^H X$$

These gps are a graded Mackey functor

Ex: $R(G) = \mathbb{C}$-Rep of G

$\mathcal{C}$ = family of cyclic subgps of G

$$\bigoplus_{C \in \mathcal{C}} R(C) \xrightarrow{\Sigma \mathrm{Ind}} R(G) \xrightarrow{\Pi \mathrm{Res}} \prod_{C \in \mathcal{C}} R(C)$$

Thm (Artin): is $\mathbb{Q}$-surjection $\qquad$ injection, $\mathbb{Q}$-surj

$$\Rightarrow \quad \underset{\overrightarrow{\mathcal{O}(G)_{\mathcal{C}}}}{\mathrm{colim}}\, R(C) \xrightarrow{\mathrm{Ind}} R(G) \xrightarrow{\mathrm{Res}} \underset{\overleftarrow{\mathcal{O}(G)_{\mathcal{C}}^{op}}}{\lim}\, R(C)$$

these maps are $\mathbb{Q}$-isos.

In Dress' terminology: the defect base of $R(-) \otimes \mathbb{Q}$ is $\mathcal{C}$.

Thm (Quillen): $\quad \mathcal{E}_{(p)} :=$ family of elementary abelian p-gps

$$H^*(BG; \mathbb{F}_p) \xrightarrow{\mathrm{Res}_{\mathcal{E}_{(p)}}} \lim_{\mathcal{O}(G)^{op}_{\mathcal{E}_{(p)}}} H^*(BE; \mathbb{F}_p)$$

- $\ker \mathrm{Res}_{\mathcal{E}_{(p)}}$ is a nilpotent ideal
- if $x \in$ RHS then $x^{p^N} \in \mathrm{Im}\, \mathrm{Res}_{\mathcal{E}_{(p)}}$ for some N (~~ca~~ can be chosen uniformly).

Relation to Sp_G

① $\exists\ KU_G$ G-equiv. complex K-theory

$X \in Top_G^{fin}$ $\quad KU_G^{G,0}(X) =$ Grothendieck gp of $\cong$-classes of $\mathbb{C}G$-vector bundles on X.

$$KU_G^{G,0}(X) = \pi_0^G(F(X_+, KU_G))$$

$$\Rightarrow \pi_{2t}^G KU_G \cong R(G)$$

$$\pi_{2t-1}^G KU_G = 0.$$

② $H\mathbb{F}_p \in Sp$ represents $H^*(-;\mathbb{F}_p)$

$$\underline{H\mathbb{F}_p} = F(EG_+, \mathrm{Inf}_G^{G/G} H\mathbb{F}_p)$$

represents

$$Top_G \ni X \longmapsto H^*(X \times_G EG; \mathbb{F}_p)$$

For $R = KU_G$, $\underline{H\mathbb{F}_p}$, $\quad \sigma_{\mathcal{F}} = \mathcal{C}, \mathcal{E}_{(p)}$ resp.

$$\underset{\overrightarrow{\mathcal{O}(G)_{\mathcal{F}}}}{\mathrm{hocolim}}\ \mathrm{Ind}_H^G \mathrm{Res}_H^G R \longrightarrow R \longrightarrow \underset{\overleftarrow{\mathcal{O}(G)_{\mathcal{F}}^{op}}}{\mathrm{holim}}\ \mathrm{CoInd}\ \mathrm{Res}\ R$$

there are spectral sequences calculating these

$$E_{0t}^2 = \underset{\overrightarrow{\mathcal{O}(G)_{\mathcal{F}}}}{\mathrm{colim}}\ \pi_*^H R \longrightarrow \pi_*^G R \longrightarrow \underset{\overleftarrow{\mathcal{O}(G)_{\mathcal{F}}^{op}}}{\lim}\ \pi_*^H R$$

Def^n: For a family of subgps $\mathcal{F}$ of G, let $\mathcal{F}\text{-nil} \subseteq Sp_G$ be the thick $\otimes$-ideal gen. by $\{G/H_+\}_{H \in \mathcal{F}}$ $= \langle A_{\mathcal{F}} \rangle_{\otimes}$ for $A_{\mathcal{F}} = F((\coprod_{H \in \mathcal{F}} G/H)_+, S^0)$

Thm (Naumann–Mathew–N.): Let $M \in Sp_G$ then TFAE:

① $M \in \mathcal{F}\text{-nil}$

② $\forall K \leq G, K \notin \mathcal{F}$: $e_{\tilde{\rho}_K}: S^0 \to S^{\tilde{\rho}_K}$ induces $M \to S^{\tilde{\rho}_K} \wedge M$ $\in \pi_{-\tilde{\rho}_K}^K \mathrm{End}\, M$ which is a nilpotent map, i.e. $\Phi^K \mathrm{End}(M) = 0$.

[NB $\Phi^K \mathrm{End}(M) = (\mathrm{End}(M) \wedge \tilde{E}P)^K \simeq (\mathrm{End}(M)[e_{\mathcal{F}_K}^{-1}])^K$]

③ Induced map $M \longrightarrow \underset{\mathcal{O}(G)^{op}_{\mathcal{F}}}{\mathrm{holim}}\, \mathrm{CoInd}_H^G \mathrm{Res}_H^G M$

is an equivalence & $\exists N \geq 0$ s.t. $\forall X \in Sp_G$

the holim s.s. $\underset{\mathcal{O}(G)_{\mathcal{F}}}{\lim}{}^s \pi_t^H F(X,M) \Rightarrow \pi_{t-s}^G F(X,M)$

collapses at E_{N+1} onto the first N filtration degrees.

Consequences:

① if $R \in Sp_G$ is a ring spectrum ($HoSp_G$) and $R \in \mathcal{F}$-nil and M is an R-module then M is $\mathcal{F}$-nil

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[rr, bend right, "id"'] & R\otimes M \arrow[r, "\mu"] & M \end{tikzcd}$$

② Say if $M \in \mathcal{F}_1$-nil and $M \in \mathcal{F}_2$-nil $\Rightarrow M \in \mathcal{F}_1 \cap \mathcal{F}_2$-nil
So $\exists$ minimal $\mathcal{F}$ s.t. M is $\mathcal{F}$-nil. This family is the <u>derived defect base</u> of M

③ $\Rightarrow$ if M is $\mathcal{F}$-nil & $X \in Sp_G$ then $F(X,M) \in \mathcal{F}$-nil
(i.e. it's a cotensor ideal too!)

~~$\Rightarrow M \in \mathcal{F}$-nil implies: $\mathrm{End}(M) \in \mathcal{F}$-nil if and only if $M \in \mathcal{F}$ nil~~ ~~(①)~~

$\Rightarrow M \in \mathcal{F}$-nil then $\mathrm{End}(M) \in \mathcal{F}$-nil $\underset{(①)}{\Leftrightarrow} M \in \mathcal{F}$-nil

③ $\Rightarrow M$ is a retract of a finite stage of the Tot tower used to construct the sseq.

$\Rightarrow \forall Z \in Sp_G$ $\mathrm{holim}(\mathrm{CoInd}\,\mathrm{Res}\, M \otimes Z) \xrightarrow{\sim} (\mathrm{holim}\,\mathrm{CoInd}\,\mathrm{Res}\, M) \otimes Z$

Thm: $\forall$ family $\mathcal{F}$ of subgroups of G, $\exists N(\mathcal{F}) \in \mathbb{N}$, $N(\mathcal{F}) \mid |G|$ s.t. $\underset{\mathcal{O}(G)_{\mathcal{F}}}{\mathrm{colim}}{}^s(-)$ and $\underset{\mathcal{O}(G)^{op}_{\mathcal{F}}}{\lim}{}^s(-)$ are $N(\mathcal{F})$-torsion for $s>0$

$\Rightarrow$ after inverting $|G|$ these sequences collapse onto the zero line.

$$\begin{tikzcd}
E\mathcal{F}_+ \wedge M \arrow[d, phantom, "\simeq" rotate=90] & & & \\
\operatorname{hocolim}_{\mathcal{O}(G)_{\mathcal{F}}} \operatorname{Ind}_H^G \operatorname{Res}_H^G M \arrow[r, "\sim"] \arrow[d, "\simeq\ (Ex)"] & M \arrow[r] \arrow[d, "\simeq"] & \tilde{E}\mathcal{F} \wedge M \simeq * \arrow[d] & \leftarrow \text{cofibre seqs} \arrow[dl] \\
E\mathcal{F}_+ \wedge F(E\mathcal{F}_+, M) \arrow[r, "\sim"] & \operatorname{holim} \operatorname{CoInd} \operatorname{Res} M \arrow[d, phantom, "\simeq" rotate=90] \arrow[r] & \tilde{E}\mathcal{F} \wedge F(E\mathcal{F}_+, M) \simeq * & \\
 & F(E\mathcal{F}_+, M) & &
\end{tikzcd}$$

If M is $\mathcal{F}$-nil, $\tilde{E}\mathcal{F} \wedge M \simeq 0$ and $\tilde{E}\mathcal{F} \wedge F(E\mathcal{F}_+, M) \simeq *$

then annotations follow.

G - finite gp $\rightsquigarrow Sp_G \rightsquigarrow Ho(Sp_G) =: SH(G)$

$\rightsquigarrow \mathrm{Spec}(SH(G)^c)$

$G=1 \Rightarrow \mathrm{Spec}(SH^c)$ well understood by Ravanel, Mitchel, Devinatz, Hopkins-Smith.

General t-t-construction (Balmer):

K essentially $\otimes\Delta$ed category $\to \underset{\underset{R_K}{\shortparallel}}{\mathrm{End}_K(\mathbb{1})}$ commutative

$\exists$ natural cts map

$$\rho_K : \mathrm{Spec}\, K \longrightarrow \mathrm{Spec}(R_K)$$
$$P \longmapsto \{f \in R_K \mid \mathrm{cone}(f) \notin P\}$$

"comparison map"

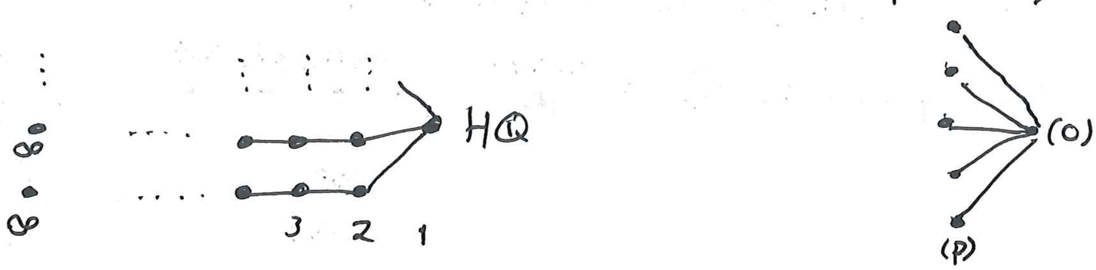
More generally, given any $\mathcal{U} \in \mathrm{Pic}(K)$ can consider $\mathrm{Hom}_K(\mathbb{1}, \mathcal{U}^{\otimes \bullet}) =: R^\bullet_{K,\mathcal{U}}$ graded-comm ring.

$$\rho^\bullet_{K,\mathcal{U}} : \mathrm{Spec}(K) \longrightarrow \mathrm{Spec}^h(R^\bullet_{K,\mathcal{U}})$$
$$P \longmapsto \{f \in R^{hom}_{K,\mathcal{U}} \mid \mathrm{cone}(f) \notin P\}$$

e.g. $\mathcal{U} = \Sigma\mathbb{1}$, ~~Hom~~ $R^\bullet_{K,\mathcal{U}}$ is graded $\mathrm{hom}^*_K(\mathbb{1},\mathbb{1})$.

<u>Example</u>: $K = SH^c \Rightarrow \mathrm{End}_K(\mathbb{1}) \cong \mathbb{Z}$

$$\rightsquigarrow \mathrm{Spec}(SH^c) \xrightarrow{\rho_{SH^c}} \mathrm{Spec}(\mathbb{Z})$$



$\mathcal{C}_{p,n} = \mathrm{Ker}(K(n-1)_* -)$

$K(0) = H\mathbb{Q}$

$K(\infty) = H\mathbb{F}_p$

What about $\mathcal{K} = SH(G)^c$

$\Rightarrow \mathrm{End}_{\mathcal{K}}(\mathbb{1}) = A(G)$ Burnside ring

$$\mathrm{Spec}(SH(G)^c) \xrightarrow{\quad \rho \quad} \mathrm{Spec}(A(G))$$

<u>Dress (1960s)</u>: $H \leq G$, $f^H : A(G) \longrightarrow \mathbb{Z}$ ring homs
$[X] \longmapsto |X^H|$

Note if $H \underset{G}{\sim} K$ then $f^H = f^K$.

$$\Rightarrow \quad \mathrm{Spec}\,\mathbb{Z} \xhookrightarrow{(f^H)^*} \mathrm{Spec}\,A(G)$$
$$(p) \longmapsto q(H,p) := (f^H)^{-1}(p)$$

<u>Thm (Dress)</u>: These copies of $\mathrm{Spec}\,\mathbb{Z}$ (one for each conj. class of subgp) cover $\mathrm{Spec}\,A(G)$.
i.e. every prime ideal of $A(G)$ is $q(H,p)$ for some $H \leq G$, $(p) \in \mathrm{Spec}\,\mathbb{Z}$.

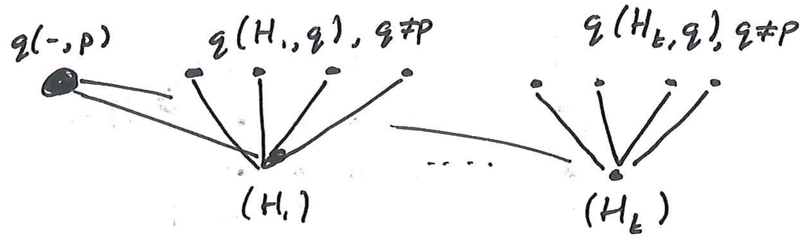
However, $q(H,p) = q(K,p') \Leftrightarrow$ $p>0$ & $p = p'$ and $O^p(H) \underset{G}{\sim} O^p(K)$

⌜$O^p(H)$ = smallest normal subgp of H whose quotient is a p-group⌟

<u>Example</u>: $p \nmid |G| \Rightarrow \forall H \leq G : O^p(H) = H$.
so $O^p(H) \underset{G}{\sim} O^p(K) \Rightarrow H \underset{G}{\sim} K$.

<u>Example</u>: G a p-gp $\Rightarrow \forall H \leq G : O^p(H) = 1$
so $O^p(H) \underset{G}{\sim} O^p(K)$ $\forall H, K \leq G$.

$\Rightarrow \mathrm{Spec}(A(G)) =$



j/w P. Balmer

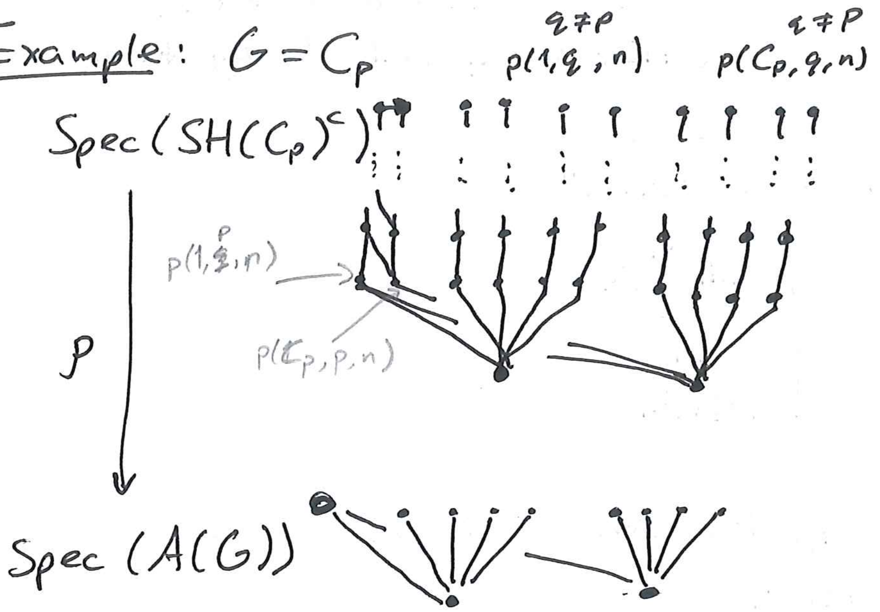
$\Phi^H : SH(G)^c \longrightarrow SH^c$ $\mathcal{H}$-functors

$\Rightarrow \mathrm{Spec}(SH^c) \xleftarrow{(\Phi^H)^*} \mathrm{Spec}(SH(G)^c)$

$\mathcal{C}_{p,n} \longmapsto P_G(H,p,n) := (\Phi^H)^{-1}(\mathcal{C}_{p,n})$

Thm: These copies of $\mathrm{Spec}(SH^c)$, one for each conj. class, cover $\mathrm{Spec}(SH(G)^c)$. Moreover, these copies are disjoint, i.e. $P_G(H,p,n) = P_G(H',p',n')$
$\iff H \underset{G}{\sim} H'$ and $\mathcal{C}_{p,n} = \mathcal{C}_{p',n'}$.

Example: $G = C_p$



The continuity of ρ implies: if $P_G(K,q,m) \subseteq P_G(H,p,n)$ then $q=p$ and $K \underset{G}{\sim}$ (a "p-subnormal" subgp of H)

i.e. $H_0 \trianglelefteq \dots \trianglelefteq H_n = H$
index p

Topology reduces to the following question:
When do we have $P_G(H,p,m) \subseteq P_G(G,p,n)$ for G a p-group, $H \leq G$?

Def^n: For G a p-gp, define $\beta(H,G,n) :=$ smallest i s.t. $P_G(H,p,n+i) \subseteq P_G(G,p,n)$.

i.e. $P_G(H,p,m) \subseteq P_G(G,p,n) \iff m \geq n + \beta(H,G,n)$

They showed:

$$0 < \beta(H, G, n) \leq \log_p \frac{|G|}{|H|} \qquad \text{for } H < G.$$

<u>Example</u>: $G = C_p \Rightarrow 0 < \beta(1, C_p, n) \leq 1$
i.e. $\beta(1, C_p, n) = 1 \quad \forall n$

$\rightsquigarrow$ diagonal lines of slope 1 in $\mathrm{Spec}(SH(C_p)^c)$

II. j/w I. Patchakoria & C. Wimmer

$SH = D(\mathbb{S}) \rightsquigarrow D(\mathbb{Z})$

$SH^{\mathbb{A}^1}(k) \rightsquigarrow DM(k)$

$SH(G) \rightsquigarrow D(\mathrm{Mackey}(G))$, or Kaledin's derived Mackey functors.
($\rightsquigarrow$ Barwick's spectral Mackey functors)

$- \rightsquigarrow \underbrace{\hphantom{xxxxxxxx}}_{\text{"linearised version"}}$

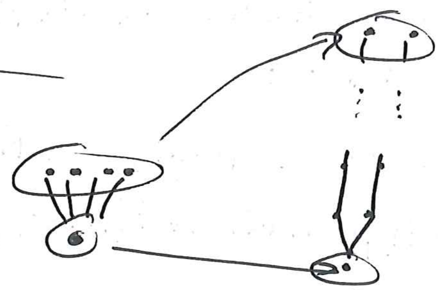
$SH = D(\mathbb{S})$ $\downarrow\uparrow$ ~~fr~~ from $\mathbb{S} \to H\mathbb{Z}$ $D(H\mathbb{Z})$	$SH^{\mathbb{A}^1}(k)$ $\downarrow\uparrow$ $DM(k)$

This project: they compute the spectrum of a linearised version of $SH(G)$

$$\begin{array}{c} SH(G) \\ \downarrow\uparrow \\ D(H\mathbb{Z}_G) \end{array} \qquad \text{where} \quad D(H\mathbb{Z}_G) = \mathrm{Ho}(\mathrm{triv}_G(H\mathbb{Z})\text{-}\mathrm{mod}_{Sp_G})$$

$G = 1$ reduces to $\begin{array}{c} SH \\ \downarrow\uparrow \\ D(H\mathbb{Z}) \end{array}$

$$\begin{array}{c} SH(G) \\ \downarrow\uparrow \\ D(H\mathbb{Z}_G) \end{array} \qquad \begin{array}{c} SH \\ \downarrow \\ D(\mathbb{Z}) \end{array}$$



$$\begin{array}{ccc} \mathrm{Spec}\,\mathbb{Z} & \to & \mathrm{Spec}(SH^c) \\ & \searrow^{\sim} & \downarrow \\ & & \mathrm{Spec}\,\mathbb{Z} \end{array}$$

Use: $H\mathbb{Z}\wedge -$ is conservative on compact spectra (by Hurewicz argument)