Masterclass: Rigidity and algebraic models in stable homotopy theory

2018-04-11

Barnes: Rational S^1-Equivariant Ring Spectra

$\otimes \mathbb{Q}$

$S^1 = SO(2) \overset{''}{\mathbb{T}} = \mathbb{T}$

$$\mathcal{O}_\mathcal{F} = \prod_{n\geq 1} \mathbb{Q}[\overset{-2}{c_n}]$$

$$\mathcal{E}^{-1}\mathcal{O}_\mathcal{F} = \operatorname{colim}_k \left(\prod_{i=1}^{k} \mathbb{Q}[c_i, c_i^{-1}] \times \prod_{j \geq k+1} \mathbb{Q}[c_j] \right) = t$$

$\mathcal{A}(S^1)$ object

$$\begin{tikzcd} N \arrow[d, "\beta"'] \\ t\otimes V \end{tikzcd}$$

N - $\mathcal{O}_\mathcal{F}$-module

$\mathcal{E}^{-1}\beta$ is a iso

V - $g\mathbb{Q}$-module

morphisms: squares

$d\mathcal{A}(S^1)$ = objects $X \xrightarrow[\partial]{\text{in } \mathcal{A}(S^1)} \Sigma X$, $\partial^2 = 0$

morphisms as in $\mathcal{A}(S^1)$ + commute with differentials

Monoidal structure:

$$\begin{pmatrix} M \\ \downarrow \\ t\otimes V \end{pmatrix} \otimes \begin{pmatrix} N \\ \downarrow \\ t\otimes W \end{pmatrix} = \begin{matrix} M \underset{\mathcal{O}_\mathcal{F}}{\otimes} N \\ \downarrow \beta\otimes\gamma \\ t\otimes (V \underset{\mathbb{Q}}{\otimes} W) \end{matrix}$$

This has an internal fuction object

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Let $\hat{\mathcal{A}}$ be $\mathcal{A}(S^1)$ without the condition on β.

$$\mathcal{A}(S^1) \underset{\Gamma}{\overset{inc}{\rightleftarrows}} \hat{\mathcal{A}} \qquad \Gamma \circ inc = id$$

$$F\left(\begin{matrix} M \\ \downarrow \\ t\otimes V \end{matrix}, \begin{matrix} N \\ \downarrow \\ t\otimes W \end{matrix} \right) = \Gamma \left(\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Hom}_{\mathcal{O}_\mathcal{F}}(M,N) \arrow[d] \\ t\otimes \operatorname{Hom}_{\mathbb{Q}}(V,W) \arrow[r] & \operatorname{Hom}_{\mathcal{O}_\mathcal{F}}(M, t\otimes W) \end{tikzcd} \right)$$

$$0 \longrightarrow \pi_*^{\mathcal{A}}(X)\otimes\pi_*^{\mathcal{A}}(Y) \longrightarrow \pi_*^{\mathcal{A}}(X\wedge Y) \longrightarrow \Sigma\,\mathrm{Tor}_{\mathcal{A}}(\pi_*^{\mathcal{A}}(X),\pi_*^{\mathcal{A}}(Y)) \longrightarrow 0$$

$$\pi_*^{\mathcal{A}}(X) = \pi_*^{S^1}(X\wedge DE\mathcal{F}_+)$$

$S^1\text{Sp}_{\mathbb{Q}}$

$$\pi_*^{S^1}(X\wedge DE\mathcal{F}_+\wedge E\mathcal{F}) = t\otimes\pi(\Phi^{S^1}X)$$

<u>Model structures on $d\mathcal{A}(S^1)$:</u>

Injective model structure:

cofibrations = monomorphisms

w.e. = H_*-isos

fibrations = surjections with injective kernel

This is not monoidal

Every object is cofibrant

$\_\otimes X$ should preserve cofibrations

<u>Exercise</u> !

<u>Definitions:</u> We say $X = \begin{pmatrix} M \\ \downarrow \\ t\otimes V\end{pmatrix}$ in $\mathcal{A}(S^1)$ is <u>strongly dualizable</u> if any of the following equivalent condition hold:

- $\underline{\mathrm{Hom}}^F\left(\begin{matrix} M \\ \downarrow\beta \\ t\otimes V'\end{matrix}\;\; \begin{matrix} N \\ \downarrow \\ t\otimes W\end{matrix}\right) \cong \underbrace{\underline{\mathrm{Hom}}^F\left(\begin{matrix} M \\ \downarrow \\ t\otimes V'\end{matrix}\;\; \begin{matrix} \mathcal{O}_{\mathcal{F}} \\ \downarrow \\ t\otimes Q\end{matrix}\right)}_{DX} \otimes \begin{matrix} N \\ \downarrow \\ t\otimes W\end{matrix}$ for all $\begin{matrix} N \\ \downarrow \\ t\otimes W\end{matrix}$ in $\mathcal{A}(S^1)$
- $DDX \cong X$
- N is finitely presented in $\mathcal{O}_{\mathcal{F}}$-modules. and β is injective
- $\cong \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(M,N)$

  $t\otimes \mathrm{Hom}(V,W)$

<u>Theorem (B)</u>: There is a symmetric monoidal model structure on $d\mathcal{A}(S^1)$ with w.e.'s the H_*-isos and gen. cofibrations:

$$I = \{ S^{n-1} \underset{\mathbb{Q}}{\otimes} P \longrightarrow D^n \underset{\mathbb{Q}}{\otimes} P \mid P \text{ is in a skel. of the strongly dualizable} \}$$

$$S^{n-1} \quad \begin{array}{cc} & \vdots \\ & 0 \\ & 0 \\ n-1 & \mathbb{Q} \\ & 0 \\ & \vdots \end{array} \longrightarrow \begin{array}{cc} \vdots & \\ 0 & \\ \mathbb{Q} & n \\ \downarrow & \\ \mathbb{Q} & n-1 \\ 0 & \\ \vdots & \end{array} \quad D^n$$

<u>Proof</u>: Smith's theorem.
We are in a Grothendieck category and the H_*-isos are accessible.
— we need $I\text{-inj} \subseteq H_*\text{-isos}$
This follows as there are enough dualizables.

$$d\mathcal{A}(S^1)_{dual} \underset{\longleftarrow}{\xrightarrow{id}} d\mathcal{A}(S^1)_{inj}$$

Joint with Greenlees, Kedziorek, Shipley

$$SO(2)Sp_{\mathbb{Q}} \rightleftarrows \text{cell-}\{T/H_+\} \left(\begin{array}{ccc} & L_{\tilde{E}\mathcal{F}} S^1Sp_{\mathbb{Q}} & \\ & \searrow {-\wedge DE\mathcal{F}_+} & \\ DE\mathcal{F}_+\text{-mod}\ S^1Sp_{\mathbb{Q}} & \xrightarrow{id} & L_{DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F}}\ DE\mathcal{F}_+\text{-mod}\ S^1Sp_{\mathbb{Q}} \end{array} \right)$$

$\boxed{\text{cell principle}}$

$\boxed{\text{Isotropy separation:}}$

$$\begin{array}{ccc} S & \longrightarrow & \tilde{E}\mathcal{F} \\ \downarrow & \lrcorner & \downarrow \\ DE\mathcal{F}_+ & \longrightarrow & DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F} \end{array}$$

in S^1Sp

$$(X, M, K)$$

$$X \to M \leftarrow K \wedge DE\mathcal{F}_+$$

$$Z \longmapsto (DE\mathcal{F}_+ \wedge Z,\ DE\mathcal{F}_+ \wedge Z,\ Z)$$

$$(-)^{!} \rightleftarrows \text{cell-}\left(\begin{array}{ccc} & Sp_{\mathbb{Q}} & \\ & \downarrow {-\wedge DEF_+^{\mathbb{T}}} & \\ DEF_+^{\mathbb{T}}\text{-mod in } Sp_{\mathbb{Q}} & \xrightarrow{id} & L_{(EF \wedge DEF_+)^{\mathbb{T}}} DEF_+^{\mathbb{T}}\text{-mod in } Sp_{\mathbb{Q}} \end{array}\right) \quad \boxed{\text{Shipley}}$$

$\boxed{\text{Fixed point adjunctios}}$

$$\simeq \text{cell-}\left(\begin{array}{ccc} & Ch_{\mathbb{Q}} & \\ & \downarrow & \\ \theta DEF_+^{\mathbb{T}}\text{-mod in } Ch_{\mathbb{Q}} & \longrightarrow & L_S\, \theta DEF_+^{\mathbb{T}}\text{-mod in } Ch_{\mathbb{Q}} \end{array}\right) \quad \boxed{\text{Formality}}$$

$$\text{cell-}\left(\begin{array}{ccc} & & Ch_{\mathbb{Q}} \\ & & \downarrow {-\otimes_{\mathbb{Q}} \mathcal{E}^{-1}\mathcal{O}_F} \\ d\mathcal{O}_F\text{-mod} & \cancel{L_{\mathcal{E}_{\mathbb{Q}}} d\mathcal{O}_{F}\text{-mod}} & \\ & \xrightarrow{\otimes \mathcal{E}^{-1}} & d\,\mathcal{E}^{-1}\mathcal{O}_F\text{-mod} \end{array}\right)$$

$$\uparrow \;\; \downarrow \Gamma$$

$$d\mathcal{A}(s^{-1})_{dual}$$

$$H_*(\theta DEF_+^{\mathbb{T}}) = \pi_*(DEF_+^{\mathbb{T}}) = \pi_*^{\mathbb{T}}(DEF_+) = \mathcal{O}_F$$

$d\mathcal{A}(\mathcal{B}^{-1})_{dual}$