Grothendieck Inequalities—From Classical to Noncommutative

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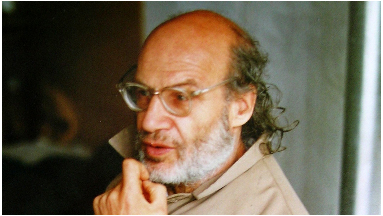
Outline

In 1956 Grothendieck published the celebrated *Résumé de la théorie métrique des produits tensoriels topologiques*, containing a general theory of tensor norms on tensor products of Banach spaces, describing several operations to generate new norms from known ones, and studying the duality theory between these norms.

Since 1968 it has had a major impact first on the development of Banach space theory, and later on, in operator algebras theory (roughly after 1978).



The highlight of the paper, now referred to as *The Résumé* is a result that Grothendieck called *The fundamental theorem on the metric theory of tensor products*, now called *Grothendieck's thm*:

Theorem (Grothendieck 1956). Let K_1, K_2 be compact sets. Let $u : C(K_1) \times C(K_2) \to \mathbb{K}$ be a bounded bilinear form, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then there exist probability measures μ_1 and μ_2 on K_1 and K_2, resp., such that

$$|u(f,g)| \leq K_G^{\mathbb{K}} \|u\| \left(\int_{K_1} |f|^2 \, d\mu_1 \right)^{1/2} \left(\int_{K_2} |g|^2 \, d\mu_2 \right)^{1/2}$$

for all $f \in C(K_1), g \in C(K_2)$, where $K_G^{\mathbb{K}}$ is a universal constant.

Remarks about Grothendieck's constant $K_G^{\mathbb{K}}$:

- $\frac{1}{2}K_G^{\mathbb{R}} \leq K_G^{\mathbb{C}} \leq 2K_G^{\mathbb{R}}$. Exact values still unknown!
- $\frac{\pi}{2} \leq K_G^{\mathbb{R}} \leq \frac{\pi}{2\log(1+\sqrt{2})} = 1.782...$

LHS is due to Grothendieck, while RHS is due to **Krivine (1977)**.

- $\frac{4}{\pi} \leq K_G^{\mathbb{C}} < 1.40491$.

LHS due to Grothendieck, RHS to **Haagerup (1987)**, who showed

$$K_G^{\mathbb{C}} \leq \pi(k_0+1)/8 < 1.40491 .$$

Here k_0 is the unique solution in the interval $[0,1]$ of the equation

$$\phi(k) = \pi(k+1)/8 ,$$

where $\phi(k) := k\int_0^{\pi/2} \frac{\cos^2 t}{\sqrt{1-k^2\sin^2 t}}\,dt$, $-1 \leq k \leq 1$.

The previously known upper bound was obtained by **Pisier (1976)**,

$$K_G^{\mathbb{C}} \leq e^{1-\gamma} \approx 1.52621 \text{, where } \gamma \text{ is Euler's constant.}$$

Little Grothendieck Inequality: Let $T : C(K) \to H$ bounded linear operator, where K is a compact set and H a Hilbert space. Then there exists a probability measure μ on K such that

$$\|T(f)\| \le \sqrt{K_G^{\mathbb{K}}}\,\|T\| \left(\int_K |f|^2\, d\mu \right)^{1/2}, \quad f \in C(K).$$

Proof: Define $u : C(K) \times C(K) \to \mathbb{C}$ by

$$u(f,g) := \langle Tf, T\bar{g} \rangle_H, \quad f, g \in C(K).$$

Then $\|u\| \le \|T\|^2$. By Grothendieck's thm, $\exists$ proba meas. μ_1, μ_2:

$$|u(f,g)| \le K_G^{\mathbb{K}} \|u\| \left(\int_K |f|^2\, d\mu_1 \right)^{1/2} \left(\int_K |g|^2\, d\mu_2 \right)^{1/2}.$$

Set $\mu := (\mu_1 + \mu_2)/2$. Then, for all $f \in C(K)$,

$$\|Tf\|^2 = u(f, \bar{f}) \le K_G^{\mathbb{K}} \|u\| \left(\int_K |f|^2\, d\mu_1 \right)^{1/2} \left(\int_K |f|^2\, d\mu_2 \right)^{1/2}$$

$$\le K_G^{\mathbb{K}} \|u\| \int_K |f|^2\, d\mu \le K_G^{\mathbb{K}} \|T\|^2 \int_K |f|^2\, d\mu. \quad \square$$

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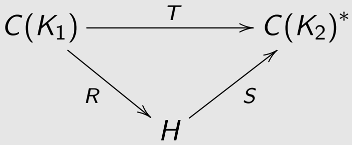
$$\begin{aligned} \|Tf\|^2 = u(f,\bar{f}) &\leq K_G^{\mathbb{K}}\|u\| \left(\int_K |f|^2\, d\mu_1\right)^{1/2} \left(\int_K |f|^2\, d\mu_2\right)^{1/2} \\ &\leq K_G^{\mathbb{K}}\|u\| \int_K |f|^2\, d\mu \leq K_G^{\mathbb{K}}\|T\|^2 \int_K |f|^2\, d\mu\,. \quad \square \end{aligned}$$

The best constants in the Little Grothendieck Inequality are known, namely, $\sqrt{4/\pi}$ (in the complex case) and $\sqrt{\pi/2}$ (in the real case).

Theorem: Any bounded linear operator $T : C(K_1) \to C(K_2)^*$ factors through a Hilbert space H,



such that $\|R\|\|S\| \leq K_G^{\mathbb{K}} \|T\|$.

Proof: Follows from Grothendieck's theorem applied to the bilinear form $u : C(K_1) \times C(K_2) \to \mathbb{C}$ defined by

$$u(f, g) :\, = (Tf)(g) , \quad f \in C(K_1) , g \in C(K_2) . \qquad \square$$

Matrix version of Grothendieck theorem (Lindenstrauss-Pelczynski):

Theorem: Let $a = [a_{ij}] \in M_n(\mathbb{K})$ be such that

$$\Big| \sum_{i,j=1}^{n} a_{ij} x_i y_j \Big| \leq \sup_i |x_i| \cdot \sup_j |y_j|, \quad (x_i), (y_i) \in \mathbb{K}^n.$$

Then, for all Hilbert spaces H (over $\mathbb{K}$),

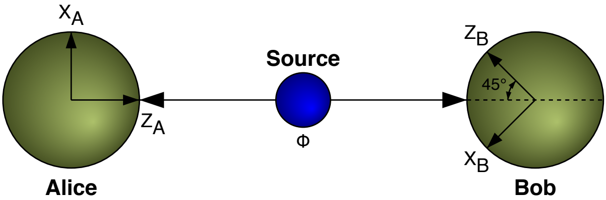
$$\Big| \sum_{i,j=1}^{n} a_{ij} \langle h_i, k_j \rangle \Big| \leq K_G^{\mathbb{K}} \sup_i \|h_i\| \cdot \sup_j \|k_j\|, \quad (h_i), (k_i) \in H^n.$$

We can interpret the $n \times n$ matrix a as a bounded bilinear form $a \colon \ell_n^\infty \times \ell_n^\infty \to \mathbb{K}$, which has norm ≤ 1 by the assumption.

The theorem concludes that norm of the bilinear form $a \colon \ell_n^\infty(H) \times \ell_n^\infty(H) \to \mathbb{K}$ is $\leq K_G^{\mathbb{K}}$.

Outline

Tsirelson: The failure of the EPR suggestion of "hidden variable" can be explained by the fact that $K_G^{\mathbb{R}} > 1$. The Grothendieck constant moreover gives an upper bound for the quantum mechanic *deviation* from the classical picture.



EPR experiment: A source emits in opposite directions two spin 1/2 particles created from one particle of spin 0. Alice and Bob can measure the spin in n different directions, and the possible outcome of a measurement is ± 1. We record the *product* of each measurement. The product is -1 if Alice and Bob measure spin in the same direction. If they measure in different directions, the outcome is no longer deterministic.

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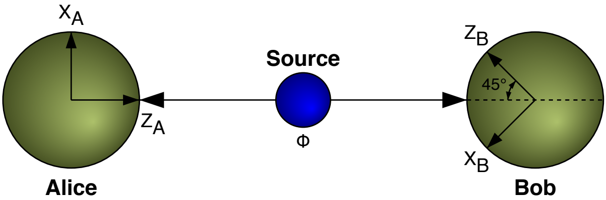

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Hidden variable model: The expected value of each pair of measurements (the covariance matrix) is here given by

$$\xi_{ij} = \int_\Omega A_i(\omega) B_j(\omega)\, d\mu(\omega), \quad i,j = 1,2,\ldots,n,$$

where (Ω, μ) is a proba space and $A_i, B_j \colon \Omega \to \{\pm 1\}$ random var.

Quantum mechanic model: Here

$$\xi_{ij} = \langle A_i B_j \psi, \psi \rangle \quad i,j = 1,2,\ldots,n,$$

where H finite dim Hilbert space, A_i, B_j *commuting* self-adjoint unitaries on H (with spectrum $\{\pm 1\}$), and $\psi \in H$ a unit vector.

Pick $a = [a_{ij}] \in M_n(\mathbb{R})$. We measure $\sum_{i,j} a_{ij}\xi_{ij} \in \mathbb{R}$.

Let $\mathrm{HV}_{\max}(a)$ and $\mathrm{QM}_{\max}(a)$ be $\max \left| \sum_{i,j} a_{ij}\xi_{ij} \right|$, where max is taken over all Ω, A_i, B_j, respectively, over all H, A_i, B_j, ψ.

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Recall: We measure $\sum_{i,j} a_{ij}\xi_{ij} \in \mathbb{R}$, where $a = [a_{ij}] \in M_n(\mathbb{R})$.
Let $\mathrm{EXP}(a)$ be the max found in experiments.

Let $\widetilde{a} = \sum_{i,j} a_{ij}\, e_i \otimes e_j \in \ell_1^n \otimes \ell_1^n$. Then:

$$\mathrm{HV}_{\max}(a) = \sup_{\phi_i=\pm 1, \psi_j=\pm 1} \left| \sum a_{ij}\phi_i\psi_j \right| = \|\widetilde{a}\|_\vee,$$

$$\mathrm{QM}_{\max}(a) = \sup_{A_i, B_j, \psi} \left| \langle A_i B_j \psi, \psi\rangle \right| = \|\widetilde{a}\|_{H'}.$$

By Grothendieck's thm (matrix version), the injective tensor norm $\|\widetilde{a}\|_\vee$ and the dual Hilbert space tensor norm $\|\widetilde{a}\|_{H'}$ satisfy:

$$\|\widetilde{a}\|_\vee \leq \|\widetilde{a}\|_{H'} \leq K_G^{\mathbb{R}} \|\widetilde{a}\|_\vee,$$

and $K_G^{\mathbb{R}} > 1$ is the best constant. Hence

$$\mathrm{HV}_{\max}(a) \leq \mathrm{QM}_{\max}(a) \leq K_G^{\mathbb{R}}\, \mathrm{HV}_{\max}(a).$$

For suitably chosen $a \in M_n(\mathbb{R})$ experiments show that

$$\mathrm{HV}_{\max}(a) < \mathrm{EXP}(a) \approx \mathrm{QM}_{\max}(a) \leq K_G^{\mathbb{R}}\, \mathrm{HV}_{\max}(a).$$

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$$\begin{aligned} \mathrm{HV}_{\max}(a) &= \sup_{\phi_i=\pm1,\psi_j=\pm1} \left|\sum a_{ij}\phi_i\psi_j\right| = \|\widetilde{a}\|_\vee, \\ \mathrm{QM}_{\max}(a) &= \sup_{A_i,B_j,\psi} \left|\langle A_iB_j\psi,\psi\rangle\right| = \|\widetilde{a}\|_{H'}. \end{aligned}$$

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Outline

The Résumé ends with a remarkable list of 6 problems that are linked together and revolve around the question When does a bounded lin operator between Banach spaces factor through a Hilbert space? Among the 6 problems was the famous *Approximation problem*, solved by **Enflo (1972)**, for which he received the promised goose from Mazur!

- The 4th problem in the Résumé was the C*-algebraic version of Grothendieck's theorem, as conjectured by Grothendieck himself. Probability measures on compact spaces are replaced by *states* on C*-algebras (i.e., positive linear functionals of norm 1.)

Conjecture (Grothendieck): Let A be a C*-algebra and let $u : A \times A \to \mathbb{C}$ be a bounded bilinear form. Then there exist $f, g \in S(A)$ such that

$$|u(a,b)| \leq k\|u\| f(|a|^2)^{1/2} g(|b|^2)^{1/2}, a, b \in A,$$

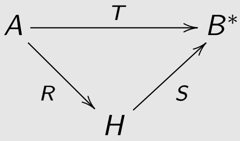
where $|x| := \Big((x^*x + xx^*)/2\Big)^{1/2}$. Here k is a universal constant.

Grothendieck Ineq (Haagerup 85, extension of Pisier 78): Let A and B be C*-algebras, and $u : A \times B \to \mathbb{C}$ a bounded bilin. form. There exist $f_1, f_2 \in S(A)$ and $g_1, g_2 \in S(B)$ such that

$$|u(a,b)| \leq \|u\| \Big(f_1(aa^*) + f_2(a^*a) \Big)^{1/2} \Big(g_1(b^*b) + g_2(bb^*) \Big)^{1/2},$$

for all $a \in A$ and $b \in B$.

Corollary (Haagerup 1985): Any bounded linear operator $T : A \to B^*$, where A and B are C*-algebras, factors through a Hilbert space H,



such that $\|R\|\|S\| \leq 2\|T\|$.

Little Grothendieck's Inequality (Haagerup 1985): Let A be a C*-algebra and H a Hilbert space. If $T : A \to H$ is a bounded linear operator, then there exist $f_1, f_2 \in S(A)$ such that

$$\|Ta\| \leq \|T\| \Big(f_1(a^*a) + f_2(aa^*) \Big)^{1/2}, \quad a \in A.$$

- It was shown by **Haagerup–Itoh (1995)** that constant 1 above is optimal.
- Major impact on operator algebra cohomology.

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Let H be a Hilbert space and $E \subseteq \mathcal{B}(H)$ a closed subspace. Then E becomes an *operator space*, equipped with norms on $M_n(E)$ inherited from $\mathcal{B}(H^n)$, $n \in \mathbb{N}$, via the isometric embeddings

$$M_n(E) \subseteq M_n(\mathcal{B}(H)) = \mathcal{B}(H^n).$$

Note that C*-algebras are operator spaces.

Ruan (1985): an abstract characterization of operator spaces.

Let E, F be operator spaces, $\phi\colon E \to F$ linear, bounded. Consider

$$\phi \otimes \mathrm{Id}_n \colon M_n(E) \to M_n(F), \quad n \in \mathbb{N}.$$

The map ϕ is called *completely bounded* (for short, c.b.) if

$$\|\phi\|_{\mathrm{cb}} := \sup_{n\in\mathbb{N}} \|\phi \otimes \mathrm{Id}_n\| < \infty.$$

ϕ is a *complete isometry* if all ϕ_m are isometries, and a *complete isomorphism* if it is an isomorphism with $\|\phi\|_{cb}, \|\phi^{-1}\|_{cb} < \infty$.

Let $\mathrm{CB}(E,F) := \{\phi : E \to F : \|\phi\|_{\mathrm{cb}} < \infty\}$.

If E is an operator space, then its dual $E^* = \mathcal{B}(E,\mathbb{C}) = \mathrm{CB}(E,\mathbb{C})$, endowed with matrix norms given by

$$M_n(E^*) := \mathrm{CB}(E, M_n(\mathbb{C})), \quad n \geq 1$$

is again an operator space, called the *operator space dual* of E.

- The *predual* $\mathcal{M}_*$ of a vN algebra $\mathcal{M}$ is an operator space with norms inherited from the isometric embedding

$$M_n(\mathcal{M}_*) \subseteq M_n(\mathcal{M}^*): \; = \mathrm{CB}(\mathcal{M}, M_n(\mathbb{C})), \quad n \in \mathbb{N}.$$

- Next we describe two (different) operator space structures on $\ell^2(\mathbb{N})$: the *row Hilbert space R* and the *column Hilbert space C*. Let $e_1, e_2, \ldots$ be the standard unit vector basis in $\ell^2(\mathbb{N})$. For each $n \in \mathbb{N}$, set for all $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in M_n(\mathbb{C})$,

$$\left\| \sum_{i=1}^k x_i \otimes e_i \right\|_{M_n(R)} = \left\| \sum_{i=1}^k x_i x_i^* \right\|^{\frac{1}{2}}, \quad \left\| \sum_{i=1}^k x_i \otimes e_i \right\|_{M_n(C)} = \left\| \sum_{i=1}^k x_i^* x_i \right\|^{\frac{1}{2}}$$

The following simple computations show that R and C are different operator spaces.

Let $x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $x_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{C})$. Then

$$\|x_1 \otimes e_1 + x_2 \otimes e_2\|_{M_2(R)} = \|x_1 x_1^* + x_2 x_2^*\|^{1/2} = \left\| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\|^{1/2} = 1,$$

while

$$\|x_1 \otimes e_1 + x_2 \otimes e_2\|_{M_2(C)} = \|x_1^* x_1 + x_2^* x_2\|^{1/2} = \left\| \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \right\|^{1/2} = \sqrt{2}.$$

Fact: $R^* \cong C$ and $C^* \cong R$ (complete isometries)

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Fact: $R^* \cong C$ and $C^* \cong R$ (complete isometries)

Theorem (Pisier): There exists a unique operator space, called OH, satisfying

(1) OH is isometric to $\ell^2(\mathbb{N})$ (as a Banach space)

(2) The canonical identification between OH and $\overline{OH^*}$ (corresponding to the canonical identification between $\ell^2(\mathbb{N})$ and $\overline{\ell^2(\mathbb{N})^*}$) is a complete isometry.

Moreover, OH is the unique operator space (up to complete isometry) satisfying (1) and (2) .

- For $n \in \mathbb{N}$, $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in M_n(\mathbb{C})$,

$$\left\| \sum_{i=1}^{k} x_i \otimes e_i \right\|_{M_n(OH)} := \left\| \sum_{i=1}^{k} x_i \otimes \overline{x_i} \right\|^{1/2}_{M_n(\mathbb{C}) \otimes \overline{M_n(\mathbb{C})}} .$$

Outline

Let A, B be C*-algebras and $u : A \times B \to \mathbb{C}$ a bounded bilin. form. There exists a unique bounded lin. operator $\widetilde{u} : A \to B^*$ such that

$$u(a, b) = (\widetilde{u}(a))(b), \ \ a \in A, b \in B.$$

The bilinear form u is called *jointly completely bounded* (j.c.b.) if $\widetilde{u} : A \to B^*$ is completely bounded, in which case we set

$$\|u\|_{\mathrm{jcb}} := \|\widetilde{u}\|_{\mathrm{cb}}\,.$$

Conjecture (Effros-Ruan 1991): Let A and B be C*-algebras and let $u : A \times B \to \mathbb{C}$ be a j.c.b. bilinear form. Then there exist $f_1, f_2 \in S(A)$ and $g_1, g_2 \in S(B)$ such that for $a \in A$ and $b \in B$,

$$|u(a,b)| \le K\|u\|_{\text{jcb}} \left(f_1(aa^*)^{\frac{1}{2}} g_1(b^*b)^{\frac{1}{2}} + f_2(a^*a)^{\frac{1}{2}} g_2(bb^*)^{\frac{1}{2}} \right)$$

where K is a universal constant.

Theorem (Haagerup-M., 2008) The Effros-Ruan conjecture holds with $K = 1$, and this is the best possible constant.

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Corollary A: Let A be a C*-algebra. If $T : A \to OH$ is a completely bounded linear map, then there exist $f_1, f_2 \in S(A)$ such that for all $a \in A$,

$$\|T(a)\| \leq \sqrt{2}\|T\|_{cb} f_1(aa^*)^{1/4} f_2(a^*a)^{1/4} .$$

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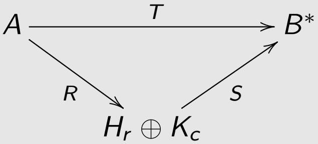
Corollary B: Let A, B be C*-algebras. Every completely bounded linear map $T : A \to B^*$ admits a factorization through $H_r \oplus K_c$, where H and K are Hilbert spaces



with cb maps R and S satisfying $\|R\|_{cb}\|S\|_{cb} \leq 2\|T\|_{cb}$.

- A version of this result, proven by **Junge–Pisier (1995)**, namely that every cb map $u : E \to F^*$, where E and F are operator spaces, factors boundedly through a Hilbert space, was a key ingredient in their proof that $B(H) \otimes_{\max} B(H) \neq B(H) \otimes_{\min} B(H)$.

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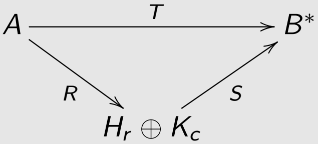

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- The following are noncommutative analogues of the classical isomorphic characterization of Hilbert spaces (also obtained as a consequence of Grothendieck's theorem!):

If X is a Banach space such that both X and its dual X^* embed into L_1-spaces, then X is isomorphic to a Hilbert space.

Corollary C: Let E be an operator space such that E and its dual E^* embed completely isomorphically into preduals $\mathcal{M}_*$ and $\mathcal{N}_*$, resp, of von Neumann alg $\mathcal{M}$ and $\mathcal{N}$. Then E is cb-isomorphic to a quotient of a subspace of $H_r \oplus K_c$, for some Hilbert spaces H and K.

Corollary D: Let E be an operator space, and let $E \subseteq A$ and $E^* \subseteq B$ be completely isometric embeddings into C*-algebras A, B such that both subsp. are cb-complemented. Then E is cb-isomorphic to $H_r \oplus K_c$, for some Hilbert spaces H and K.