(Prismatic algebra: interaction between e.g. stable homotopy theory, rep. theory, algebraic geometry ...)

Def^n: A triangulated category is an add. cat $\mathcal{T}$ together w/ $\Sigma : \mathcal{T} \xrightarrow{\simeq} \mathcal{T}$ & a distinguished class of (exact) triangles $X \to Y \to Z \to \Sigma X$ s.t.

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & * \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

(TR0) $X \xrightarrow{1} X \to 0 \to \Sigma X$ exact. Any $\Delta \simeq$ to exact one is exact.

(TR2) $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ exact $\iff$ $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$ exact

(TR1) Any $f : X \to Y$ fits into an exact Δ $X \xrightarrow{f} Y \to Z \to \Sigma X$

(TR3) $\forall$

$$\begin{tikzcd} X \arrow[r] \arrow[d, "a"'] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed, "\exists"] & \Sigma X \arrow[d, "\Sigma a"] & \text{exact} \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' & \text{exact} \end{tikzcd}$$

$\exists$ fill-in

Example:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] & \Sigma X \end{tikzcd}$$

an exact Δ

$Z = \operatorname{cone}(f)$

Or with 2 maps:

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X_1 \arrow[r, "f_1"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & 0 \arrow[d] & \\ 0 \arrow[r] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \Sigma X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & 0 \arrow[d] \\ & 0 \arrow[r] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \Sigma X_1 \arrow[r] \arrow[d] & \Sigma \operatorname{cone} f_0 \\ & & 0 \arrow[r] & \Sigma X_2 & \end{tikzcd}$$

cone f_0

cone$(f_1 f_0)$

cone (f_1)

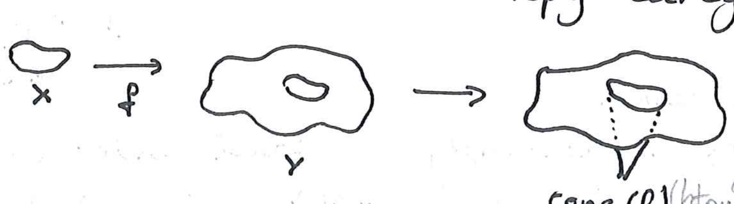
$\rightsquigarrow$ octahedral axiom + higher versions

(TR1^n) + (TR3^n) with "n-triangles".

Examples:

1) St. htpy thry: $\mathcal{T} = SH$, stable homotopy category

$\otimes = \wedge$



(htpy cat) finite pointed CW cx

SH^c = Spanier-Whitehead cat.

2) Equivariant st. htpy thy $SH(G)$ $\otimes = \wedge$
G finite gp (compact Lie) $SH(G) = \ldots$ G-CW cx

3) Algebraic geometry; X a q-compact & q-sep. scheme
$\rightsquigarrow D_{qc}(\mathcal{O}_X\text{-Mod}) =: D(X)$ $\otimes = \otimes_{\mathcal{O}_X}$, $D(X)^c = D^{perf}(X)$

Ex: $X = \operatorname{Spec}(R) \Rightarrow D(X) = D(R\text{-Mod})$ $D(R)^c = K^b(R\text{-proj})$

4) Modular rep^n theory: G finite gp, k field, char $k = p \mid |G|$

$$\frac{kG\text{-Mod}}{kG\text{-Proj}} = kG\text{-StMod}$$

$$\begin{tikzcd} & M \arrow[r, "f"] \arrow[d] & N \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & I \arrow[r] & \operatorname{cone}(f) \arrow[r] & \Sigma M \end{tikzcd}$$

$\otimes = \otimes_k$

$(kG\text{-StMod})^c = kG\text{-stmod}$

"Heller suspension"

5) (G-) C^*-algebras: KK^G (Kasparov K-theory) $\otimes = \hat{\otimes}_{\mathbb{C}}$

6) Motivic theories $DM(k)$ or $SH^{\mathbb{A}^1}(k)$
$\updownarrow$ ex. 3 $\quad$ $\updownarrow$ ex. 1

motives X, Y
$X \otimes Y = X \times_k Y$

for k perfect, $DM(k)^c =$ subcat. gen. by smooth schemes $/k$

Tensor: ($\rightsquigarrow$ symm. mon. cat + additive struct. compatible)
$\otimes : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$, $\mathbb{1} \in \mathcal{T}$, $\mathbb{1} \otimes - \cong \mathrm{Id}$ symm. mon.
compatible w/ Δs:

- $\forall X \in \mathcal{T}$, the functor $X \otimes - : \mathcal{T} \to \mathcal{T}$ is exact, i.e. additive functor that preserves Σ & Δs
- $\forall X, Y \in \mathcal{T}$:

$$\begin{tikzcd} \Sigma X \otimes \Sigma Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ominus"] & \Sigma(X \otimes \Sigma X) \arrow[d] \\ \Sigma(\Sigma X \otimes Y) \arrow[r] & \Sigma^2(X \otimes Y) \end{tikzcd}$$

commutes up to (fixed) sign; $-$ is the normal choice!

Terminology:

1) Ex: given $f: X \to Y$, the cone of f is unique up to (non-unique) isomorphism. $\mathcal{C} \subseteq \mathcal{T}$ closed under cones if $X, Y \in \mathcal{C} \Rightarrow \operatorname{cone}(f) \in \mathcal{C}$

2) When $\mathcal{T}$ admits arbitrary $\coprod$, we have a subclass of compact objects $\mathcal{T}^c \subset \mathcal{T}$, i.e. those $x \in \mathcal{T}$ s.t. $\operatorname{Hom}_{\mathcal{T}}(x, -)$ commutes w/ $\coprod$, i.e.

$$\begin{tikzcd} \forall\, x \arrow[r] \arrow[dr, dashed, "\exists"'] & \coprod Y_i \\ & \coprod_{\text{finite}} Y_i \arrow[u, hook] \end{tikzcd}$$

3) full replete $\mathcal{J} \subset \mathcal{T}$ is called

- triangulated if non-empty & closed under cone (& hence $\Sigma^{\pm}$)
- thick if triangulated & $X \oplus Y \in \mathcal{J} \Rightarrow X, Y \in \mathcal{J}$

Ex: $\ker(\mathcal{T} \xrightarrow{ex} \mathcal{T}')$. conversely $\mathcal{T} \longrightarrow \mathcal{T}/\mathcal{J}$ Verdier quotient

- localising if closed under $\amalg$
- $\otimes$-ideal if $\mathcal{T} \otimes \mathcal{J} \subseteq \mathcal{J}$ (⚠ : $\mathcal{J}$ $\otimes$-id in $\mathcal{T}^c$ would mean $\mathcal{T}^c \otimes \mathcal{J} \subseteq \mathcal{J}$)

4) An object $x \in \mathcal{T}$ is called rigid if $\exists$ a 'dual' $x^\vee \in \mathcal{T}$ and an adjunction $\circlearrowleft$ $\mathcal{T} \underset{x^\vee \otimes -}{\overset{x \otimes -}{\rightleftarrows}} \mathcal{T}$
$(\mathbb{1} \xrightarrow{coev} x^\vee \otimes x, \; x \otimes x^\vee \xrightarrow{ev} \mathbb{1})$

(rigid = strongly dualisable $\approx$ "finite")

Ex: if x rigid then x is a direct summand of $x \otimes x^\vee \otimes x$.

Ex: if x rigid and $x^{\otimes n} \in \mathcal{J}$ thick $\otimes$-ideal then $x \in \mathcal{J}$.

We say $\mathcal{K} := \mathcal{T}^c$ is rigid if all $x \in \mathcal{K}$ are.

$D(kG\text{-mod}) \rightsquigarrow k$ rigid but not compact
$\mathcal{T} = K^b(kG\text{-proj})$

Lecture 2 - Equivariant homotopy theory (Noel)

G finite group, Top_G = category of G-spaces w/ equivariant maps

- G-weak eq. $f: X \to Y$ ($\in \mathrm{Top}_G$) $\iff \forall H \leq G$: $f^H : X^H \to Y^H$ is a w.eq.

- $EG \to *$ is a G-w.e. iff $|G| = 1$
(EG: contractible free G-space)

- A G-CW-cx is a G-space w/ a filtration $\{X_N\}_{N \geq -1}$ by G-subspaces where $X_{-1} = \emptyset$ and X_{N+1} is a pushout

$$\begin{tikzcd} \coprod G/H \times S^N \arrow[r] \arrow[d] & X_N \arrow[d] \\ \coprod G/H \times D^{N+1} \arrow[r] & X_{N+1} \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

Ex: Let V be a $\mathbb{R}$-rep^n of C_p. Construct a C_p-CW approx. of $S(V) = \{v \in V \mid \|v\| = 1\}$

Top_G admits the structure of a closed cartesian monoidal combinatorial model category (for combinatorial need "space" = sSet)

Given a gp inclusion $H \hookrightarrow G$ we get functors

$Res^G_H : Top_G \to Top_H$ w/ left adjoint $Ind^G_H : Top_H \to Top_G$

$X \mapsto G \times_H X$

right adjoint $CoInd^G_H : Top_H \to Top_G$

$X \mapsto Top_H(G, X)$

Given a surj. $G \twoheadrightarrow G/N$ we get inflation functor

$Inf^{G/N}_G : Top_{G/N} \to Top_G$ w/ right adjoint $(-)^N : Top_G \to Top_{G/N}$

$X \mapsto Top_G(G/N, X) \cong X^N$

Very important full subcategory $\mathcal{O}(G) \subseteq Top_G$, the <u>orbit category</u>, spanned by transitive orbits G/H, $H \leq G$.

Given a G-space X we get a functor $\mathcal{O}(G)^{op} \to Top$

$\frac{G}{H} \mapsto Top_G(\frac{G}{H}, X) \cong X^H$.

Ex: construct the natural action of $W_G H := N_G H / H$ on X^H.

<u>Thm</u>: This functor induces a Quillen eq.

(Elmendorf)

$$Top_G \longrightarrow Fun(\mathcal{O}(G)^{op}, Top)$$
$$X \longmapsto Top_G(-, X)$$

(proj. model structure)

w/ left adjoint

$$F(G/e) \longleftarrow F$$

Ex: $BG = G$ as cat. w/ 1 object

$\simeq$ full subcat spanned by G/e in $\mathcal{O}(G)^{op}$

This gives ~~ries~~ rise to restriction functor

$$Top_G \simeq Fun_G(\mathcal{O}(G)^{op}, Top) \xrightarrow{i^*_{triv}} Fun(BG, Top)$$

the w.eqs here are the G-maps that non-equivariantly are w. eq.

e.g. $EG \to *$

We will abuse notation & identify these htpy theories w/ their associated ∞-cats. (in particular sSets)

$i^* = i^*_{triv}$ has a left adjoint $i_!$ and right adjoint i_*

$$\begin{array}{ccc} i_! i^* X \to X & \longrightarrow & i_* i^* X \\ \text{is} & & \text{is} \\ EG \times X & & Top(EG, X) \\ & & \downarrow (-)^G \\ & & Top_G(EG, X) =: X^{hG} \end{array}$$

A <u>family</u> $\mathcal{F}$ of subgps of G is a subset of subgps of G closed under subconjugation

Def^n: $\mathcal{O}(G)_{\mathcal{F}} \subseteq \mathcal{O}(G)$ is full subcat spanned by $\{G/H\}_{H \in \mathcal{F}}$

We again get restriction functors
$$Fun(\mathcal{O}(G)^{op}, Top) \xrightarrow{j^*} Fun(\mathcal{O}(G)_{\mathcal{F}}^{op}, Top)$$
w/ (derived) adjoints inducing maps
$$E\mathcal{F} \times X \longrightarrow X \longrightarrow Top(E\mathcal{F}, X), \quad E\mathcal{F} = j_! j^* (\underset{\substack{\uparrow \\ \text{constant} \\ \text{functor @ } *.}}{*})$$

$$\Rightarrow (E\mathcal{F})^H = \begin{cases} \emptyset & \text{if } H \notin \mathcal{F} \\ * & \text{otherwise} \end{cases}$$

Rmk: All of this goes through (w/ obvious changes) for $Top_{G,*}$, based G-spaces (use e.g. G/H_+)

$Top^{fin}_{G,*} \subseteq Top_{G,*}$ full subcat. spanned by finite G-CW cxes.
& $Top^{\omega}_{G,*}$ is the closure of $Top^{fin}_{G,*}$ under retracts.

Ex: $\underset{\substack{\uparrow \text{reg. rep.,} \\ = \rho_G}}{\mathbb{R}[G]} \rightsquigarrow \mathbb{R}[G]^+ =: S^{\rho_G}$ one-pt compactification
$$S^{\rho_G} \in \cancel{Top^*_{G,fin}}\ Top^{fin}_{G,*}$$

Let $\Sigma^{\rho_G} = S^{\rho_G} \wedge (-) : Top^{\omega}_{G,*} \longrightarrow Top^{\omega}_{G,*}$

Def^n: $Sp_G^\omega = \underrightarrow{\mathrm{colim}} \left(Top_{G,*}^\omega \xrightarrow{\Sigma^{\rho_G}} Top_{G,*}^\omega \longrightarrow \cdots \right)$

$\uparrow$ compact G-spectra.

objects $\text{"}\dfrac{X}{S^{k\rho_G}}\text{"}$ homs $Sp_G^\omega\left(\text{"}\dfrac{X}{S^{k\rho_G}}\text{"}, \text{"}\dfrac{Y}{S^{m\rho_G}}\text{"}\right)$

$$\simeq \underset{N \geq \max(k,m)}{\underrightarrow{\mathrm{colim}}} Top_{G,*}\left(S^{(N-k)\rho_G} \wedge X, S^{(N-m)\rho_G} \wedge Y\right)$$

$Sp_G := \mathrm{Ind}\, Sp_G^\omega \subseteq \mathrm{Fun}((Sp_G^\omega)^{op}, Top)$

smallest full subcat containing representable presheaves & closed under filtered colimits.

The compact objects are Sp_G^ω. Moreover, $\times$-monoidal struct on Top_G induces symm. mon. struct. on $Top_{G,*}$ $(= \wedge)$

induces ———— " ———— on Sp_G (")

(needs result of Marco Robalo: $Sp_G \underset{\otimes}{\simeq} Top_{G,*}[\Sigma^{-\rho_G}]$)

$\uparrow$ colimit in $\mathcal{P}r^L$

We have functors:

$\mathrm{Res}^G_H : Sp_G \longrightarrow Sp_H$

$\mathrm{Inf}^{G/N}_G : Sp_{G/N} \longrightarrow Sp_G$

$\Phi^N : Sp_G \longrightarrow Sp_{G/N}$

$N \trianglelefteq G$

all symm. mon. & restrict to functors between compact objects.

$\uparrow$ induced by $(-)^N : Top_{G,*} \to Top_{G/N,*}$