RATIONAL EQUIVARIANT COHOMOLOGY THEORIES COPENHAGEN WORKSHOP 9-13 APRIL 2018

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1. Preamble

My main aim is to describe the general form of the triangulated category of G-equivariant cohomology theories for a compact Lie group G in a way that makes it easy to do calculations. This means I will spend a lot of time doing algebra, and much less time talking about model categories and Quillen equivalences. This is despite the fact that the theory of model categories is absolutely fundamental to proving the main results. This involves the work of many people but especially Dave Barnes, Magdalena Kedziorek and Brooke Shipley. There is a skeleton bibliography at the end, and a few citations in the text.

I will not be giving a systematic introduction to stable equivariant homotopy category. This is partly because it is well done elsewhere (probably best in [27]), partly because it takes a while to do it, but mainly because it is not directly relevant to my aim. However I will say enough to explain the context and its relevant formal properties ([28] gives a first introduction to the formalities). Beyond algebraic topology and homotopy theory, the other prerequisites are a little commutative algebra and rather more homological algebra.

Background on homotopical foundations and model categories will be provided in other talks and the support sessions.

2. The plan

Lecture 1. Equivariant cohomology theories: *The lecture will introduce the context, describe the landscape and outline the plan for the week.*

Some examples. K, MU, Borel cohomology. The Localization Theorem. Representability. Formal properties (restriction and its adjoints, categorical and geometric fixed points).

Rationalization. The Burnside ring. Splitting for finite groups. Haeberly's example (and a simpler example) to show that K-theory is not ordinary if G is not finite. The conjecture about algebraic models and its status, including mention of monoidal issues.

Morita approach. Calculating maps and Adams spectral sequences.

Lecture 2. Free G-spectra: *Free G-spectra provide the perfect start* [21, 22]*. First, many subtleties of G-spectra do not arise in the free case. Second, the algebraic models are easy to describe. Third, the proof strategy has the same form as the general case. Fourth, this case is a basic ingredient in the general case.*

The Adams spectral sequence. Mention and dismiss Morita. Approach through change of rings and fixed points. Shiplification [34]. The Cellularization Principle [23]. Algebraic torsion functors.

Lecture 3. The circle group: *The main point is to describe the algebraic model for rational circle-equivariant cohomology theories* [10]*, showing where it comes from and why it is so easy to work with. On the other hand, it is rich enough to include a construction of rational equivariant elliptic cohomology* [12] *associated to an elliptic curve C and it contains an island modelling the category of sheaves over C.*

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Isotropy separation. Splicing and the Tate construction. The standard abelian model, and mention of the torsion model. Homological algebra. Circle-equivariant elliptic cohomology.

Lecture 4. The abelian model for the torus: *Algebraic models follow the usual pattern that one dimensional objects can be dealt with by 'algebra' whilst higher dimensional objects really need to be thought of 'geometrically'. The entire lecture will be spent describing the algebraic model. From the grass roots up, it is just formed by using the Burnside ring, the Localization Theorem and the case of free spectra. From the top down it is a sheaf over the space of subgroups assembled from this data.* [13, 16]

Euler classes. Coefficient systems. Flags and the Localization Theorem. Collecting subgroups with the same identity component. Adelic cohomology. The Adams spectral sequence and formality of cells.

Lecture 5. The algebraic model of rational torus equivariant cohomology theories: *This lecture will outline the proof* [26] *that the category of rational G-spectra for a torus G is Quillen equivalent to differential objects in $\mathcal{A}(G)$. The focus will be on the homotopical ingredients rather than on the verification that the underlying model categories have the necessary formal properties.*

The sphere as an isotropic pullback. Modules over diagrams of rings. The diagram of rings and its formality. Proof of the Quillen equivalence.

Prospects....onward and upward: *This heading describes what might naturally come next. What happens for other compact Lie groups? More generally, how would one attempt to build algebraic models of other model categories?*

The Balmer spectrum of rational G-spectra is the poset of subgroups under cotoral inclusion (with the f-topology) [17]. The abelian model of toral G-spectra [15] (the necessity of flags). Discrete adelic models [19, 18, 1] and towards the full model for general compact Lie groups.

A selective bibliography

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