Chromatic cohomology of finite general linear groups

Neil Strickland
(with Sam Marsh and Sam Hutchinson)

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The problem

Let E be Morava E-theory of height $n > 0$ at a prime $p > 2$.
Many things are known about E^0BG for finite groups G.

- The full structure is known for abelian groups, symmetric groups and various other groups.
- The Hopkins-Kuhn-Ravenel generalised character theory gives a clear description of $\mathbb{Q} \otimes E^0BG$ for any G.
- This determines the 0th chromatic stratum precisely; there are approximate descriptions of the other strata in similar terms.
- In the common case where $E^1BG = 0$, the ring E^0BG has a natural inner product making it a Frobenius algebra.
- There is an extensive theory of the relationship between E^0BG and the λ-ring structure of the representation ring $R(G)$.

Here we take $G = GL_d(F)$, where F is a finite field of characteristic $\neq p$.
The ring $E^0BGL_d(F)$ was described by Tanabe, but we are looking for a more explicit answer. The first interesting case $d = p$ was done in the thesis of Sam Marsh. Most of the general case is in the thesis of Sam Hutchinson.

(The case where F has characteristic p is also interesting, but much harder.)

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Morava E-theory

- Morava E-theory is a generalised cohomology theory giving a graded ring E^*X for every space X.
- $E^* = E^*(\text{point}) = \mathbb{Z}_p[\![u_1, \dots, u_{n-1}]\!][u^{\pm 1}]$ with $|u_i| = 0$ and $|u| = -2$.
- $E^*BS^1 = E^*\mathbb{C}P^\infty \simeq E^*[\![t]\!]$ with $|t| = 0$.
- It is often natural to formulate results in terms of the formal scheme $X_E = \operatorname{spf}(E^0X)$ (similar to the ordinary scheme $\operatorname{spec}(E^0X)$) rather than directly in terms of E^0X.
- The formal scheme $\mathbb{G} = (BS^1)_E$ has a natural abelian group structure.
- For finite abelian groups A we have $BA_E = \operatorname{Hom}(A^*, \mathbb{G}) = \operatorname{Tor}(A, \mathbb{G})$, where $A^* = \operatorname{Hom}(A, S^1)$ is the character group.
- More concretely,

$$E^0BC_{p^m} = E^0[\![t]\!]/[p^m](t) = E^0\{t^i \mid 0 \leq i < p^{nm}\},$$

  where n is the height.
- We also have $BU(d)_E = \mathbb{G}^d/\Sigma_d$. This can be identified with $\operatorname{Div}_d^+(\mathbb{G})$, the moduli scheme for effective divisors of degree d on $\mathbb{G}$.
- There is a dual version $E_*^\vee(X)$ and quotient theories $K^*(X)$ and $K_*(X)$ with $K^0(\text{point}) = \mathbb{Z}/p$.

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- For finite abelian groups A we have $BA_E = \operatorname{Hom}(A^*, \mathbb{G}) = \operatorname{Tor}(A, \mathbb{G})$, where $A^* = \operatorname{Hom}(A, S^1)$ is the character group.
- More concretely,

$$E^0BC_{p^m} = E^0[\![t]\!]/[p^m](t) = E^0\{t^i \mid 0 \leq i < p^{nm}\},$$

  where n is the height.
- We also have $BU(d)_E = \mathbb{G}^d/\Sigma_d$. This can be identified with $\operatorname{Div}_d^+(\mathbb{G})$, the moduli scheme for effective divisors of degree d on $\mathbb{G}$.
- There is a dual version $E_*^\vee(X)$ and quotient theories $K^*(X)$ and $K_*(X)$ with $K^0(\text{point}) = \mathbb{Z}/p$.

Finite general linear groups

- Let F be a finite field of characteristic not equal to p.
- To simplify bookkeeping, we will assume that $|F| = q$ with $v_p(q-1) = r > 0$ so $q = 1 \pmod{p^r}$ but $q \neq 1 \pmod{p^{r+1}}$. This implies that $v_p(q^m - 1) = v_p(m) + r$ for all $m > 0$.
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Theorem

The inclusion $GL_1(\overline{F})^d \to GL_d(\overline{F})$ induces $GL_d(\overline{F})_E \simeq \mathbb{H}^d/\Sigma_d \simeq \mathrm{Div}_d^+(\mathbb{H})$.

Equivalently,

$$E^0(BGL_1(\overline{F})^d) = E^0[\![x_1, \ldots, x_d]\!],$$

and $E^0 BGL_d(\overline{F})$ is the subring of symmetric functions, generated by elementary symmetric functions $c_1, \ldots, c_d$.

Proof.

This is built into the foundations of étale homotopy theory.
The main point is that one can build a torsion-free local ring $\overline{W}$ (the Witt ring of $\overline{F}$) with residue field $\overline{F}$.
One can then choose an embedding $\overline{W} \to \mathbb{C}$.
Using the fact that $|F|$ is coprime to p, one can check that the maps

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Recall that the group $\Gamma = \mathrm{Gal}(\overline{F}/F)$ is generated by the Frobenius map ϕ.

Theorem (Tanabe)

The elements

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form a regular sequence, and

$$E^0 BGL_d(F) = \frac{E^0[\![c_1, \dots, c_d]\!]}{(\phi^*(c_1) - c_1, \dots, \phi^*(c_d) - c_d)} = (E^0 BGL_d(\overline{F}))_\Gamma.$$

Equivalently, we have $BGL_d(F)_E = \mathrm{Div}_d^+(\mathbb{H})^\Gamma$.

In many respects this is very satisfactory, but there are many natural questions that cannot be answered without more detailed algebraic analysis.

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Groupoids

- Let $\mathcal{V}$ be the groupoid of finite dimensional vector spaces over F, and their isomorphisms. Then $B\mathcal{V} \simeq \coprod_d BGL_d(F)$.
- We write $\overline{\mathcal{V}}$ for the corresponding groupoid for $\overline{F}$, so $B\overline{\mathcal{V}} \simeq \coprod_d BGL_d(\overline{F})$.
- Now $B\overline{\mathcal{V}}_E = \coprod_d \mathrm{Div}_d^+(\mathbb{H}) = \mathrm{Div}^+(\mathbb{H})$, and the functor $V \mapsto \overline{F} \otimes_F V$ gives $B\mathcal{V}_E = \mathrm{Div}^+(\mathbb{H})^\Gamma$.
- The functors $\oplus, \otimes \colon \mathcal{V}^2 \to \mathcal{V}$ make $B\mathcal{V}$ a commutative semiring in the homotopy category of spaces.This in turn makes $B\mathcal{V}_E$ a commutative semiring in the category of formal schemes.This matches an obvious commutative semiring structure on $\mathrm{Div}^+(\mathbb{H})^\Gamma$.
- Alternatively, $E_*^\vee(B\mathcal{V})$ and $K_*(B\mathcal{V})$ are Hopf rings.
- Some other groupoids are also relevant, for example

  $$\mathcal{L} = \{(X, L) \mid X \text{ is a finite set, and } L \text{ is an } F\text{-linear line bundle over } X\}.$$

  This has $B\mathcal{L} \simeq \coprod_d E\Sigma_d \times_{\Sigma_d} BGL_1(F)^d$.
  There is a functor $\pi \colon \mathcal{L} \to \mathcal{V}$ given by $\pi(X, L) = \bigoplus_x L_x$.
- The index of $\Sigma_d \wr GL_1(F)^d$ in $GL_d(F)$ has index coprime to p, so $B\mathcal{L} \to B\mathcal{V}$ gives an epimorphism in E-cohomology. Earlier work on symmetric groups gives a good understanding of $E^0 B\mathcal{L}$.

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- Put $\Theta^* = \mathbb{Z}_p^n$, and regard it as a groupoid with one object.
- Hopkins, Kuhn and Ravenel defined a ring L which is an extension of $\mathbb{Q} \otimes E^0$ with Galois group $\mathrm{Aut}(\Theta^*)$.
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- HKR constructed isomorphisms

$$L \otimes_{E^0} E^0 B\mathcal{G} \simeq \mathrm{Map}([\Theta^*, \mathcal{G}], L) \qquad L \otimes_{E_0} E_0^\vee B\mathcal{G} \simeq L\{[\Theta^*, \mathcal{G}]\}.$$

- $E_0^\vee B\mathcal{G}$ has a natural inner product, which becomes $\langle[\alpha], [\beta]\rangle = |\operatorname{Iso}(\alpha, \beta)|$ on $L\{[\Theta^*, \mathcal{G}]\}$.
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More about (co)algebraic structures

- The functors $\oplus, \otimes \colon \mathcal{V}^2 \to \mathcal{V}$ give products on $E_0^\vee B\mathcal{V}$ and on $K_0 B\mathcal{V}$ and on $L \otimes_{E^0} E_0^\vee B\mathcal{V} = L\{\mathrm{Rep}(\Theta^*, F)\}$.
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- The functors $\oplus, \otimes \colon \mathcal{V}^2 \to \mathcal{V}$ give products on $E_0^\vee B\mathcal{V}$ and on $K_0 B\mathcal{V}$ and on $L \otimes_{E^0} E_0^\vee B\mathcal{V} = L\{\mathrm{Rep}(\Theta^*, F)\}$.
- These are just $[U] * [W] = [U \oplus W]$ and $[U] \circ [W] = [U \otimes W]$.
- We can grade everything with $GL_d(F)$ in degree d; then $|a * b| = |a| + |b|$.
- $K_0 B\mathcal{V}$ embeds in $K_0 B\overline{\mathcal{V}} = K_0[K_0 BGL_1(\overline{F})]$, which is polynomial under $*$; so $K_0 B\mathcal{V}$ has no $*$-nilpotents. If $K_0 B\mathcal{V}$ had a coproduct that interacted correctly with the product and grading, we could conclude that $K_0 B\mathcal{V}$ and $E_0^\vee B\mathcal{V}$ are polynomial under $*$.
- The diagonal $\delta \colon \mathcal{V} \to \mathcal{V}^2$ gives a coproduct $[V] \mapsto [V] \otimes [V]$. This is compatible with the two products, giving a Hopf ring. But it does not interact correctly with the grading.
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Harish Chandra induction

- A variant: instead of using the transfer for $GL_i \times GL_j \to GL_{i+j}$, use the parabolic subgroup P_{ij}, the projection $P_{ij} \to GL_i \times GL_j$ and the inclusion $P_{ij} \to GL_i \times GL_j$. ("Harish Chandra induction").
- This gives another product $[U] \times [W] = |\mathrm{Hom}_{\Theta^*}(W, U)|^{-1}[U \oplus W]$ and coproduct $\psi_\times([V]) = \sum_{U \leq V}[U] \otimes [V/U]$. These still do not give a Hopf algebra structure.
- One can find $e \in E^0 B\mathcal{V}$ with character values $e(U) = |\mathrm{End}_{\Theta^*}(U)|$. This is 1 mod the maximal ideal in $E^0 B\mathcal{V}_d$, and $p > 2$, so it has a square root. This satisfies

$$(\sqrt{e(U)}[U]) \times (\sqrt{e(W)}[W]) = \sqrt{e(U \oplus W)}[U \oplus W],$$

  so $(E_0^\vee B\mathcal{V}, \times) \simeq (E_0^\vee B\mathcal{V}, *)$.
- The failure of the Hopf algebra axiom is measured by $u \in (E^0 B\mathcal{E})^\times$ for an auxiliary groupoid $\mathcal{E}$. This has character values of the form $|\mathrm{Hom}_{\Theta^*}(A, B)| \in q^{\mathbb{N}} \subseteq 1 + p^r\mathbb{Z}$. However, $u \neq 1$ in $(E/I_n)^0 B\mathcal{E}$.

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Trellises

- $[\Theta^*, \mathcal{L}]$ is the set of isomorphism classes of pairs (X, L), where X is a finite Θ^*-set, and L is a Θ^*-equivariant F-linear line bundle over X.
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Irreducibles

- Let W be an irreducible F-linear representation of Θ^*. Then $\mathrm{End}_{F[\Theta^*]}(W)$ is a field (by Schur's Lemma) and a finite extension of F, so we can choose an embedding in $\overline{F}$, unique up to the action of Γ.
- Let $\omega\colon \Theta^* \to \overline{F}^\times$ be a continuous homomorphism. Then the set $W = \mathrm{span}_F(\omega(\Theta^*))$ is a finite subfield of $\overline{F}$, and we can use ω to give an action of Θ^* on W, making it an irreducible representation.
- These constructions give a bijection $\mathrm{Irr}(\Theta^*; F) \simeq \mathrm{Hom}(\Theta^*, \overline{F}^\times)/\Gamma = \Phi/\Gamma$.
- This in turn gives $\mathrm{Rep}(\Theta^*; F) = \mathrm{Div}^+(\Phi)^\Gamma$, meshing nicely with Tanabe's $(B\mathcal{V})_E = \mathrm{Div}^+(\mathbb{H})^\Gamma$.
- For $m > 0$, the irreducibles of dimension p^m correspond to orbits $\Gamma\phi = \phi + p^r\mathbb{Z}_p\phi$ where ϕ has order precisely p^{m+r}.
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- Problem: find a closed subscheme of $E^0BGL_{p^m}(F) = \mathrm{Div}^+_{p^m}(\mathbb{H})^\Gamma$ that corresponds to $\mathrm{Irr}_{p^m}(\Theta^*; F)$ in generalised character theory.

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- Let W be an irreducible F-linear representation of Θ^*.
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Irreducibles in formal group theory

There is a cyclic subgroup $U_m \leq GL_{p^m}(F)$ of order p^{m+r}, so $E^0 BU_m \simeq E^0[\![x]\!]/[p^{m+r}](x)$.
Now $[p^{m+r}](x)$ factors as $g_m(x)\,[p^{m+r-1}](x)$, and we put $D_m = E^0[\![x]\!]/g_m(x)$.
This still has an action of Γ, and we put $X_m = \mathrm{spf}(D_m^\Gamma)$.
In a different language: $\mathrm{spf}(D_m) = \mathrm{Level}(U_m^*, \mathbb{G})$ and $X_m = \mathrm{Level}(U_m^*, \mathbb{G})/\Gamma$.

We also put

$$y = \prod \{\Gamma - \text{orbit of } x\} = \prod_{i=0}^{p^m-1} [q^i](x) \in D_m^\Gamma.$$

One can check that the set $\{y^i \mid 0 \leq i < p^{(m+r-1)n-m}(p^n-1)\}$ is a basis for D_m^Γ over E^0, and that D_m^Γ is a regular local ring.

We can regard U_m as a groupoid with one object, and there is an evident functor $i\colon U_m \to \mathcal{V}$ sending the unique object to F_{p^m}.
There is an isomorphism $\overline{F} \otimes_F F_{p^m} \to \prod_{i=0}^{p^m-1} \overline{F}$ given by

$$a \otimes b \mapsto (ab,\ a\,\phi(b),\ a\,\phi^2(b), \dots, a\,\phi^{p^m-1}(b)).$$

Using this, we find that the element $c_{p^m} \in E^0 BGL_{p^m}(\overline{F})$ maps to $y \in D_m^\Gamma$.
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The semiring $\mathrm{Rep}^+(\Theta^*, F)$ is a set (not a formal scheme), and it splits as

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Question: is there an analogous splitting

$$BGL_{p^m}(F)_E = X_m \amalg W_m \text{ of formal schemes, or}$$
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Answer: no, $E^0BGL_{p^m}(F)$ is a local ring, and does not split as a product. It does split after rationalising, by HKR.

This is a common phenomenon in this kind of algebra.Instead of splittings $A = B \times C$, we often have $B = A/I$ and $C = A/J$ with $I = \mathrm{ann}(J)$ and $J = \mathrm{ann}(I)$, which makes I a C-module and J a B-module.In the best cases I will be free of rank one over C and/or J will be free of rank one over B.

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$$\mathrm{Rep}^+_{p^m}(\Theta^*; F) = \mathrm{Irr}_{p^m}(\Theta^*; F) \amalg \mathrm{Red}_{p^m}(\Theta^*; F) = \mathrm{Mon}(U_m^*, \Theta) \amalg \mathrm{Red}_{p^m}(\Theta^*; F).$$

Question: is there an analogous splitting

$$BGL_{p^m}(F)_E = X_m \amalg W_m \text{ of formal schemes, or}$$
$$E^0 BGL_{p^m}(F) = D_m^\Gamma \times C_m \text{ of rings?}$$

Answer: no, $E^0 BGL_{p^m}(F)$ is a local ring, and does not split as a product. It does split after rationalising, by HKR.

This is a common phenomenon in this kind of algebra.Instead of splittings $A = B \times C$, we often have $B = A/I$ and $C = A/J$ with $I = \mathrm{ann}(J)$ and $J = \mathrm{ann}(I)$, which makes I a C-module and J a B-module.In the best cases I will be free of rank one over C and/or J will be free of rank one over B.

Example: $A = R[t]/(f(t)g(t))$, $B = R[t]/f(t)$, $C = R[t]/g(t)$ where $f(t)$ and $g(t)$ are monic polynomials.

Splitting and amalgamation

We have seen that D^{Γ}_m is the quotient of the ring $A = E^0BGL_{p^m}(F)$ by an ideal J say.Here A and D^{Γ} are both Frobenius algebras over E^0.From this it follows automatically that J and $\mathrm{ann}(J)$ are both E^0-module summands in A, and that $\mathrm{ann}^2(J) = J$.Moreover, $\mathrm{ann}(J)$ is a free module of rank one over D^{Γ}_m.

We know that $E_0^{\vee}B\mathcal{V}$ is polynomial, and it follows by self-duality that $E^0B\mathcal{V}$ is polynomial under the transfer product, and we have

$$J = \mathrm{img}(\mathrm{tr}\colon E^0(BGL_{p^{m-1}}(F)^p) \to E^0(BGL_{p^m}(F))),$$

so $\mathrm{Ind}_{p^m}(E^0B\mathcal{V}) = D^{\Gamma}_m$.

Problem: find an explicit generator for $\mathrm{ann}(J)$.

In the case $m = 1$, the element $c_p(\psi^p(\mathrm{Taut}))$ is the required generator, but the proof is elaborate. We do not know whether a similar formula works for $m > 1$.

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Consider instead the ideals $I = \ker(E^0 BGL_d(F) \to E^0(BGL_{d-1}(F))$ and

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