# $G$-COHOMOLOGY THEORIES AND $G$-SPECTRA A FORMULARY.

J.P.C.GREENLEES

## 1. $G$-spectra

Equivariant cohomology theories are represented by $G$-spectra. These should be thought of as formed from stable spaces by closing under suitable direct limits.

There exists a category of $G$-spectra. Actually, there are many constructions.These are all known to have equivalent homotopy categories. If you want just one construction, use orthogonal spectra but here we describe basic properties enjoyed by all.

The relationship between spectra and cohomology theories is as follows:

For any (suitable) equivariant cohomology theory $E^*_G(X)$ there is a $G$-spectrum $E$ so that

$$E^*_G(X) = [X, E]^*_G.$$

Indeed the category of $G$-equivariant cohomology theories and stable natural transformations is the homotopy category of $G$-spectra. The associated homology theory is defined by

$$E^G_*(X) = [S^0, E \wedge X]^G_*$$

so we need a tensor product $\wedge$. Formal properties of the category of spectra ensure there is a good theory of duality (Alexander, Lefschetz, Poincaré).

For some purposes one wants to consider equivariant theories for all subgroups $H$ of $G$ together:

$$\underline{E}^G_*(X) = \left\{ E^H_*(X) \mid H \subseteq G \right\},$$

These groups are all related by restrictions and transfers, with formal properties making them into Mackey functors, but we will not emphasize this structure.

## 2. Change of groups

We will have to introduce a number of change of groups adjunctions. You will have seen them before in algebraic contexts, and some of them also for spaces. Some of them are substantial theorems, but we list them here for convenience. You can get a very long way by just using these properties.

Suppose given a group $G$, a subgroup $H$, a normal subgroup $N$ and a quotient group $Q = G/N$. We also need a $G$-space $X$, an $H$-space $Y$ and a $Q$-space $Z$.

- **induction-restriction:**

  $$[G_+ \wedge_H Y, X]^G = [Y, X]^H$$

  So, for any $G$-equivariant cohomology theory, we may define an $H$-equivariant theory by

  $$E^*_G(G_+ \wedge_H Y) = E^*_H(Y).$$

- **restriction-coinduction:**

  $$[X, F_H(G_+, Y)]^G = [X, Y]^H.$$

- **inflation-categorical fixed points:**
$$[Z, X]^G = [Z, X^N]^Q.$$
- **free-inflation:** if $T$ is an $N$-free $G$-space,
$$[T, Z]^G = [T/N, Z]^Q,$$
so, for suitable cohomology theories (there are many like this, but it is a condition ('split')),
$$E_G^*(T) = E_Q^*(T/N)$$
- **inflation-free (Adams Isomorphism):** if $T$ is an $N$-free $G$-space,
$$[Z, T]^G = [Z, (S^{LN} \wedge T)/N]^Q,$$
where $LN$ is the tangent space at the identity of $N$ with the conjugation $G$-action.
- **Wirthmüller Isomorphism:** Induction and coinduction are related by
$$F_H(G_+, Y) \simeq G_+ \wedge_H (S^{LG/H} \wedge Y)$$
where $LG/H$ is the tangent $H$-space at the identity left coset of $G/H$.

## 3. Fixed points

There are three sorts of $N$-fixed points that are relevant. All give functors from $G$-spectra to $Q$-spectra.

- **Geometric fixed points ($\Phi^N X$) (Adams)**: this will be most important to us. It has the properties
  - $\Phi^N(\Sigma^\infty A) \simeq \Sigma^\infty(A^N)$ for a based $G$-space $A$
  - $\Phi^N(X_1 \wedge X_2) \simeq (\Phi^N X_1) \wedge (\Phi^N X_2)$
  - $\Phi^N$ preserves homotopy direct limits.
  - $[X_1, X_2 \wedge E[N \subseteq]]^G \cong [\Phi^N X_1, \Phi^N X_2]^Q$
  - A spectrum $X$ is $G$-contractible if $\Phi^H X$ is non-equivariantly contractible for all subgroups $H$.
- **Categorical fixed points, $X^N$ (Lewis-May)**. This has already come up as the right adjoint of inflation, and detection of weak equivalence is built into the definition of most models. It does not preserve smash products or commute with suspension spectra.
  - $[Z, X]^G = [Z, X^N]^Q$
  - $\Phi^N$ preserves homotopy direct limits.
  - A spectrum $X$ is $G$-contractible if $X^H$ is non-equivariantly contractible for all subgroups $H$.
- **Homotopy fixed points $X^{hN}$**. By definition $X^{hN} = F(EG_+, X)^N$. Used because of the three facts
  - A $G$-map $X_1 \longrightarrow X_2$ which is a non-equivariant equivalence induces a weak equivalence $X_1^{hG} \xrightarrow{\simeq} X_2^{hG}$
  - there is a descent spectral sequence
$$H^s(G; \pi_*(X)) \Rightarrow \pi_*(X^{hG}),$$
  - in favourable cases the map $X^G \longrightarrow X^{hG}$ is some form of algebraic completion (or even an equivalence).

School of Mathematics and Statistics, Hicks Building, Sheffield S3 7RH. UK.
*E-mail address*: john.greenlees@warwick.ac.uk